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CONVEX FUNCTION AND ITS SECANT

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Abstract. The paper deals with the applications of convex functions to convex and affine combinations. We use combinations that have the common center. As a result of this approach, the Jensen type inequalities are obtained in the discrete and integral form. This issue can also be considered with the functions which are not necessarily convex.

Keywords: Affine combination; Convex combination; Jensen's inequality.

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1. Introduction

In summary form, we present the concept of convexity and affinity by using binomial combinations in a real linear space $\mathscr{X}$.

Let $x,y \in \mathscr{X}$ be points and let $\kappa,\lambda \in \mathbb{R}$ be coefficients. Their binomial combination

$$\kappa x + \lambda y \tag{1}$$

is convex if $\kappa,\lambda \geq 0$ and if $\kappa+\lambda = 1$. If $c = \kappa x + \lambda y$, then the point c itself is called the combination center. A subset of $\mathscr{X}$ is convex if it contains all binomial convex combinations of its points. The convex hull $\operatorname{conv}\mathscr{S}$ of a set $\mathscr{S} \subseteq \mathscr{X}$ is the smallest convex set which contains

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$\mathscr{S}$, and it consists of all binomial convex combinations of points of $\mathscr{S}$. A function f defined on the convex set $\mathscr{C} \subseteq \mathscr{X}$ is convex if the inequality

$$f(\kappa x+\lambda y) \leq \kappa f(x)+\lambda f(y) \tag{2}$$

holds for all binomial convex combinations $\kappa x+\lambda y$ of pairs of points $x, y \in \mathscr{C}$.

Using the adjective affine instead of convex, requiring that the condition $\kappa+\lambda=1$ holds for coefficients, and requiring that the equality holds in equation (2), we get a characterization of the affinity.

Implementing mathematical induction, we can prove that all of the above properties concerning binomial combinations apply to n-membered combinations for any positive integer n.

We present the discrete and integral form of the famous Jensen's inequality. In 1905, applying mathematical induction to the convex combination, Jensen has obtained in [4] the following discrete inequality.

Discrete form of Jensen's inequality. *Let $\mathscr{C}$ be a convex set of a real linear space, and let $\sum_{i=1}^{n} \kappa_i x_i$ be a convex combination of points $x_i \in \mathscr{C}$.*

Then the inequality

$$f\left(\sum_{i=1}^{n} \kappa_i x_i\right) \leq \sum_{i=1}^{n} \kappa_i f(x_i). \tag{3}$$

holds for every convex function $f: \mathscr{C} \rightarrow \mathbb{R}$.

In 1906, working on transition to integrals, Jensen has stated in [5] the another form. Version with the measurable set is as follows.

Integral form of Jensen's inequality. *Let $\mathscr{S}$ be a measurable set of a space of positive measure μ so that $\mu(\mathscr{S})>0$, let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $g: \mathscr{S} \rightarrow \mathbb{R}$ be an integrable function so that $g(\mathscr{S}) \subseteq \mathscr{I}$.*

Then the inequality

$$f\left(\frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} g(x)\, d\mu\right) \leq \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} f(g(x))\, d\mu \tag{4}$$

holds for every convex function $f: \mathscr{I} \rightarrow \mathbb{R}$ such that $f(g)$ is integrable.

The extension of Jensen's inequality to affine combinations, as well as the integral form of that extension, can be found in [12]. Some new Jensen type inequalities were presented in [10].

2. Two Basic Inequalities

The aim of this short section is to repeat the derivation of two basic inequalities concerning a convex function and their secant line by using binomial affine, and especially convex combinations.

In what follows, we will use bounded intervals of real numbers, closed interval $[a,b]$ and open interval (a,b), with endpoints a less than b.

Using endpoints a and b, every number $x \in \mathbb{R}$ can be uniquely represented by the binomial affine combination

$$x = \frac{b-x}{b-a}a + \frac{x-a}{b-a}b. \tag{5}$$

This combination is convex if, and only if, the number x belongs to the closed interval $[a,b]$. Basic inequalities for convex functions can be obtained by applying the representing formula in equation (5).

Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval containing the segment $[a,b]$, let $f : \mathscr{I} \to \mathbb{R}$ be a function, and let $f^{\sec}_{\{a,b\}} : \mathbb{R} \to \mathbb{R}$ be the function of the secant line passing through the graph points $(a, f(a))$ and $(b, f(b))$. Applying the affinity of $f^{\sec}_{\{a,b\}}$ to the affine combination in (5), we get the equation

$$f^{\sec}_{\{a,b\}}(x) = \frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b). \tag{6}$$

If the function f is convex, then we have two basic relations between the values $f(x)$ and $f^{\sec}_{\{a,b\}}(x)$,

$$f(x) \leq f^{\sec}_{\{a,b\}}(x) \ \text{ for } x \in [a,b], \tag{7}$$

and

$$f(x) \geq f^{\sec}_{\{a,b\}}(x) \ \text{ for } x \in \mathscr{I} \setminus (a,b). \tag{8}$$

To obtain the inequality in equation (7) concerning $x \in [a,b]$, we apply the convexity of f to $f(x)$ respecting equation (5), and use equation (6),

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b) = f^{\mathrm{sec}}_{\{a,b\}}(x). \tag{9}$$

To obtain the inequality in equation (7) concerning $x \in \mathscr{I} \setminus (a,b)$, we suppose that $x \leq a$. Applying equation (5) to the point $a \in [x,b]$, we get the convex combination

$$a = \frac{b-a}{b-x} x + \frac{a-x}{b-x} b. \tag{10}$$

Using equation (6), and applying the convexity of f to $f(a)$ respecting the above combination, we get

$$f^{\mathrm{sec}}_{\{a,b\}}(x) \leq \frac{b-x}{b-a} \left(\frac{b-a}{b-x} f(x) + \frac{a-x}{b-x} f(b) \right) + \frac{x-a}{b-a} f(b) = f(x). \tag{11}$$

The case $x \geq b$ is handled in the same way.

The review of fundamental and particular inequalities, as well as the extended concept of convex functions, was done in the book [8].

3. Main results

The main results are related to Jensen's inequality, its generalizations and refinements. As mentioned in the Introduction, Jensen has proved his inequality by applying mathematical induction. On the other hand, we emphasize an importance of the secant and its affinity in proving inequalities whose terms include affine combinations.

The basic statement of this section is the following lemma applicable to many other inequalities.

Lemma 3.1. *Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $f : \mathscr{I} \to \mathbb{R}$ be a convex function. Let $[a,b] \subseteq \mathscr{I}$ be a bounded closed subinterval, let $\sum_{i=1}^{n} \kappa_i x_i$ be a convex combination of points $x_i \in \mathscr{I} \setminus (a,b)$, and let $\alpha a + \beta b$ be the unique affine combination such that*

$$c = \sum_{i=1}^{n} \kappa_i x_i = \alpha a + \beta b. \tag{12}$$

If $c \in [a,b]$, then

$$f\left(\sum_{i=1}^{n} \kappa_i x_i\right) \le \alpha f(a) + \beta f(b) \le \sum_{i=1}^{n} \kappa_i f(x_i). \tag{13}$$

If $c \in \mathscr{I} \setminus (a,b)$, then

$$\alpha f(a) + \beta f(b) \le f\left(\sum_{i=1}^{n} \kappa_i x_i\right) \le \sum_{i=1}^{n} \kappa_i f(x_i). \tag{14}$$

Proof. Let us firstly prove the inequality in equation (13). Relying on equation (7), the affinity of $f^{\sec}_{\{a,b\}}$, and then equation (8), we get the series of inequalities

$$\begin{aligned} f\left(\sum_{i=1}^{n} \kappa_i x_i\right) &\le f^{\sec}_{\{a,b\}}\left(\sum_{i=1}^{n} \kappa_i x_i\right) \\ &= \sum_{i=1}^{n} \kappa_i f^{\sec}_{\{a,b\}}(x_i) \\ &\le \sum_{i=1}^{n} \kappa_i f(x_i). \end{aligned} \tag{15}$$

Extending the above series of inequalities by inserting the equality

$$f^{\sec}_{\{a,b\}}\left(\sum_{i=1}^{n} \kappa_i x_i\right) = f^{\sec}_{\{a,b\}}(\alpha a + \beta b) = \alpha f(a) + \beta f(b), \tag{16}$$

we obtain the series which contains the double inequality in equation (13).

