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ON SOME EQUALITIES AND INEQUALITIES OF CONTINUOUS G-FRAMES IN HILBERT C*-MODULES

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Abstract. Continuous g-frame in Hilbert C*-module is a generalization of g-frame in Hilbert C*- module. This generalization is a natural generalization of continuous and discrete g-frame and frame in Hilbert space too. Some equalities and inequalities of continuous g-frames in Hilbert C*-module has been verified. In this paper we present some others equalities and inequalities of continuous g-frames in Hilbert C*-module. These equalities and inequalities are generalization of some equalities and inequalities in frame and g-frame in Hilbert C*-modules and Hilbert space. Also continuous g-frame identity in Hilbert C*- modules is discussed.

Keywords: frame; g-frame; continuous g-frame; Hilbert C*-module.

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1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [8] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies, Grossmann and Meyer [7], and popularized from then on. For basic results on frames, see [4, 5, 6, 11]. If H be a Hilbert space, and I a set which is finite or countable. A system

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$\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \tag{1}$$

for all $f \in H$. The constants A and B are called frame bounds. If $A = B$ we call this frame a tight frame and if A = B =1 it is called a Parseval frame.

Wenchang Sun [16] introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context.

In other hand, the concept of frames especially the g-frames was introduced in Hilbert C*-modules, and some of their properties were investigated in [9, 12, 13]. Frank and Larson [9] defined the standard frames in Hilbert C*-modules in 1998 and got a series of result for standard frames in finitely or countably generated Hilbert C*-modules over unital C*-algebras. As for Hilbert C*-module, it is a generalizations of Hilbert spaces in that it allows the inner product to take values in a C*-algebra rather than the field of complex numbers. There are many differences between Hilbert C*-modules and Hilbert spaces. For example, we know that any closed subspaces in a Hilbert space has an orthogonal complement, but it is not true for Hilbert C*-module. And we can't get the analogue of the Riesz representation theorem of continuous functionals in Hilbert C*-modules generally. Thus it is more difficult to make a discussion of the theory of Hilbert C*-modules than that of Hilbert spaces in general. We refer readers to [14] for more details on Hilbert C*-modules. In [13] and [17], the authors made a discussion of some properties of g-frame in Hilbert C*-module in some aspects.

The concept of a generalization of frames to a family indexed by some locally compact space endowed with a Radon measure was proposed by G. Kaiser [11] and independently by Ali, Antoine and Gazeau [1]. These frames are known as continuous frames.

In this paper, we mainly derive several equalities and inequalities of continuous g-frames in Hilbert C*-modules. The continuous g-frames in Hilbert C*-modules were firstly proposed by Rashidi and Nazari in [15], which are an extension to g-frames and continuous frames in Hilbert space and Hilbert C*-modules. Equalities on discrete frames appeared originally in [3] just when the authors studied the optimal decomposition of a Parseval frame in a Hilbert space.

Later on, Găvruta in [10] developed some inequalities about discrete frames on the basis of the work in [3] of Balan et al.

The paper is organized as follows.In Sections 2 we recall the basic definitions and some notations about continuous g-frames in Hilbert C*-module, we also give some basic properties of g-frames which we will use in the later sections. In Section 3, we extend some important equalities and inequalities of frame in Hilbert spaces to continuous frames and continuous g-frames in Hilbert C*-modules.

2. **Preliminaries**

In the following we review some definitions and basic properties of Hilbert C*-modules and g-frames in Hilbert C*-module, we first introduce the definition of Hilbert C*-modules.

