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ESSENTIAL NORM OF WEIGHTED COMPOSITION OPERATORS ON BARGMANN-FOCK SPACES

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Abstract. Let φ be an entire self-map of the n-dimensional Euclidean complex space $\mathbb{C}^n$ and ψ be an entire function on $\mathbb{C}^n$. A weighted composition operator induced by φ with weight ψ is given by $(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$, for $z \in \mathbb{C}^n$ and f is entire function on $\mathbb{C}^n$. In this paper, we study weighted composition operators between Bargmann-Fock spaces $\mathscr{F}_\alpha^p(\mathbb{C}^n)$ and $\mathscr{F}_\alpha^p(\mathbb{C}^n)$ for $0 < p,q < \infty$. Using Carleson-type measures techniques, we characterize the boundedness and compactness of these operators, when $0 < p,q < \infty$. We also obtained an estimate of the essential norm of these operators, when $1 < p \leq q < \infty$. The results written in terms of a certain Berezin-type integral transform on $\mathbb{C}^n$.

Keywords: Weighted Composition Operators; Compact Operators; Entire Functions; Carleson Measures; Bargmann-Fock Spaces.

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1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidean space, and let dv be the usual Lebesgue volume measure on $\mathbb{C}^n$. For a fixed $\alpha > 0$, we consider the Gaussian measure on $\mathbb{C}^n$

$$dv_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2} dv(z).$$

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For any $p>0$, the Bargmann-Fock (Fock space) $\mathscr{F}^p_\alpha(\mathbb{C}^n)$ consists of all functions f such that

$$\|f\|^p_{p,\alpha}=\int_{\mathbb{C}^n}|f(z)|^p e^{\frac{-\alpha p}{2}|z|^2}dv(z),$$

is finite; that is $f\in\mathscr{F}^p_\alpha(\mathbb{C}^n)$ if and only if $f(z)e^{\frac{-\alpha}{2}|z|^2}\in L^p(\mathbb{C}^n,dv)$.

Recall that if $z=(z_1,z_2,\cdots,z_n)$ and $w=(w_1,w_2,\cdots,w_n)$ are points in $\mathbb{C}^n$, we write $\langle z,w\rangle=\sum_{j=1}^n z_j\overline{w_j}$ and $|z|^2=\langle z,z\rangle$. It is well known that the space $\mathscr{F}^2_\alpha$ of Gaussian square-integrable entire functions on $\mathbb{C}^n$ is a closed subspace of $L^2(\mathbb{C}^n,dv_\alpha)$ and it has the Bergman reproducing kernel property

$$f(z)=\int_{\mathbb{C}^n}f(w)K_\alpha(z,w)dv_\alpha(w)=\langle f(\cdot),K_\alpha(\cdot,z)\rangle,$$

for all $f\in\mathscr{F}^2_\alpha$ and $z\in\mathbb{C}^n$. Moreover, the reproducing kernel function of $\mathscr{F}^2_\alpha$ is given by

$$K_w(z)=K_\alpha(z,w)=e^{\alpha\langle z,w\rangle}.$$

Let k_w denote the normalized reproducing kernel function, which is given by

$$k_w(z)=\frac{K_\alpha(z,w)}{\|K_\alpha(w,w)\|_{2,\alpha}}=e^{\alpha\langle z,w\rangle-\frac{\alpha}{2}|w|^2}.$$

For any $z\in\mathbb{C}^n$ and $r>0$ we use

$$B(z,r)=\{w\in\mathbb{C}^n:|w-z|<r\}$$

to denote the Euclidean ball centered at z with radius r. Let μ be a positive Borel measure on $\mathbb{C}^n$, the average of μ on $B(z,r)$ is $\frac{\mu(B(z,r))}{v(B(z,r))}$. Since the Lebesgue volume $v(B(z,r))=\int_{B(z,r)}dv\simeq r^{2n}$ is a constant over all $z\in\mathbb{C}^n$, we call $\mu(B(z,r))$ an averaging function of μ. For $t>0$, we define the t-Berezin transform of μ as follows

$$\tilde{\mu}_t(z)=\int_{\mathbb{C}^n}e^{\frac{-\alpha t}{2}|z-w|^2}d\mu(w).$$

Since $k_z(w)=e^{\alpha w\bar{z}-\frac{\alpha}{2}|z|^2}$, we can write

$$\tilde{\mu}_t(z)=\int_{\mathbb{C}^n}|k_z(w)|^t e^{\frac{-\alpha t}{2}|w|^2}d\mu(w).$$

Note that if $t=2$ and $\alpha=0$, we get the tradition definition of Berezin transform in Bergman spaces.