Let us now prove the inequality in equation (14). Using secant's inequality in equation (8) and Jensen's inequality in equation (3), we get

$$\alpha f(a) + \beta f(b) = f^{\sec}_{\{a,b\}}(c) \le f(c) = f\left(\sum_{i=1}^{n} \kappa_i x_i\right) \le \sum_{i=1}^{n} \kappa_i f(x_i) \tag{17}$$

ending the proof. □

The n-membered convex and binomial affine combination in Lemma 3.1 have the common center c. The coefficients of the combination $\alpha a + \beta b$ with the center c are determined with fractions

$$\alpha = \frac{b-c}{b-a}, \beta = \frac{c-a}{b-a} \tag{18}$$

by the representing formula in equation (5). The point c belongs to the interval $\mathscr{I}$ because it is also the center of the convex combination of points in $\mathscr{I}$.

To visually explain the inequalities in equations (13) and (14) by the planar figure, we highlight several points. First we take the graph points $A=(a,f(a))$ and $B=(b,f(b))$. The affine combination

$$S=\alpha A+\beta B \tag{19}$$

belongs to the interval $\mathscr{J}=\{(x,f^{\mathrm{sec}}_{\{a,b\}}(x)):x\in\mathscr{I}\}$ of the secant line through A and B. Then we take the graph points $P_i=(x_i,f(x_i))$, and their convex hull $\mathscr{P}=\operatorname{conv}\{P_1,\dots,P_n\}$, which is the convex polygon inscribed into the epigraph of the function f. The convex combination

$$P=\sum_{i=1}^{n}\kappa_i P_i \tag{20}$$

belongs to the polygon $\mathscr{P}$. The points S, P and $C=(c,f(c))$ have the common abscissa c. Comparing the order of these three points on the line $x=c$, we notice the following two options. The order

$$C\preceq S\preceq P \tag{21}$$

holds for $S\in[A,B]$, and the order

$$S\preceq C\preceq P \tag{22}$$

holds for $S\in\mathscr{J}\setminus(A,B)$. The equations (21)-(22) are equivalent to the equations (13)-(14), and they are visually shown in Figure 1.

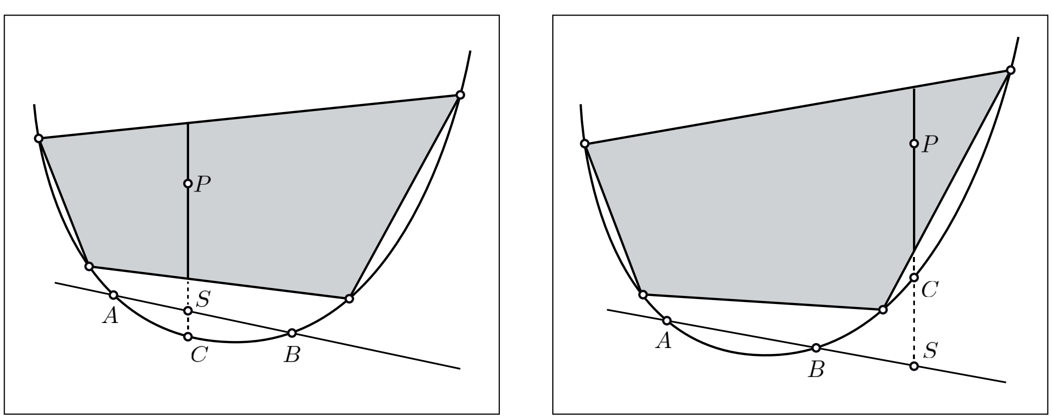

FIGURE 1. Visual presentation of equations (21) and (22)

It is interesting that the Jensen inequality is a consequence of the inequality in equation (13). To demonstrate that, let us take a convex function $f : \mathscr{I} \to \mathbb{R}$ and a convex combination $c = \sum_{i=1}^{n} \kappa_i x_i$ of points $x_i \in \mathscr{I}$. If the set $\{x_1, \dots, x_n\}$ is singleton, then every $x_i = c$, so in this case the trivial equality

$$f(c) = f\left(\sum_{i=1}^{n} \kappa_i x_i\right) = \sum_{i=1}^{n} \kappa_i f(x_i) = f(c) \tag{23}$$

presents the Jensen inequality. If the set $\{x_1, \dots, x_n\}$ is not singleton, then it is equal to the set $\{x_{i_1}, \dots, x_{i_m}\}$ where $x_{i_1} < \dots < x_{i_m}$. Then $c \in [x_{i_j}, x_{i_{j+1}}]$ for some $1 \le j \le m$. Taking $a = x_{i_j}$ and $b = x_{i_{j+1}}$, we have achieved that all $x_i \in \mathscr{I} \setminus (a, b)$, thus we can apply Lemma 3.1 and get the inequality in equation (13) containing the Jensen inequality.