Definition 2.1. *Let A be a C*-algebra with involution * . An inner product A-module (or pre Hilbert A-module) is a complex linear space H which is a left A-module with map* $\langle .,. \rangle : H \times H \to A$ *which satisfies the following properties:*
1) $\langle \alpha f + \beta g, h \rangle = \alpha < f,h > + \beta < g,h >$ *for all* $f,g,h \in H$ *and* $\alpha, \beta \in \mathbf{C}$*;*
2) $\langle af, g \rangle = a\langle f, g \rangle$ *for all* $f,g \in H$ *and* $a \in A$*;*
3) $\langle f, g \rangle = \langle g, f \rangle^*$ *for all* $f,g \in H$*;*
4) $\langle f, f \rangle \geq 0$ *for all* $f \in H$ *and* $\langle f, f \rangle = 0$ *iff* $f = 0$*.*
For $f \in H$*,we define a norm on H by* $\|f\|_H = \|\langle f, f \rangle\|_A^{1/2}$*. If H is complete with this norm, it is called a Hilbert C*-module over A or a Hilbert A-module.*

An element a of a C*-algebra A is positive if a* = a and spectrum of a is a subset of positive real number. We write $a \geq 0$ to mean that a is positive. It is easy to see that $\langle f, f \rangle \geq 0$ for every $f \in H$, hence we define $|f| = \langle f, f \rangle^{1/2}$.

Frank and Larson in [9] defined the standard frames in Hilbert C*-modules. If H be a Hilbert C*-module, and I a set which is finite or countable. A system $\{f_i\}_{i \in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\langle f,f\rangle \leq \sum_{i\in I}\langle f,f_i\rangle\langle f_i,f\rangle \leq B\langle f,f\rangle \tag{2}$$

for all $f \in H$. The constants A and B are called frame bounds.

Khosravi and Khosravi in [13] defined g-frame in Hilbert C*-module. Let U and V be two Hilbert C*-module and $\{V_i : i \in I\}$ is a sequence of subspaces of V, where I is a subset of Z and $End_A^*(U,V_i)$ is the collection of all adjointable A-linear maps from U into V_i i.e. $\langle Tf,g\rangle = \langle f,T^*g\rangle$ for all $f,g \in H$ and $T \in End_A^*(U,V_i)$. We call a sequence $\{\Lambda_i \in End_A^*(U,V_i) : i \in I\}$ a generalized frame, or simply a g-frame, for Hilbert C*-module U with respect to $\{V_i : i \in I\}$ if there are two positive constants A and B such that

$$A\langle f,f\rangle \leq \sum_{i\in I}\langle \Lambda_i f,\Lambda_i f\rangle \leq B\langle f,f\rangle \tag{3}$$

for all $f \in U$. The constants A and B are called g-frame bounds. If $A = B$ we call this g-frame a tight g-frame and if A = B =1 it is called a Parseval g-frame.

Let $(M;\mathscr{S};\mu)$ be a measure space, U and V be two Hilbert C*-module, $\{V_m : m \in M\}$ is a sequence of subspaces of V and $End_A^*(U,V_m)$ is the collection of all adjointable A-linear maps from U into V_m.

Definition 2.2. *(see* [15]*) We call a net* $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ *a continuous generalized frame, or simply a continuous g-frame, for Hilbert C*-module U with respect to* $\{V_m : m \in M\}$ *if:*

(a) for any $f \in U$*, the function* $\tilde{f} : M \to V_m$*defined by* $\tilde{f}(m) = \Lambda_m f$ *is measurable,*

(b) there is a pair of constants $0 < A,B$ *such that, for any* $f \in U$*,*

$$A\langle f,f\rangle \leq \int_M \langle \Lambda_m f,\Lambda_m f\rangle d\mu(m) \leq B\langle f,f\rangle. \tag{4}$$

The constants A and B are called continuous g-frame bounds. If $A = B$ *we call this continuous g-frame a continuous tight g-frame and if A = B =1 it is called a continuous Parseval g-frame. If only the right-hand inequality of (6) is satisfied, we call* $\{\Lambda_m : m \in M\}$ *the continuous g-Bessel for U with respect to* $\{V_m : m \in M\}$ *with Bessel bound B.*

If $M=\mathbf{N}$ and μ is the counting measure, the continuous g-frame for U with respect to $\{V_m : m\in M\}$ is a g-frame for U with respect to $\{V_m : m\in M\}$.