Suppose φ is an entire function maps $\mathbb{C}^n$ into itself and ψ is an entire function on $\mathbb{C}^n$, the weighted composition operator $W_{\psi,\varphi}$ is defined on the space $H(\mathbb{C}^n)$ of all entire functions on $\mathbb{C}^n$ by

$$(W_{\psi,\varphi}f)(z) = \psi(z)C_\varphi f(z) = \psi(z)f(\varphi(z)),$$

for all $f \in H(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. The composition operator C_φ is a weighted composition operator with the weight function ψ identically equal to 1. It is well known that the composition operator $C_\varphi f = f \circ \varphi$ defines a linear operator C_φ which acts boundedly on spaces of entire functions on $\mathbb{C}^n$.

These operators have been studied on many spaces of analytic functions and entire functions as well. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of $W_{\psi,\varphi}$ in terms of the function-theoretic properties of the induced maps φ and ψ. On the spaces of analytic functions, we refer to the monographs [4], [8], [9], [17], [28], and [29] for the overview of the field as of the early 1990s. See [21] and [27] for the studies on the spaces of entire functions.

Recently, boundedness and compactness of (weighted) composition operators on Fock-type spaces have been studied by the authors of these papers [3], [19], [22], [23], and [25]. The authors of those papers used classical techniques which were used by many authors in Bergman and Hardy spaces, see for example [4], [5], [6], [15], [17], [18], [28], and [29]. In this paper we use similar techniques, in fact we use Carleson measures techniques to characterize the boundedness and compactness of a weighted composition operator $W_{\psi,\varphi}$ acting from $\mathscr{F}^p_\alpha(\mathbb{C}^n)$ to $\mathscr{F}^q_\alpha(\mathbb{C}^n)$, for $0 < p, q < \infty$. Then we provide an upper and a lower bound estimates of the essential norm of $W_{\psi,\varphi}$ when $1 < p \leq q < \infty$. Our results will be expressed in terms of the integral operator $B_{\varphi,p}(|\psi|)$, which we define next. Let φ be an entire self-map of $\mathbb{C}^n$, and let ψ be an entire function of $\mathbb{C}^n$, then for $w \in \mathbb{C}^n$ and $0 < p < \infty$ we define

$$B_{\varphi,p}(|\psi(w)|) = \int_{\mathbb{C}^n} |k_w(\varphi(z))|^p |\psi(z)|^p e^{\frac{-\alpha p}{2}|z|^2} dv(z).$$

2. Preliminaries

The essential norm of a bounded operator T, denote by $\|T\|_e$, is its distance in the operator norm from the space of compact operators. Thus, for any $W_{\psi,\varphi}:\mathscr{F}^p_\alpha\to\mathscr{F}^q_\alpha$ we define

$$\|W_{\psi,\varphi}\|_e=\inf_{K\in\mathbb{K}}\|W_{\psi\varphi}-K\|_{q,\alpha},$$

where $\mathbb{K}=\mathbb{K}(\mathscr{F}^p_\alpha,\mathscr{F}^q_\alpha)$ is the space of compact operators from $\mathscr{F}^p_\alpha$ into $\mathscr{F}^q_\alpha$. It is well known that $W_{\psi,\varphi}$ is compact if and only if $\|W_{\psi,\varphi}\|_e=0$, so that estimates of essential norm lead for $W_{\psi,\varphi}$ to be compact. The essential norm has been studied by many authors in spaces of analytic functions, see for example [1], [5], [6], [14], [15], [18], [20], [24], [26] and the references in these papers. Also it has been studied by many authors in Fock-type spaces, see for example [19], [22], [23], and [25]. Moreover, the essential norm of Toeplitz operator induced by bounded symbols has been studied in [12] on generalized Fock spaces.

The following lemma gives an upper bound of the essential norm of the weighted composition operator $W_{\psi,\varphi}:\mathscr{F}^p_\alpha\to\mathscr{F}^q_\alpha$, for $1<p\le q<\infty$, in terms of the operators R_j which we define next. For each entire function f on $\mathbb{C}^n$, there is a unique sequence $\{p_k(z)\}$ of homogenous polynomials of degree k such that f can be written as $f(z)=\sum_{k=0}^{\infty}p_k(z)$. For each $j\in\mathbb{N}$ define the operators R_j as $R_jf(z)=\sum_{k=j}^{\infty}p_k(z)$. It is known ([23], Proposition 2.3) that for each $f\in\mathscr{F}^p_\alpha(\mathbb{C}^n)$, $\lim_{j\to\infty}\|R_jf\|_{p,\alpha}=0$. Moreover, by the principle of uniform boundedness, $\sup_{j\in\mathbb{N}}\|R_j\|<\infty$. This lemma is a classical result; see for example ([4], Lemma 3.16), ([5], Lemma 2) and ([18], proposition 5.1) in context of Hardy and Bergman spaces, and in context of Fock-type spaces see for example [22] when $n=1$, $q=\infty$ and see [19] when $n\ge 2$, $q=\infty$. The proof is similar to these previous results, so we omit the proof.