Furthermore, the formula in equation (13) can be used for refinements of Jensen's inequality.

Corollary 3.2. *Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $f : \mathscr{I} \to \mathbb{R}$ be a convex function. Let $[a, b] \subseteq \mathscr{I}$ be a bounded closed subinterval, let $\sum_{i=1}^{n} \kappa_i x_i$ be a convex combination of points $x_i \in \mathscr{I} \setminus (a, b)$, and let $\sum_{j=1}^{m} \lambda_j y_j$ be a convex combination of points $y_j \in [a, b]$. Assume that these two combinations have the common center c.*

Then

$$f\left(\sum_{i=1}^{n} \kappa_i x_i\right) \le \sum_{j=1}^{m} \lambda_j f(y_j) \le \sum_{i=1}^{n} \kappa_i f(x_i). \tag{24}$$

Proof. The point c belongs to $[a, b]$ because it is the center of the convex combination of points $y_j \in [a, b]$. Let $\alpha a + \beta b$ be the unique binomial convex combination whose center falls into the point c. Using Jensen's and chord's inequality, and the right inequality in equation (13), we get the series of inequalities

$$\begin{aligned} f\left(\sum_{i=1}^{n} \kappa_i x_i\right) &= f\left(\sum_{j=1}^{m} \lambda_j y_j\right) \le \sum_{j=1}^{m} \lambda_j f(y_j) \\ &\le \alpha f(a) + \beta f(b) \le \sum_{i=1}^{n} \kappa_i f(x_i) \end{aligned} \tag{25}$$

containing equation (24). □

The inequality in equation (24) can be used to obtain the Hermite-Hadamard inequality. A special case of equation (24), with $x_i \in [a,b]$ instead of y_j, is the inequality

$$f(\alpha a+\beta b) \leq \sum_{i=1}^{n} \kappa_i f(x_i) \leq \alpha f(a)+\beta f(b). \tag{26}$$

In order to get the integral sums, we fix a positive integer n and take the equidistant points $x_{ni} = \big((n-i)/n\big)a + \big(i/n\big)b$ for $i=0,1,\ldots,n$. Then we determine two convex combinations whose centers fall into the same point,

$$c_n = \frac{n-1}{2n}a + \frac{n+1}{2n}b = \sum_{i=1}^{n} \frac{x_{ni}-x_{n\,i-1}}{b-a} x_{ni}. \tag{27}$$

Applying equation (26) to the above combinations, it follows that

$$f\left(\frac{n-1}{2n}a+\frac{n+1}{2n}b\right) \leq \sum_{i=1}^{n} \frac{x_{ni}-x_{n\,i-1}}{b-a} f(x_{ni}) \leq \frac{n-1}{2n}f(a)+\frac{n+1}{2n}f(b),$$

and letting n to infinity, we obtain the Hermite-Hadamard inequality

$$f\left(\frac{1}{2}a+\frac{1}{2}b\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{1}{2}f(a)+\frac{1}{2}f(b). \tag{28}$$

In 1883, studying convex functions, Hermite has attained in [3] the important double inequality in equation (28). In 1893, not knowing Hermite's result, Hadamard has gotten in [2] the left inequality in equation (28). An interesting historical story about the inequality name can be read in [9]. Extension of the Hermite-Hadamard inequality to convex functions of two variables was presented in [6].

Convexity of the function f is not decisive for the validity of inequalities in equations (13) and (14). It is sufficient that the function f is bellow the secant (more precisely, that f is not above the secant) on the interval $[a,b]$, and above the secant (more precisely, that f is not bellow the secant) on the remaining part of the domain.

Corollary 3.3. *Let the assumptions of Lemma be fulfilled with a function* $f : \mathscr{I} \to \mathbb{R}$ *satisfying the inequalities in equations* (7) *and* (8)*, and which need not necessarily be convex.*

Then the function f *satisfies the double inequalities in equations* (13) *and* (14).