Let $\{\Lambda_m : m\in M\}$ be a continuous g-frame for U with respect to $\{V_m : m\in M\}$. Define the continuous g-frame operator S on U by

$$Sf=\int_M \Lambda_m^*\Lambda_m f d\mu(m).$$

Proposition 2.3. *(see* [15]*) The frame operator S is a bounded, positive, self-adjoint, and invertible.*

Proposition 2.4. *(see* [15]*) Let $\{\Lambda_m : m\in M\}$ be a continuous g-frame for U with respect to $\{V_m : m\in M\}$ with continuous g-frame operator S with bounds A and B. Then $\{\tilde{\Lambda}_m : m\in M\}$ defined by $\tilde{\Lambda}_m=\Lambda_m S^{-1}$ is a continuous g-frame for U with respect to $\{V_m : m\in M\}$ with continuous g-frame operator S^{-1} with bounds $1/B$ and $1/A$.That is called continuous canonical dual g-frame of $\{\Lambda_m : m\in M\}$.*

Next Theorem show that the continuous g-frame is equivalent to which the middle of (2.3) is norm bounded. The reader can find proof in [15].

Theorem 2.5. *Let $\Lambda_m\in End_A^*(U,V_m)$ for any $m\in M$. Then $\{\Lambda_m : m\in M\}$ is a continuous g-frame for U with respect to $\{V_m : m\in M\}$ if and only if there exist constants $A,B>0$ such that for any $f\in U$*

$$A\|f\|^2\le\left\|\int_M\langle\Lambda_m f,\Lambda_m f\rangle d\mu(m)\right\|\le B\|f\|^2. \tag{5}$$

We define

$$\bigoplus_{m\in M}V_m=\left\{g=\{g_m\} : g_m\in V_m \text{ and } \left\|\int_M |g_m|^2 d\mu(m)\right\|<\infty\right\}.$$

For any $f=\{f_m : m\in M\}$ and $g=\{g_m : m\in M\}$, if the A-valued inner product is defined by $\langle f,g\rangle=\int_M\langle f_m,g_m\rangle d\mu(m)$, the norm is defined by $\|f\|=\|\langle f,f\rangle\|^{\frac{1}{2}}$, then $\bigoplus_{m\in M}V_m$ is a Hilbert A-module (see [14]).

Let $\{\Lambda_m\in End_A^*(U,V_m) : m\in M\}$ is a continuous g-frame for U with respect to $\{V_m : m\in M\}$, we define synthesis operator $T:\bigoplus_{m\in M}V_m\to U$ by, $T(g)=\int_M\Lambda_m^* g_m d\mu(m)$ for all $g=\{g_m :$

$m \in M\} \in \bigoplus_{m\in M} V_m$. So analysis operator is defined for map $F : U \to \bigoplus_{m\in M} V_m$ by $F(f) = \{\Lambda_m : m \in M\}$ for any $f \in U$.

3. Main Results

Some equalities for frames involving the real parts of some complex numbers have been established in [10]. These equalities generalized in [17] for g-frames in Hilbert C*-modules. In [15] there are some generalizations in a more general setting. In this section we continue that work and get some others equalities and inequalities in a different setting.

In [2], the authors verified a longstanding conjecture of the signal processing community: a signal can be reconstructed without information about the phase. While working on efficient algorithms for signal reconstruction, the authors of [3] established the remarkable Parseval frame equality given below.

Theorem 3.1. *If $\{f_j : j \in J\}$ is a Parseval frame for Hilbert space H, then for any $K \subset J$ and $f \in H$, we have*

$$\sum_{j\in K} |\langle f, f_j\rangle|^2 - \left\| \sum_{j\in K} \langle f, f_j\rangle f_j \right\|^2 = \sum_{j\in K^c} |\langle f, f_j\rangle|^2 - \left\| \sum_{j\in K^c} \langle f, f_j\rangle f_j \right\|^2. \tag{6}$$

Theorem 3.1 was generalized to alternate dual frames [10]. If $\{f_j : j \in J\}$ is a frame, then frame $\{g_j : j \in J\}$ is called alternate dual frame of $\{f_j : j \in J\}$ if for any $f \in H$, $f = \sum_{j\in J} \langle f, g_j\rangle f_j$.