Lemma 2.1. *Let $1<p\le q<\infty$, let φ be an entire self-map of $\mathbb{C}^n$, and let ψ be and entire function on $\mathbb{C}^n$. If $W_{\psi,\varphi}$ is bounded from $\mathscr{F}^p_\alpha$ into $\mathscr{F}^q_\alpha$, then*

$$\|W_{\psi,\varphi}\|_e\le\liminf_{j\to\infty}\|W_{\psi,\varphi}R_j\|_{q,\alpha}.$$

The following lemma is proved in ([22], Lemma 3) for $n=1$ and ([23], Lemma 3.2) for $n\ge 2$.

Lemma 2.2. *Let $1 < p < \infty$ and q be the conjugate exponent of p, i.e $1/p + 1/q = 1$. Then for each $f \in \mathscr{F}_\alpha^p(\mathbb{C}^n)$, there exists a positive constant C such that*

$$|R_j f(w)| \le C \|f\|_{p,\alpha} \sum_{k=j}^{\infty} \frac{\alpha^k |w|^k}{k!} \left\{ \left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+n} \Gamma\left(\frac{2k}{q}+n\right) \right\}^{1/q},$$

for all $w \in \mathbb{C}^n$ and all positive integers j.

Carleson measures were first introduced by Carleson [2], who studied positive Borel measures μ on the unit disk that satisfy for any function f in the Hardy space $H^p(\mathbb{D})$ the condition

$$\int_{\mathbb{D}} |f(z)|^p d\mu(z) \le C \int_0^{2\pi} |f(e^{it})|^p dt,$$

as a tool to study interpolating sequences and the corona problem. These measures have been extended and found many applications in the study of composition operators in various spaces of functions, for example see [4], [8], [9], [28] and [29] for the study of Carleson measures in Hardy and Bergamn spaces, and see [10], [11], [13], and [16] for the study of Carleson measures in Fock-type spaces. Now we are ready to define (p,q)-Fock Carleson measure and vanishing (p,q)-Fock Carleson measure as well.

Definition 2.3. *Let $0 < p, q < \infty$ and let μ be a positive Borel measure. We say μ is a (p,q)-Fock Carleson measure if there exists a constant C such that for all $f \in \mathscr{F}_\alpha^p$,*

$$\int_{\mathbb{C}^n} |f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\mu(z) \le C \|f\|_{p,\alpha}^q.$$

Moreover, we say μ is a vanishing (p,q)-Fock Carleson measure if

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} |f_j(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\mu(z) = 0$$

for any bounded sequence $\{f_j\}$ in $\mathscr{F}_\alpha^p$ that converges to zero uniformly on compact subsets of $\mathbb{C}^n$ as $j \to \infty$.

The following two lemmas are from [11], they characterize (vanishing) (p,q)-Fock Carleson measures in terms of the t-Berezin transform $\tilde{\mu}_t$ and the averaging function $\mu(B(\cdot, r))$.

Lemma 2.4. *Let $0 < p \le q < \infty$, and let $\mu \ge 0$. Then the following statements are equivalent.*

(1) μ is a (p,q)-Fock Carleson measure;

(2) $\tilde{\mu}_t$ is bounded on $\mathbb{C}^n$ for some (or any) $t > 0$;

(3) $\mu(B(\cdot,\delta))$ *is bounded on* $\mathbb{C}^n$ *for some (or any)* $\delta > 0$*;*

(4) For some (or any) $r > 0$*, the sequence* $\{\mu(B(a_k,r))\}_{k=1}^{\infty}$ *is bounded.*

Lemma 2.5. *Let* $0 < q \leq p < \infty$*, and let* $\mu \geq 0$*. Set* $s = \frac{p}{q}$ *and* s' *to be the conjugate exponent of* s*. Then the following statements are equivalent.*

(1) μ *is a* (p,q)*-Fock Carleson measure;*

(2) μ *is a vanishing* (p,q)*-Fock Carleson measure;*

(3) $\tilde{\mu}_t \in L^{s'}(dv)$ *for some (or any)* $t > 0$*;*

(4) $\mu(B(\cdot,\delta)) \in L^{s'}(dv)$ *for some (or any)* $\delta > 0$*;*

(5) For some (or any) $r > 0$*,* $\sum_{k=1}^{\infty} \mu(B(a_k,r))^{s'} < \infty$*.*

Further information on boundedess and compactness of weighted composition operators come from using a pull-back measure o $\mathbb{C}^n$, which we define next. Suppose φ is an entire self-map of $\mathbb{C}^n$. For each $p > 0$, we define for any $\psi \in \mathscr{F}_{\alpha}^{p}(\mathbb{C}^n)$ a positive Borel measure $\mu_{\varphi,p}$ on $\mathbb{C}^n$ by

$$\mu_{\varphi,p}(E) = \int_{\varphi^{-1}(E)} |\psi(z)|^p e^{\frac{-\alpha p}{2}|z|^2} dv(z),$$

where E is a Borel subset of $\mathbb{C}^n$.