The barycenter of a set $\mathscr{S} \subseteq \mathbb{R}$ respecting a measure μ on $\mathbb{R}$ so that $\mu(\mathscr{S}) > 0$ is the point

$$c = \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} x \, d\mu. \tag{29}$$

If $\mathscr{S}$ is the interval, then its barycenter c belongs to $\mathscr{S}$. If g is an integrable function on the set $\mathscr{S}$, then every affine function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equality

$$f\left(\frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} g(x) \, d\mu\right) = \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} f(g(x)) \, d\mu. \tag{30}$$

This equality can be easily verified by using the function equation $f(x) = \kappa x + \lambda$ where κ and λ are real constants.

Theorem 3.4. *Let μ be a positive measure on $\mathbb{R}$. Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $f : \mathscr{I} \to \mathbb{R}$ be an integrable convex function. Let $[a,b] \subseteq \mathscr{I}$ be a bounded closed subinterval so that the set $\mathscr{S} = \mathscr{I} \setminus (a,b)$ has a positive μ-measure, and let $\alpha a + \beta b$ be the unique affine combination such that*

$$c = \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} x \, d\mu = \alpha a + \beta b. \tag{31}$$

If $c \in [a,b]$, then

$$f\left(\frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} x \, d\mu\right) \leq \alpha f(a) + \beta f(b) \leq \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} f(x) \, d\mu. \tag{32}$$

If $c \in \mathscr{I} \setminus (a,b)$, then

$$\alpha f(a) + \beta f(b) \leq f\left(\frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} x \, d\mu\right) \leq \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} f(x) \, d\mu. \tag{33}$$

Proof. The inequalities in (32) and (33) can be verified by applying the procedure of proving Lemma 3.1. The key formula is

$$f^{\mathrm{sec}}_{\{a,b\}}\left(\frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} x \, d\mu\right) = \frac{1}{\mu(\mathscr{S})} \int_{\mathscr{S}} f^{\mathrm{sec}}_{\{a,b\}}(x) \, d\mu \tag{34}$$

based on the affinity of $f^{\mathrm{sec}}_{\{a,b\}}$ and equation (30). $\square$

Connections between convex combination centers and set barycenters were also discussed in [11].

Corollary 3.5. *Let the assumptions of Theorem 3.4 be fulfilled with an integrable function $f : \mathscr{I} \to \mathbb{R}$ satisfying the inequalities in equations* (7) *and* (8)*, and which need not necessarily be convex.*

Then the function f satisfies the double inequalities in equations (32) *and* (33).

4. Applications to Means

The applications of the function convexity and concavity can be realized by using quasi-arithmetic means.

Let $\sum_{i=1}^{n} \kappa_i x_i$ be a convex combination of points $x_i \in \mathscr{I}$, and let $\varphi : \mathscr{I} \to \mathbb{R}$ be a strictly monotone continuous function. The quasi-arithmetic mean of the above convex combination respecting the function φ is the point defined by

$$M_\varphi(x_i, \kappa_i) = \varphi^{-1}\left(\sum_{i=1}^{n} \kappa_i \varphi(x_i)\right), \tag{35}$$

where the abbreviation $M_\varphi(x_i, \kappa_i)$ replaces the full mark $M_\varphi(x_1, \dots, x_n, \kappa_1, \dots, \kappa_n)$.

The term in parentheses belongs to the interval $\varphi(\mathscr{I})$, and therefore the quasi-arithmetic mean $M_\varphi(x_i, \kappa_i)$ belongs to the interval $\mathscr{I}$. It is well known that quasi-arithmetic means satisfy the invariant property

$$M_\varphi(x_i, \kappa_i) = M_{\kappa\varphi+\lambda}(x_i, \kappa_i) \tag{36}$$

where $\kappa \neq 0$ and λ are real constants.

Quasi-arithmetic means may also be applied to affine combinations, in the case where the combination center belongs to the interval $\mathscr{I}$.

The order of two quasi-arithmetic means M_φ and M_ψ can be determined if the composite function $\psi(\varphi^{-1})$ is convex or concave. We say that the function ψ is φ-convex (respectively φ-concave) if the composite function $\psi(\varphi^{-1})$ is convex (respectively concave).

In the next theorem, we will use the quasi-arithmetic means for convex and affine combinations of Lemma 3.1.