Theorem 3.2. *If $\{f_j : j \in J\}$ is a frame for Hilbert space H and $\{g_j : j \in J\}$ is an alternate dual frame of $\{f_j : j \in J\}$, then for any $K \subset J$ and $f \in H$, we have*

$$Re\left(\sum_{j\in K} \langle f, g_j\rangle \overline{\langle f, f_j\rangle}\right) - \left\| \sum_{j\in K} \langle f, g_j\rangle f_j \right\|^2 = Re\left(\sum_{j\in K^c} \langle f, g_j\rangle \overline{\langle f, f_j\rangle}\right) - \left\| \sum_{j\in K^c} \langle f, g_j\rangle f_j \right\|^2. \tag{7}$$

X. Zhu, G. Wuthe in [18] generalized equality (3.2) to a more general form which does not involve the real parts of the complex numbers.

Theorem 3.3. *If $\{f_j : j \in J\}$ is a frame for Hilbert space H and $\{g_j : j \in J\}$ is an alternate dual frame of $\{f_j : j \in J\}$, then for any $K \subset J$ and $f \in H$, we have*

$$\left(\sum_{j\in K}\langle f,g_j\rangle\overline{\langle f,f_j\rangle}\right)-\left\|\sum_{j\in K}\langle f,g_j\rangle f_j\right\|^2=\overline{\left(\sum_{j\in K^c}\langle f,g_j\rangle\overline{\langle f,f_j\rangle}\right)}-\left\|\sum_{j\in K^c}\langle f,g_j\rangle f_j\right\|^2. \tag{8}$$

Now, we extended this equality to continuous g-frames and g-frames in Hilbert C*-modules and Hilbert spaces. Let H be a Hilbert C*-module. If $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ be a continuous g-frame for U with respect to $\{V_m : m \in M\}$, then continuous g-frame $\{\Gamma_m \in End_A^*(U,V_m) : m \in M\}$ is called alternate dual continuous g-frame of $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ if for any $f \in H$, $f = \int_M \Lambda_m^*\Gamma_m f d\mu(m)$.

Lemma 3.4. *(See* [17]*) Let H be a Hilbert C*-module. If $P,Q \in End_A^*(H,H)$ are two bounded A-linear operators in H and $P+Q = I_H$, then we have*

$$P - P^*P = Q^* - Q^*Q.$$

Now, we present main theorem of this section. In following, some result of this theorem for the discrete case will be present.

Theorem 3.5. *Let $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ be a continuous g-frame, for Hilbert C*-module U with respect to $\{V_m : m \in M\}$ and continuous g-frame $\{\Gamma_m \in End_A^*(U,V_m) : m \in M\}$ is alternate dual continuous g-frame of $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in H$, we have*

$$\int_K \langle\Lambda_m^*\Gamma_m f,f\rangle d\mu(m) - \left|\int_K \Lambda_m^*\Gamma_m f d\mu(m)\right|^2 = \left(\int_{K^c} \langle\Lambda_m^*\Gamma_m f,f\rangle d\mu(m)\right)^* - \left|\int_{K^c} \Lambda_m^*\Gamma_m f d\mu(m)\right|^2.$$

Proof. For any measurable subset $K \subset M$, let the operator U_K be defined for any $f \in H$ by $U_K f = \int_K \Lambda_m^*\Gamma_m f d\mu(m)$.