It is known that, by using ([7], Theorem III.10.4), for any $g \in \mathscr{F}_{\alpha}^{p}(\mathbb{C}^n)$ the change of variable formula

$$\int_{\mathbb{C}^n} |g(z)|^p d\mu_{\varphi,p}(z) = \int_{\mathbb{C}^n} |\psi(z)|^p |g(\varphi(z))|^p e^{\frac{-\alpha p}{2}|z|^2} dv(z).$$

3. Main results

In this section, we characterize the boundedness and compactness of $W_{\psi,\varphi} : \mathscr{F}_{\alpha}^{p}(\mathbb{C}^n) \to \mathscr{F}_{\alpha}^{q}(\mathbb{C}^n)$, for $0 < p,q < \infty$. Then we give an upper and a lower estimates for the essential norm of $W_{\psi,\varphi}$, for $1 < p \leq q < \infty$, in terms of the integral operator $B_{\varphi,q}(|\psi|)$.

Theorem 3.1. *Let* $0 < p \leq q < \infty$*, let* φ *be an entire self-map of* $\mathbb{C}^n$*, and let* $\psi \in \mathscr{F}_{\alpha}^{q}$*. Then the following are equivalent.*

(1) $W_{\psi,\varphi} : \mathscr{F}_{\alpha}^{p} \to \mathscr{F}_{\alpha}^{q}$ *is bounded;*

(2) $\lambda_{\varphi,q}$ *is a* (p,q)*-Fock Carleson measure, where* $d\lambda_{\varphi,q}(z) = e^{\frac{\alpha q}{2}|z|^2} d\mu_{\varphi,q}(z)$*;*

(3) $M = \sup_{w\in\mathbb{C}^n} B_{\varphi,q}(|\psi(w)|) < \infty$.

Proof. First, we prove that (3) implies (2). For a fixed $w \in \mathbb{C}^n$, set $f_w(z) = e^{\alpha\langle z,w\rangle} e^{\frac{-\alpha}{2}|z|^2}$. It is known that for each $z \in B(w,r)$ we have

$$\left| e^{\alpha\langle z,w\rangle - \frac{\alpha}{2}|z|^2} \right|^q \geq e^{\frac{\alpha q}{2}|w|^2 - \frac{\alpha q}{2} r^2}.$$

Then,

$$\begin{aligned} e^{\frac{\alpha q}{2}|w|^2} e^{\frac{-\alpha q}{2} r^2} \lambda_{\varphi,q}(B(w,r)) &\leq \int_{B(w,r)} |f_w(z)|^q d\lambda_{\varphi,q}(z) \\ &\leq e^{\frac{\alpha q}{2}|w|^2} \int_{\mathbb{C}^n} e^{\frac{-\alpha q}{2}|z-w|^2} d\lambda_{\varphi,q}(z) \\ &= e^{\frac{\alpha q}{2}|w|^2} \int_{\mathbb{C}^n} |k_w(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z) \\ &= e^{\frac{\alpha q}{2}|w|^2} \int_{\mathbb{C}^n} |k_w(z)|^q d\mu_{\varphi,q}(z) \\ &= e^{\frac{\alpha q}{2}|w|^2} \int_{\mathbb{C}^n} |k_w(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z). \end{aligned}$$

Hence, we get

$$e^{\frac{-\alpha q}{2} r^2} \lambda_{\varphi,q}(B(w,r)) \leq B_{\varphi,q}(|\psi(w)|). \tag{1}$$

Thus, by our hypothesis we get

$$e^{\frac{-\alpha q}{2} r^2} \lambda_{\varphi,q}(B(w,r)) \leq M.$$

Therefore, by Lemma 2.4, we get $\lambda_{\varphi,q}$ is a (p,q)-Fock Carleson measure.

Next we prove that (2) implies (1). By our hypothesis (2) and Definition 2.3, for all $f \in \mathscr{F}^p_\alpha(\mathbb{C}^n)$ there exists a constant $C > 0$ such that

$$\int_{\mathbb{C}^n} |f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z) \leq C\|f\|^q_{p,\alpha}.$$

On the other hand,

$$\begin{aligned} \int_{\mathbb{C}^n} |f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z) &= \int_{\mathbb{C}^n} |f(z)|^q d\mu_{\varphi,q}(z) \\ &= \int_{\mathbb{C}^n} |f(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\ &= \|W_{\varphi,q}(f)\|^q_{q,\alpha}. \end{aligned}$$

This gives the operator $W_{\psi,\varphi}$ is bounded from $\mathscr{F}^p_\alpha(\mathbb{C}^n)$ into $\mathscr{F}^q_\alpha(\mathbb{C}^n)$.