Theorem 4.1. *Let $\mathscr{I} \subseteq \mathbb{R}$ be an interval, and let $\varphi, \psi : \mathscr{I} \to \mathbb{R}$ be strictly monotone continuous functions such that ψ is φ-convex and increasing. Let $[a,b] \subseteq \mathscr{I}$ be a bounded closed subinterval, let $\sum_{i=1}^{n} \kappa_i x_i$ be a convex combination of points $x_i \in \mathscr{I} \setminus (a,b)$, and let $\alpha a + \beta b$ be the unique affine combination such that*

$$c = M_{\varphi}(x_i, \kappa_i) = M_{\varphi}(a, b, \alpha, \beta). \tag{37}$$

If $c \in [a,b]$, then

$$M_{\varphi}(x_i, \kappa_i) \le M_{\psi}(a, b, \alpha, \beta) \le M_{\psi}(x_i, \kappa_i). \tag{38}$$

If $c \in \mathscr{I} \setminus (a,b)$, then

$$M_{\psi}(a, b, \alpha, \beta) \le M_{\varphi}(x_i, \kappa_i) \le M_{\psi}(x_i, \kappa_i). \tag{39}$$

Proof. By acting with the function φ to the equality in equation (37), it follows that

$$\varphi(c) = \sum_{i=1}^{n} \kappa_i \varphi(x_i) = \alpha \varphi(a) + \beta \varphi(b). \tag{40}$$

Let us prove the inequality in equation (38) assuming that $c \in [a,b]$. Applying the inequality in equation (13) to the points $\varphi(x_i)$ and the convex function $f = \psi(\varphi^{-1}) : \varphi(\mathscr{I}) \to \mathbb{R}$, we get

$$f\left(\sum_{i=1}^{n} \kappa_i \varphi(x_i)\right) \le \alpha f(\varphi(a)) + \beta f(\varphi(b)) \le \sum_{i=1}^{n} \kappa_i f(\varphi(x_i)). \tag{41}$$

Acting with the increasing function ψ^{-1} to the above inequality, and taking into account that $f(\varphi(a)) = \psi(a)$, $f(\varphi(b)) = \psi(b)$ and $f(\varphi(x_i)) = \psi(x_i)$, we get the inequality

$$\varphi^{-1}\left(\sum_{i=1}^{n} \kappa_i \varphi(x_i)\right) \le \psi^{-1}\big(\alpha \psi(a)) + \beta \psi(b)\big) \le \psi^{-1}\left(\sum_{i=1}^{n} \kappa_i \psi(x_i)\right) \tag{42}$$

representing those required in equation (38).

In a similar way, we can prove the inequality in equation (39). □

In the above theorem, we discussed the case that the function ψ is φ-convex and increasing. Relying on this case, we can easily prove the following cases.

Corollary 4.2. *Let φ and ψ be functions satisfying the conditions of Theorem 4.1.*

If either ψ is φ-convex and increasing or φ-concave and decreasing, then the inequalities in equations (38) *and* (39) *are valid.*

If either ψ is φ-convex and decreasing or φ-concave and increasing, then the reverse inequalities in equations (38) *and* (39) *are valid.*

A special case of the quasi-arithmetic mean in equation (35) are power means depending on real exponents r. Precisely, we use the functions

$$\varphi_r(x) = \begin{cases} x^r & , \ r \neq 0 \\ \ln x & , \ r = 0 \end{cases} \tag{43}$$

where the variable x takes on only positive values. The power mean of order r of the convex combination $\sum_{i=1}^{n} \kappa_i x_i$ of points $x_i \in (0,\infty)$ is the point

$$M_r(x_i,\kappa_i) = \begin{cases} \left(\sum_{i=1}^{n} \kappa_i x_i^r\right)^{\frac{1}{r}} & , \ r \neq 0 \\ \exp\left(\sum_{i=1}^{n} \kappa_i \ln x_i\right) & , \ r = 0 \end{cases} \tag{44}$$

where the abbreviation $M_r(x_i,\kappa_i)$ replaces the full mark $M_r(x_1,\ldots,x_n,\kappa_1,\ldots,\kappa_n)$. In order to facilitate the application of power means, one can use the formula

$$M_{-r}(x_i^{-1},\kappa_i) = \big(M_r(x_i,\kappa_i)\big)^{-1}. \tag{45}$$

The application of Theorem 4.1 to power means is as follows.