Then it is easy to prove that the operator U_K is well defined and the integral $\int_K \Lambda_m^*\Gamma_m f d\mu(m)$ it's finite. By definition alternate dual continuous g-frame $U_K + U_{K^c} = 1$. Thus, by Lemma 5.4

we have

$$\begin{aligned}
&\int_K \langle \Lambda_m^* \Gamma_m f, f\rangle d\mu(m) - \left| \int_K \Lambda_m^* \Gamma_m f d\mu(m) \right|^2 \\
&\qquad = \int_K \langle \Lambda_m^* \Gamma_m f, f\rangle \, d\mu(m) - \langle U_K f, U_K f\rangle \\
&\qquad = \langle U_K f, f\rangle - \langle U_K^* U_K f, f\rangle = \langle U_{K^c}^* f, f\rangle - \langle U_{K^c}^* U_{K^c} f, f\rangle \\
&\qquad = \langle f, U_{K^c} f\rangle - \langle U_{K^c} f, U_{K^c} f\rangle \\
&\qquad = \left(\int_{K^c} \langle \Lambda_m^* \Gamma_m f, f\rangle d\mu(m) \right)^* - \left| \int_{K^c} \Lambda_m^* \Gamma_m f d\mu(m) \right|^2 .
\end{aligned}$$

Hence the theorem holds. The proof is completed.

Corollary 3.6. *Let $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$ be a discrete g-frame, for Hilbert C*-module U with respect to $\{V_j : j \in J\}$ and discrete g-frame $\{\Gamma_j \in End_A^*(U, V_j) : j \in J\}$ is alternate dual discrete g-frame of $\{\Lambda_j \in End_A^*(U, V_j) : j \in J\}$, then for any subset $K \subset J$ and $f \in H$, we have*

$$\sum_{j\in K} \langle \Lambda_m^* \Gamma_m f, f\rangle - \left| \sum_{j\in K} \Lambda_m^* \Gamma_m f \right|^2 = \left(\sum_{j\in K^c} \langle \Lambda_m^* \Gamma_m f, f\rangle \right)^* - \left| \sum_{j\in K^c} \Lambda_m^* \Gamma_m f \right|^2 .$$

Corollary 3.7. *Let $\{\Lambda_j \in L(U, V_j) : j \in J\}$ be a g-frame, for Hilbert space U with respect to $\{V_j : j \in J\}$ and g-frame $\{\Gamma_j \in L(U, V_j) : j \in J\}$ is alternate dual g-frame of $\{\Lambda_j \in L(U, V_j) : j \in J\}$, then for any measurable subset $K \subset J$ and $f \in H$, we have*

$$\sum_{j\in K} \langle \Lambda_m^* \Gamma_m f, f\rangle - \left| \sum_{j\in K} \Lambda_m^* \Gamma_m f \right|^2 = \overline{\left(\sum_{j\in K^c} \langle \Lambda_m^* \Gamma_m f, f\rangle \right)} - \left| \sum_{j\in K^c} \Lambda_m^* \Gamma_m f \right|^2 .$$

Corollary 3.8. *Let $\{\Lambda_m \in L(U, V_m) : m \in M\}$ be a continuous g-frame, for Hilbert space U with respect to $\{V_m : m \in M\}$ and continuous g-frame $\{\Gamma_m \in L(U, V_m) : m \in M\}$ is alternate dual continuous g-frame of $\{\Lambda_m \in L(U, V_m) : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in U$, we have*

$$\int_K \langle \Lambda_m^* \Gamma_m f, f\rangle d\mu(m) - \left| \int_K \Lambda_m^* \Gamma_m f d\mu(m) \right|^2 = \overline{\left(\int_{K^c} \langle \Lambda_m^* \Gamma_m f, f\rangle d\mu(m) \right)} - \left| \int_{K^c} \Lambda_m^* \Gamma_m f d\mu(m) \right|^2 .$$

Corollary 3.9. *Let $\{f_m \in H : m \in M\}$ be a continuous frame for Hilbert C*-module H and continuous frame $\{g_m \in H : m \in M\}$ is alternate dual continuous frame of $\{f_m \in H : m \in M\}$,* *then for any measurable subset $K \subset M$ and $f \in H$, we have*