Finally, we prove that (1) implies (3). For a fixed $w \in \mathbb{C}^n$, set $f_w(z) = k_w(z)$. Then by the boundedness of $W_{\psi,\varphi}$, there exists a positive constant C such that

$$\|W_{\psi,\varphi}(f_w)\|^q_{q,\alpha} \leq C\|f_w\|^q_{q,\alpha}.$$

Thus,

$$\int_{\mathbb{C}^n} |f_w(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \leq C.$$

Taking supremum over $w \in \mathbb{C}^n$, we get

$$\sup_{w\in\mathbb{C}^n} B_{\varphi,q}(|\psi(w)|) < \infty,$$

which completes the proof.

The previous Theorem 3.1 and Lemma 2.4 give that boundedness of $W_{\psi,\varphi}$ is equivalent to $\tilde{\lambda}_{\varphi,q}(z)$ is bounded in $\mathbb{C}^n$. On one hand we have

$$\begin{aligned}
\tilde{\lambda}_{\varphi,q}(z) &= \int_{\mathbb{C}^n} e^{\frac{-\alpha q}{2}|w-z|^2} d\lambda_{\varphi,q}(w) \\
&= \int_{\mathbb{C}^n} |k_z(w)|^q e^{\frac{-\alpha q}{2}|w|^2} d\lambda_{\varphi,q}(w) \\
&= \int_{\mathbb{C}^n} |k_z(w)|^q d\mu_{\varphi,q}(w) \\
&= \int_{\mathbb{C}^n} |k_z(\varphi(w))|^q |\psi(w)|^q e^{\frac{-\alpha q}{2}|w|^2} dv(w) \\
&= B_{\varphi,q}(|\psi(z)|).
\end{aligned}$$

It is clear that $B_{\varphi,q}(|\psi(z)|) = \|W_{\psi,\varphi}(k_z)\|^q_{q,\alpha}$. Therefore, boundedness of $W_{\psi,\varphi}$ implies

$$\|B_{\varphi,q}(|\psi|)\|_{L^\infty} \leq \|W_{\psi,\varphi}\|^q_{q,\alpha}.$$

On the other hand, by Lemma 2.1 of [13], for each $r > 0$ there exists $C > 0$ such that for any entire function f on $\mathbb{C}^n$ we have

$$\sup_{z\in B(a,r)} |f(z)e^{-\phi(z)}|^p \leq C_2 \int_{B(a,2r)} |f(w)|^p e^{\frac{-\alpha p}{2}|w|^2} dv(w). \tag{2}$$

Now consider the r-lattice $\{a_k\}$ in $\mathbb{C}^n$ and a positive integer m such that every point in $\mathbb{C}^n$ belongs to at most m sets in $\{B(a_k, 2r)\}$. Using the covering property, estimate (2), and the fact

$\sum_{k=1}^{\infty} b_k^l \leq \left(\sum_{k=1}^{\infty} b_k\right)^l$ whenever $1 \leq l < \infty$ and $b_k \geq 0$ for all k, we obtain

$$\begin{aligned}
\|W_{\psi,\varphi}(f)\|_{q,\alpha}^q &= \int_{\mathbb{C}^n} |f(\psi(z))|^q |u(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\
&= \int_{\mathbb{C}^n} |f(z)|^q d\mu_{\varphi,q}(z) = \int_{\mathbb{C}^n} |f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z) \\
&\leq \sum_{k=1}^{\infty} \int_{B(a_k,r)} \left| f(z) e^{\frac{-\alpha}{2}|z|^2} \right|^q d\lambda_{\varphi,q}(z) \\
&\leq \sum_{k=1}^{\infty} \lambda_{\varphi,q}(B(a_k,r)) \left(\sup_{z \in B(a_k,r)} \left| f(z) e^{\frac{-\alpha}{2}|z|^2} \right|^p \right)^{q/p} \\
&\leq C_2 \sum_{k=1}^{\infty} \lambda_{\varphi,q}(B(a_k,r)) \left(\int_{B(a_k,2r)} \left| f(w) e^{\frac{-\alpha}{2}|w|^2} \right|^p dv(w) \right)^{q/p} \\
&\leq C_2 \sup_{k \geq 1} \lambda_{\varphi,q}(B(a_k,r)) \left(\sum_{k=1}^{\infty} \int_{B(a_k,2r)} \left| f(w) e^{\frac{-\alpha}{2}|w|^2} \right|^p dv(w) \right)^{q/p} \\
&\leq C_2 m^{q/p} \sup_{k \geq 1} \lambda_{\varphi,q}(B(a_k,r)) \|f\|_{p,\alpha}^q.
\end{aligned}$$

Using estimate (1) there exists a constant $C^* > 0$ such that

$$\|W_{\psi,\varphi}(f)\|_{q,\alpha}^q \leq C^* \|B_{\varphi,q}(|\psi|)\|_{L^\infty}.$$

The above argument gives the proof of the following Theorem 3.2.