Corollary 4.3. *Let $\mathscr{I} = (0,\infty)$ be the interval of positive real numbers, and let $r \le s$ be real numbers. Let $[a,b] \subset \mathscr{I}$ be a bounded closed subinterval, let $\sum_{i=1}^{n} \kappa_i x_i$ be a convex combination of points $x_i \in \mathscr{I} \setminus (a,b)$, and let $\alpha a + \beta b$ be the unique affine combination such that*

$$c = M_r(x_i,\kappa_i) = M_r(a,b,\alpha,\beta). \tag{46}$$

If $c \in [a,b]$, then

$$M_r(x_i,\kappa_i) \le M_s(a,b,\alpha,\beta) \le M_s(x_i,\kappa_i). \tag{47}$$

If $c \in \mathscr{I} \setminus (a,b)$, then

$$M_s(a,b,\alpha,\beta) \le M_r(x_i,\kappa_i) \le M_s(x_i,\kappa_i). \tag{48}$$

Proof. The proof of inequalities in (47) and (48) takes place through several cases depending on functions $\varphi = \varphi_r$ and $\psi = \varphi_s$.

In the case $0 < r \leq s$, using functions $\varphi(x) = x^r$ and $\psi(x) = x^s$, we have the composite function $\psi(\varphi^{-1}(x)) = x^{s/r}$ which is convex because $s/r \geq 1$. The function ψ is φ-convex and increasing, and the required inequalities in equations (47) and (48) follow from Theorem 4.1.

In the case $r \leq s < 0$, the function ψ is φ-concave and decreasing, and the required inequalities follow from Corollary 4.2.

In the case $0 = r < s$, using functions $\varphi(x) = \ln x$ and $\psi(x) = x^s$, we have the composite function $\psi(\varphi^{-1}(x)) = e^{sx}$ which is convex. The function ψ is φ-convex and increasing, and the required inequalities follow from Theorem 4.1.

In the case $r < s = 0$, the function ψ is φ-convex and increasing, and the required inequalities follow from Theorem 4.1. $\square$

At the end, we apply Corollary 4.3 to the harmonic-geometric ($r = -1$, $s = 0$) and geometric-arithmetic ($r = 0$, $s = 1$) mean inequality.

In terms of the harmonic-geometric mean inequality, assuming that

$$c = \sum_{i=1}^{n} \kappa_i x_i^{-1} = \alpha a^{-1} + \beta b^{-1}, \tag{49}$$

it follows that the inequality

$$\left(\sum_{i=1}^{n} \kappa_i x_i^{-1} \right)^{-1} \leq a^{\alpha} b^{\beta} \leq \prod_{i=1}^{n} x_i^{\kappa_i} \tag{50}$$

holds for $c \in [a,b]$, and that the inequality

$$a^{\alpha} b^{\beta} \leq \left(\sum_{i=1}^{n} \kappa_i x_i^{-1} \right)^{-1} \leq \prod_{i=1}^{n} x_i^{\kappa_i} \tag{51}$$

holds for $c \in \mathscr{I} \setminus (a,b)$.

Regarding the geometric-arithmetic mean inequality, assuming that

$$c = \prod_{i=1}^{n} x_i^{\kappa_i} = a^{\alpha} b^{\beta}, \tag{52}$$

it follows that the inequality

$$\prod_{i=1}^{n} x_i^{\kappa_i} \leq \alpha a + \beta b \leq \sum_{i=1}^{n} \kappa_i x_i \tag{53}$$

holds for $c \in [a,b]$, and that the inequality

$$\alpha a + \beta b \leq \prod_{i=1}^{n} x_i^{\kappa_i} \leq \sum_{i=1}^{n} \kappa_i x_i \tag{54}$$

holds for $c \in \mathscr{I} \setminus (a,b)$.

A good approach to the theory of means can be found in the book [1]. Applications of Jensen's inequality to different forms of quasi-arithmetic means, as well as their refinements, were considered in the paper [7].

Further Research

Further research of the problem considered in this paper may be continued with the functions of several variables. The basic problem is, what the bounded interval of real numbers should be replaced with in higher dimensional Euclidean spaces. To continue studying, we can implement the simplexes (simplices) as the smallest convex sets.

Conflict of Interests

The author declares that there is no conflict of interests.

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