(9)
$$\int_K \langle f, g_m\rangle \langle f, f_m\rangle^* d\mu(m) - \left|\int_K \langle f, g_m\rangle f_m d\mu(m)\right|^2 = \left(\int_{K^c} \langle f, g_m\rangle \langle f, f_m\rangle^* d\mu(m)\right)^* - \left|\int_{j\in K^c} \langle f, g_m\rangle f_m d\mu(m)\right|^2.$$

Theorem 3.10. *(see* [15]*) Let $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ be a continuous λ-tight g-frame, for Hilbert C*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $K, L \subset M, K \bigcap L = \phi$ and $f \in U$, we have*

$$|S_{K\bigcup L} f|^2 - |S_{K^c\setminus L} f|^2 = |S_K f|^2 - |S_{K^c} f|^2 + 2\lambda \int_L \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m). \tag{10}$$

Corollary 3.11. *Let $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ be a continuous λ-tight g-frame, for Hilbert C*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $M_i \subset M, 1 \le i \le N$, where $N \ge 2$ is a positive integer, and $M_i \bigcap M_j = \phi$ for $i \neq j$, $M = \bigcup_{i=1}^N M_i$ and $f \in U$, we have*

$$\begin{aligned}|S_{(\cup_{i=N_1}^{N_4} M_i)} f|^2 + |S_{(\cup_{i=N_2}^{N_3} M_i)^c} f|^2 =& |S_{(\cup_{i=N_2}^{N_3} M_i)} f|^2 + |S_{(\cup_{i=1}^{N_1-1} M_i \bigcup \cup_{i=N_4+1}^{N} M_i)} f|^2 \\ &+ 2\lambda \int_{(\cup_{i=N_1}^{N_2-1} M_i \bigcup \cup_{i=N_3+1}^{N_4} M_i)} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m),\end{aligned}$$

where $N_i, 1 \le i \le 4$ are positive integers satisfying $1 \le N_1 \le N_2 < N_3 < N_4 \le N-1$.

Proof. In (3.5), replace K and L by $\cup_{i=N_2}^{N_3} M_i$ and $\cup_{i=N_1}^{N_2-1} M_i \bigcup \cup_{i=N_3+1}^{N_4} M_i$, respectively. Then we get the result.

The following theorem is another version of equality on continuous g-frames in Hilbert C*-modules. The coefficients take values from the space

$$\ell^\infty(A) = \left\{ \{a_m\}_{m\in M} : sup\| \int_M \langle a_m, a_m\rangle d\mu(m)\| < \infty \right\}.$$

Theorem 3.12. *Let $\{\Lambda_m \in End_A^*(U,V_m) : m \in M\}$ be a continuous λ-tight g-frame, for Hilbert C*-module U with respect to $\{V_m : m \in M\}$, then for any $\{a_m\}_{m\in M} \in \ell^\infty(A)$ and $f \in U$, we have*

$$\left|\int_M (1_A - a_m)\Lambda_m^*\Lambda_m f d\mu(m)\right|^2 - \left|\int_M a_m \Lambda_m^*\Lambda_m f d\mu(m)\right|^2 = \lambda \int_M (1_A - a_m)|\Lambda_m f|^2 d\mu(m) - \lambda \int_M a_m^* |\Lambda_m f|^2 d\mu(m). \tag{11}$$

Proof. Since $\{a_m\}_{m\in M} \in \ell^\infty(A)$ there exists a positive constant R in A, such that $|a_m| \leq R$ for any $m \in M$. Now, let

$$S_1(f) = \int_M (1_A - a_m)\Lambda_m^*\Lambda_m f d\mu(m), \quad S_2(f) = \int_M a_m\Lambda_m^*\Lambda_m f d\mu(m), \ \forall f \in U$$