Theorem 3.2. *Let $0 < p \leq q < \infty$, let φ be an entire self-map of $\mathbb{C}^n$, and let $\psi \in \mathscr{F}_\alpha^q(\mathbb{C}^n)$. If $W_{\psi,\varphi}$ is bounded from $\mathscr{F}_\alpha^p$ into $\mathscr{F}_\alpha^q$ then there exists a positive constant C^* such that*

$$\|B_{\varphi,q}(|\psi|)\|_{L^\infty} \leq \|W_{\psi,\varphi}\|_{q,\alpha}^q \leq C^* \|B_{\varphi,q}(|\psi|)\|_{L^\infty}.$$

The next Theorem 3.3 gives an upper and a lower bounds of the essential norm of a weighted composition operator $W_{\psi,\varphi}$ in terms of the integral operator $B_{\varphi,q}(|\psi|)$.

Theorem 3.3. *Let $1 < p \leq q < \infty$, let φ be an entire self-map of $\mathbb{C}^n$, and let $\psi \in \mathscr{F}_\alpha^q(\mathbb{C}^n)$. If $W_{\psi,\varphi}$ is bounded from $\mathscr{F}_\alpha^p$ into $\mathscr{F}_\alpha^q$, then there exists a positive constant C such that*

$$\limsup_{|w| \to \infty} B_{\varphi,q}(|\psi(w)|) \leq \|W_{\psi,\varphi}\|_e^q \leq C \limsup_{|w| \to \infty} B_{\varphi,q}(|\psi(w)|).$$

Proof. First, we prove the lower estimate. Let K be a compact operator acting from $\mathscr{F}_\alpha^p$ into $\mathscr{F}_\alpha^q$. For $w \in \mathbb{C}^n$, set $f_w(z) = k_w(z)$. It is clear that $\{f_w\}$ converges to 0 on any compact subset of $\mathbb{C}^n$ as $|w| \to \infty$. Hence $\|Kf_w\|_q \to 0$ as $|w| \to \infty$. Therefore,

$$\begin{aligned}
\|W_{\psi,\varphi} - K\| &\geq \limsup_{|w|\to\infty} \|(W_{\psi,\varphi} - K)f_w\|_{q,\alpha} \\
&\geq \limsup_{|w|\to\infty} \left(\|W_{\psi,\varphi} f_w\|_{q,\alpha} - \|Kf_w\|_{q,\alpha}\right) \\
&= \limsup_{|w|\to\infty} \|W_{\psi,\varphi} f_w\|_{q,\alpha}.
\end{aligned}$$

By taking infimum over all compact operators K, we get

$$\begin{aligned}
\|W_{\psi,\varphi}\|_e^q &\geq \limsup_{|w|\to\infty} \|W_{\psi,\varphi} f_w\|_{q,\alpha}^q \\
&= \limsup_{|w|\to\infty} \int_{\mathbb{C}^n} |k_w(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\
&= \limsup_{|w|\to\infty} B_{\varphi,q}(|\psi(w)|).
\end{aligned}$$

Now, we prove the upper estimate. Fix $r > 0$ and take $f \in \mathscr{F}_\varphi^p(\mathbb{C}^n)$ with $\|f\|_{p,\alpha} \leq 1$. By using Lemma 2.1, we get

$$\begin{aligned}
\|W_{\psi,\varphi}\|_e &\leq \liminf_{j\to\infty} \|W_{\psi,\varphi} R_j\| \\
&\leq \liminf_{j\to\infty} \sup_{\|f\|_{p,\alpha}\leq 1} \|(W_{\psi,\varphi} R_j) f\|_{q,\alpha}. \qquad (3)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|(W_{\psi,\varphi} R_j) f\|_{q,\alpha}^q &= \int_{\mathbb{C}^n} |R_j f(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\
&= \int_{\mathbb{C}^n} |R_j f(z)|^q d\mu_{\varphi,q}(z) \\
&= \int_{\mathbb{C}^n} |R_j f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z) \\
&= \left(\int_{\{|z|\geq r\}} + \int_{\{|z|<r\}}\right) |R_j f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z) \\
&= I_1 + I_2.
\end{aligned}$$

Since $W_{\psi,\varphi}$ is bounded, by using Theorem 3.2 and the fact $\sup_j \|R_j\| < \infty$, there exist positive constants C_1 and C_2 such that

$$\begin{aligned} I_1 &\leq C_1 \|R_j f\|_{p,\alpha} \sup_{|z|\geq r} B_{\varphi,q}(|\psi(z)|) \\ &\leq C_1 C_2 \sup_{|z|\geq r} B_{\varphi,q}(|\psi(z)|). \end{aligned}$$

Again use Theorem 3.2 then use Lemma 2.2, there exist positive constants C_3 and C_4 such that

$$\begin{aligned} I_2 &\leq C_3 \sup_{|z|\geq r} B_{\varphi,q}(|\psi(z)|) \int_{\{|z|<r\}} |R_j f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\ &\leq C_3 C_4 \|f\|_{p,\alpha}^p (H_j)^p \sup_{|z|<r} B_{\varphi,q}(|\psi(z)|) \int_{\{|z|<r\}} e^{\frac{-\alpha q}{2}|z|^2} dv(z), \end{aligned}$$

where

$$H_j = \sum_{k=j}^{\infty} \frac{\alpha^k r^k}{k!} \left\{ \left(\frac{2}{q\alpha}\right)^{\frac{qk}{2}+n} \Gamma\left(\frac{qk}{2}+n\right) \right\}^{1/q}.$$

By Stirling's formula, we get

$$H_j \simeq \sum_{k=j}^{\infty} \frac{\alpha^k r^k}{k!} \left(\frac{2}{q\alpha}\right)^{k/2} \left(\frac{qk}{2}+n\right)^{\frac{k}{2}+\frac{n}{q}-\frac{1}{2q}} e^{-k/2}.$$

By ratio test, last series is convergent. Hence, $\lim_{j\to\infty} H_j = 0$. Therefore, I_2 goes to zero for large enough j. From I_1, I_2 and estimate (3) we get

$$\begin{aligned} \|W_{\psi,\varphi}\|_e^q &\leq \liminf_{j\to\infty} \sup_{\|f\|_{p,\alpha}\leq 1} \|(W_{\psi,\varphi} R_j) f\|_{q,\alpha}^q \\ &\leq C_1 C_2 \sup_{|z|\geq r} B_{\varphi,q}(|\psi(z)|). \end{aligned}$$

Letting $r \to \infty$, we get

$$\|W_{\psi,\varphi}\|_e^q \leq C_1 C_2 \limsup_{|z|\to\infty} B_{\varphi,q}(|\psi(z)|).$$

This completes the proof.

The following is an immediate corollary of Theorem 3.3, using the fact that $W_{\psi,\varphi}$ is compact if and only if $\|W_{\psi,\varphi}\|_e = 0$.

Corollary 3.4. *Let $1 < p \leq q < \infty$, let φ be an entire self-map of $\mathbb{C}^n$, and let $\psi \in \mathscr{F}_\alpha^q(\mathbb{C}^n)$. Then a bounded $W_{\psi,\varphi}$ from $\mathscr{F}_\alpha^p$ into $\mathscr{F}_\alpha^q$ is compact if and only if*

$$\limsup_{|z|\to\infty} B_{\varphi,q}(|\psi(z)|) = 0.$$

The following Theorem 3.5 characterizes the boundedness and compactness of $W_{\psi,\varphi}$ when $0 < q < p < \infty$.

Theorem 3.5. *Let $0 < q < p < \infty$, let φ be an entire self-map of $\mathbb{C}^n$, and let $\psi \in \mathscr{F}_\alpha^q$. Then the following are equivalent.*

(1) $W_{\psi,\varphi} : \mathscr{F}_\alpha^p \to \mathscr{F}_\alpha^q$ is bounded;

(2) $W_{\psi,\varphi} : \mathscr{F}_\alpha^p \to \mathscr{F}_\alpha^q$ is compact;

(3) $B_{\varphi,q}(|\psi|) \in L^{p/(p-q)}(dv)$.

Proof. Suppose that $W_{\psi,\varphi}$ is bounded. Then for any $f \in \mathscr{F}_\alpha^p$ there exists a constant $C > 0$ such that $\|W_{\psi,\varphi}f\|_{q,\alpha}^q \leq C\|f\|_{p,\alpha}^q$. Therefore,

$$\begin{aligned} C\|f\|_{p,\alpha}^q &\geq \int_{\mathbb{C}^n} |f(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\ &= \int_{\mathbb{C}^n} |f(z)|^q d\mu_{\varphi,q}(z) \\ &= \int_{\mathbb{C}^n} |f(z)|^q e^{\frac{-\alpha q}{2}|z|^2} d\lambda_{\varphi,q}(z). \end{aligned}$$

Thus, $\lambda_{\varphi,q}$ is a (p,q)-Fock Carleson measure. By Lemma 2.5, this is equivalent to $\tilde{\lambda}_{\varphi,q} \in L^{p/(p-q)}(dv)$. From the proof of Theorem 3.2, we have $\tilde{\lambda}_{\varphi,q}(z) = B_{\varphi,q}(|\psi(z)|)$. This gives the equivalence of (1) and (3).

On the other hand, using Lemma 2.5 gives $\lambda_{\varphi,q}$ is a vanishing (p,q)-Fock Carleson measure. Then for any bounded sequence $\{f_j\}$ in $\mathscr{F}_\alpha^p$ that converges to zero uniformly on each compact subset of $\mathbb{C}^n$ as $j \to \infty$, we have

$$\lim_{j\to\infty} \int_{\mathbb{C}^n} \left| f_j(z) e^{\frac{-\alpha}{2}|z|^2} \right|^q d\lambda_{\varphi,q}(z) = 0.$$

Similar to previous argument this is equivalent to $\lim\limits_{j\to\infty} \|W_{\psi,\varphi} f_j\|_{q,\alpha}^q = 0$, which gives the compactness of $W_{\psi,\varphi}$. This gives the equivalence of (1) and (2).