and for any measurable subset $K \subset M$ and $f \in U$,

$$S_K(f) = \int_K (1_A - a_m)\Lambda_m^*\Lambda_m f d\mu(m)$$

then we have

$$\begin{aligned}
\|S_K f\|^4 = \|\langle S_K f, S_K f\rangle\|^2 &= \left\| \left\langle \int_K (1_A - a_m)\Lambda_m^*\Lambda_m f d\mu(m), S_K f \right\rangle \right\|^2 \\
&= \left\| \left\langle \int_K (1_A - a_m)\Lambda_m f d\mu(m), \Lambda_m S_K f \right\rangle \right\|^2 \\
&\leq \left\| \int_K \langle (1_A - a_m)\Lambda_m f, (1_A - a_m)\Lambda_m f\rangle d\mu(m) \right\| . \left\| \int_K \langle \Lambda_m S_K f, \Lambda_m S_K f\rangle d\mu(m) \right\| \\
&\leq \lambda \|S_K f\|^2 \left\| \int_K \langle (1_A - a_m)\Lambda_m f, (1_A - a_m)\Lambda_m f\rangle d\mu(m) \right\| .
\end{aligned}$$

Therefore

$$\|S_K(f)\|^2 = \left\| \int_K (1_A - a_m)\Lambda_m^*\Lambda_m f d\mu(m) \right\|^2 \leq \lambda \left\| \int_K \langle (1_A - a_m)\Lambda_m f, (1_A - a_m)\Lambda_m f\rangle d\mu(m) \right\| .$$

Since K is arbitrary, so the operator $S_1(f)$ is well defined for any $f \in U$. Similarly we know that $S_2(f)$ is well defined for any $f \in U$. It is easy to check that $S_1(f) + S_2(f) = \int_M \Lambda_m^*\Lambda_m f d\mu(m) = \lambda f$ for any $f \in U$, so $\lambda^{-1}S_1 + \lambda^{-1}S_2 = I_U$, by Lemma 3.4 $S_1^*S_1 - S_2^*S_2 = \lambda S_1 - \lambda S_2^*$. Therefore, we have

$$\begin{aligned}
\left| \int_M (1_A - a_m)\Lambda_m^*\Lambda_m f d\mu(m) \right|^2 - \left| \int_M a_m\Lambda_m^*\Lambda_m f d\mu(m) \right|^2 &= \langle S_1 f, S_1 f\rangle - \langle S_2 f, S_2 f\rangle \\
&= \langle S_1^*S_1 f, f\rangle - \langle S_2^*S_2 f, f\rangle = \langle (S_1^*S_1 - S_2^*S_2)f, f\rangle = \langle (\lambda S_1 - \lambda S_2^*)f, f\rangle \\
&= \lambda \int_M (1_A - a_m)|\Lambda_m f|^2 d\mu(m) - \lambda \int_M a_m^*|\Lambda_m f|^2 d\mu(m).
\end{aligned}$$

Corollary 3.13. *Let $\{f_m : m \in M\}$ be a continuous λ-tight frame for Hilbert C*-module H, then for any $\{a_m\}_{m\in M} \in \ell^\infty(A)$ and $f \in U$ we have*

$$\left|\int_M (1_A - a_m)\langle f, f_m\rangle f_m d\mu(m)\right|^2 - \left|\int_M a_m\langle f, f_m\rangle f_m d\mu(m)\right|^2$$
$$= \lambda \int_M (1_A - a_m)\langle f, f_m\rangle\langle f_m, f\rangle d\mu(m) - \lambda \int_M a_m^*\langle f, f_m\rangle\langle f_m, f\rangle d\mu(m).$$

Theorem 3.14. *(see [15]) Let $\{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ be a continuous Parseval g-frame, for Hilbert C*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in U$, we have*

$$\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \left|\int_{K^c} \Lambda_m^*\Lambda_m f d\mu(m)\right|^2 = \int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m) + \left|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\right|^2 \geq \frac{3}{4}\langle f, f\rangle.$$

Corollary 3.15. *Let $\{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ be a continuous Parseval g-frame for Hilbert C*-module U with respect to $\{V_m : m \in M\}$, then for any measurable subset $K \subset M$ and $f \in U$, we have*

$$\left\|\int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| + \left\|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\right\|^2 \geq \frac{3}{4}\|f\|^2. \tag{12}$$