The following Theorem 3.6 gives an estimate of the $\|W_{\psi,\varphi}\|_{q,\alpha}$ in terms of the $L^{p/(p-q)}$-norm of $B_{\psi,\varphi}(|\psi|)$.

Theorem 3.6. *Let $0<q<p<\infty$, let φ be an entire self-map of $\mathbb{C}^n$, and let $\psi \in \mathscr{F}^q_\alpha(\mathbb{C}^n)$. If $W_{\psi,\varphi}$ is bounded from $\mathscr{F}^p_\alpha$ into $\mathscr{F}^q_\alpha$ then there exists a positive constant C such that*

$$\|B_{\varphi,q}(|\psi|)\|_{L^{p/(p-q)}} \leq \|W_{\psi,\varphi}\|^q_{q,\alpha} \leq C\|B_{\varphi,q}(|\psi|)\|_{L^{p/(p-q)}}.$$

Proof. First, from the proof of Theorem 3.2, we have for any $z \in \mathbb{C}^n$

$$\begin{aligned} B_{\varphi,q}(|\psi(z)|) &= \|W_{\psi,\varphi}k_z\|^q_{q,\alpha} \\ &\leq \|W_{\psi,\varphi}\|^q \|k_z\|^q_{p,\alpha}. \end{aligned}$$

Since boundedness of $W_{\psi,\varphi}$ equivalent to $B_{\varphi,q}(|\psi|) \in L^{p/(p-q)}$, this implies that

$$\|B_{\varphi,q}(|\psi|)\|_{L^{p/(p-q)}} \leq \|W_{\psi,\varphi}\|^q_{q,\alpha}.$$

On the other hand, from the proof of Theorem 3.1, for any $z \in B(w,r)$ we have

$$|k_w(z)|^q \geq e^{\frac{\alpha q}{2}|z|^2} e^{\frac{-\alpha q}{2}r^2}.$$

Then,

$$\begin{aligned} e^{\frac{-\alpha q}{2}r^2}\int_{B(w,r)} e^{\frac{\alpha q}{2}|z|^2} d\mu_{\varphi,q}(z) &\leq \int_{B(w,r)} |k_w(z)|^q d\mu_{\varphi,q}(z) \\ &\leq \int_{\mathbb{C}^n} |k_w(\varphi(z))|^q |\psi(z)|^q e^{\frac{-\alpha q}{2}|z|^2} dv(z) \\ &= B_{\varphi,q}(|\psi(w)|). \end{aligned} \tag{4}$$

It is clear that, for any $f \in \mathscr{F}^p_\alpha$

$$\|W_{\psi,\varphi}(f)\|^q_{q,\varphi} = \int_{\mathbb{C}^n} |f(z)|^q d\mu_{\varphi,q}(z). \tag{5}$$

By Lemma 2.1 of [13], for any entire function f on $\mathbb{C}^n$ we have the following point estimate

$$\left| f(z) e^{\frac{-\alpha}{2}|z|^2}\right|^q \leq C \int_{B(z,r)} |f(w)|^q e^{\frac{-\alpha q}{2}|w|^2} dv(w). \tag{6}$$

By using equation (5), estimate (6), Fubini's theorem, and $\chi_{B(z,r)}(w) = \chi_{B(w,r)}(z)$, then using inequality (4) we get

$$\begin{aligned}\|W_{\psi,\varphi}(f)\|_{q,\alpha}^q &\leq C\int_{\mathbb{C}^n} e^{\frac{\alpha q}{2}|z|^2} d\mu_{\varphi,q}(z)\int_{\mathbb{C}^n}\chi_{B(z,r)}(w)|f(w)|^q e^{\frac{-\alpha q}{2}|w|^2}dv(w)\\ &\leq C\int_{\mathbb{C}^n}|f(w)|^q e^{-\frac{\alpha q}{2}|w|^2}dv(w)\int_{\mathbb{C}^n}\chi_{B(w,r)}(z)e^{\frac{\alpha q}{2}|z|^2}d\mu_{\varphi,q}(z)\\ &\leq Ce^{\frac{\alpha q}{2}r^2}\int_{\mathbb{C}^n}|f(w)|^q e^{-\frac{\alpha q}{2}|w|^2}B_{\varphi,q}(|\psi(w)|)dv(w).\end{aligned}$$

Applying Hölder's inequality with exponent p/q we get

$$\|W_{\psi,\varphi}f\|_{q,\alpha}^q \leq Ce^{\frac{\alpha q}{2}r^2}\|f\|_{p,\alpha}^q\|B_{\varphi,q}(|\psi|)\|_{L^{p/(p-q)}},$$

which gives the desired result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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