Let $\Lambda = \{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ be a continuous Parseval g-frame for Hilbert C*-module U with respect to $\{V_m : m \in M\}$. The inequality (3.7) in Corollary 3.15 leads us to define, for any measurable subset $K \subset M$ and $f \in U$,

$$\nu_+(\Lambda, K) = sup_{f\neq 0}\frac{\|\int_{K^c}\langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\| + \|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\|^2}{\|f\|^2};$$

$$\nu_-(\Lambda, K) = inf_{f\neq 0}\frac{\|\int_{K^c}\langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\| + \|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\|^2}{\|f\|^2}.$$

Theorem 3.16. *$\nu_+(\Lambda, K)$ and $\nu_-(\Lambda, K)$ have the following properties:*
(1) $\frac{3}{4} \leq \nu_-(\Lambda, K) \leq \nu_+(\Lambda, K) \leq 1$;
(2) $\nu_-(\Lambda, M) = \nu_+(\Lambda, M),\ \nu_-(\Lambda, \emptyset) = \nu_+(\Lambda, \emptyset)$.

Proof. (1) By inequality (3.7) we know that the first inequality holds. For the third inequality, we just need to show $\|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\|^2 \leq \|\int_K\langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\|$ for any measurable subset

$K \subset M$. In fact, for any $f \in U$, we have

$$\begin{aligned}\left\|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\right\|^2 &= sup_{g\in U,\|g\|=1}\left\|\left\langle \int_K \Lambda_m^*\Lambda_m f d\mu(m), g\right\rangle\right\|^2\\ &= sup_{g\in U,\|g\|=1}\left\|\int_K \langle \Lambda_m f, \Lambda_m g\rangle d\mu(m)\right\|^2\\ &\leq sup_{g\in U,\|g\|=1}\left\|\int_K \langle \Lambda_m g, \Lambda_m g\rangle d\mu(m)\right\|.\left\|\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\|\\ &\leq sup_{g\in U,\|g\|=1}\|g\|^2.\left\|\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| = \left\|\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\|.\end{aligned}$$

Hence

$$\begin{aligned}\left\|\int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| + \left\|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\right\|^2 &\leq \left\|\int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\|\\ &\quad + \left\|\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\|\\ &= \|f\|^2.\end{aligned}$$

(2) follows directly by definition $\nu_-(\Lambda, K)$ and $\nu_+(\Lambda, K)$.

Theorem 3.17. *Let $\Lambda = \{\Lambda_m \in End_A^*(U, V_m) : m \in M\}$ be a continuous Parseval g-frame for Hilbert C*-module U with respect to $\{V_m : m \in M\}$. Then for any measurable subset $K \subset M$ and $f \in U$, the following statements are equivalent:*
(1) $\nu_-(\Lambda, K) = \nu_+(\Lambda, K) = 1$;
(2) $\|\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\| = \|\int_K \Lambda_m^*\Lambda_m f d\mu(m)\|^2$.

Proof. (1)$\Rightarrow$ (2) Suppose that (1) holds. Since Λ is a continuous Parseval g-frame, we have

$$\left\|\int_{K^c} \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| + \left\|\int_K \langle \Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| = \|f\|^2$$

for any $f \in U$. It follows that

$$\begin{aligned}\left\|\int_K \langle\Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| &= \|f\|^2 - \left\|\int_{K^c} \langle\Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| \\ &= \left\|\int_{K^c} \langle\Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| + \left\|\int_K \Lambda_m^* \Lambda_m f d\mu(m)\right\|^2 \\ &\quad - \left\|\int_{K^c} \langle\Lambda_m f, \Lambda_m f\rangle d\mu(m)\right\| \\ &= \left\|\int_K \Lambda_m^* \Lambda_m f d\mu(m)\right\|^2,\end{aligned}$$

namely, (2) holds.

(2)$\Rightarrow$ (1) It follows immediately from the definition.

Conflict of Interests

The authors declare that there is no conflict of interests.

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