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A COMMON FIXED POINT THEOREM IN FUZZY METRIC SPACE USING EA PROPERTY

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Abstract: In this paper, we prove common fixed point theorems for four self-maps by using weakly compatibility, without appeal to continuity in fuzzy metric space. As a consequence, a multitude of recent fixed point theorems of the existing literature are sharpened and enriched. Our results extend, generalized several fixed point theorems on metric and fuzzy metric spaces.

Keywords: fixed point; fixed point theorem; weakly compatible; EA property.

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1. Introduction:

The evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy sets by Zadeh [12], in 1965, as a new way to represent the vagueness in everyday life. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. With the concept of fuzzy sets, the fuzzy metric space was introduced by Kramosil and Michalek [6]. Grabiec [3] proved the contraction principle in the setting of the fuzzy metric space which was further generalization of results by Subrahmanyam [10] for a pair of commuting mappings. Also, George and Veeramani [2] modified the notion of fuzzy metric spaces with the help of continuous t-norm, by generalizing the concept of probabilistic metric space to fuzzy situation. In 1999, Vasuki [11] introduced the concept of R-weak commutatively of mappings in fuzzy metric space and Pant [7] introduced the notion of reciprocal continuity of mappings in metric

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spaces. Also, Jungck and Rhoades [5] defined a pair of self mappings to be weakly compatible if they commute at their coincidence points. Aamri and Moutawakil [1] generalized the notion of non compatible mapping in metric space by E. A. property.

We prove common fixed point theorem for four mappings satisfying the general contractive condition along with the definition of EA property and weakly compatible mapping in the fuzzy metric spaces.

Role of E.A. property in proving common fixed point theorems can be concluded by following,

(1) It buys containment of ranges without any continuity requirements.

(2) It minimizes the commutatively conditions of the maps to the commutatively at their points of coincidence.

(3) It allows replacing the completeness requirement of the space with a more natural condition of closeness of the range.

2. PRELIMINARIES

Definition 2.1 (T norm) [8]

A binary operation $*:[0,1]x[0,1] \rightarrow [0,1]$ is a continuous T norm if$\{[0,1], *\}$ is an abelian topological monoid with unit 1 such that $a* b \leq c * d$, whenever $a \leq c$ and $b \leq d$, for each a, b, c, $d \in [0, 1]$.

Examples [2.1]: 1) $a* b = a b$ 2) $a* b = \min(a, b)$

Definition 2.2 (Fuzzy Metric Space) [2]

A 3-tuple (X, M, $*$) is said to be a fuzzy metric space if X is an arbitrary set, $*$is a continuous T-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all x, y, z $\in$ X, s, t > 0,

(f1) $M(x, y, t) > 0$;

(f2) $M(x, y, t) = 1$ if and only if $x = y$;

(f3) $M(x, y, t) = M(y, x, t)$;

(f4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(f5) $M(x, y, \cdot): (0,\infty) \rightarrow (0, 1]$ is continuous.

Here M(x, y, t) denote the degree of nearness between x and y with respect to t.

Example [2.2]: (Induced Fuzzy Metric) Let (X, d) be a metric space. Denote $a * b = a b$ for all a, b ∈ [0, 1] and let M be fuzzy sets on $X^2 \times (0,\infty)$ defined as $M(x, y, t) = \frac{t}{t + d(x, y)}$ Then

(X, M, ∗) is a fuzzy metric space.

Definition 2.3 (Coincidence Point)[5]

Let X be a set, f and g self maps of X. A point x ∈ X is called a coincidence point of f and g iff fx = gx. We shall call w = fx = gx a point of coincidence of f and g.

Definition 2.4 (Weakly Compatible Mapping)[5]

A pair of maps *S* and *T* is called weakly compatible pair if they commute at coincidence points.

Definition 2.5 (E. A. Property)[1]

Let f and g be two self-maps of a fuzzy metric space (X, M, ∗). We say that f and g satisfy the property E. A. property if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = z$ For some z ∈ X.

Definition 2.6 (Common E. A. Property) [1]

Let A,B, S, T : X → X where X is a fuzzy metric space, then the pair {A, S} and {B, T} said to satisfy common E. A. property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ For some z ∈ X.

3. Main Result:

Theorem 3.1 Let (X,M,∗) be a fuzzy metric space and A, B, S and T be self mapping of X satisfying the following conditions

(1) A(X) ⊂ T(X) and B(X) ⊂ S(X) and S (X) is closed

(2) $$M(Ax,By,t) \geq \varphi\left(\min\left\{ \begin{cases} M(Sx,Ty,t) \\ Sup_{t_1+t_2} = \frac{2}{k}\, t \min\begin{cases} M(Sx,Ay,t_1) \\ M(Ty,By,t_2) \end{cases} \\ Sup_{t_3+t_4} = \frac{2}{k}\, t \max\begin{cases} M(Sx,By,t_3) \\ M(Ty,Ax,t_4) \end{cases} \end{cases} \right\} \right) \quad \text{..........(3.1.1)}$$

For all x, y ∈ X, t>0 and for some 1≤K<2 . Suppose that the pair (A, S) and (B,T) satisfies Common E.A. property and (A,S) and (B,T) are weakly compatible. Also suppose that S(X) and T(X) is a closed subset of X. Then A, B,S, and T have a unique fixed point in X.

Proof: Since (A,T) and (B,S) satisfies Common E.A. property

Therefore there exist a sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} M(Ay_n \ , z, t) = \lim_{n\to\infty} M(Bx_n \, , z, t) = \lim_{n\to\infty} M(Sy_n \ , z, t) = \lim_{n\to\infty} M(Tx_n \, , z, t) = 1$

$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = z$

For some $z \in$ X and every t >0

Suppose that S(X) is closed subset of X so there exist $u \in$ X such that Su=z.

Let Au=z ,if not then

Put $x = u$ and $y = x_n$ in equation (3.1.1)

$$M(Au, Bx_n \, , t) \geq \varphi\left(\min\left\{ \begin{cases} M(Su, Tx_n \, , t) \\ Sup_{t_{1+t_2}} = \frac{2}{k}\ t\ min\begin{cases} M(Su, Au, t_1) \\ M(Tx_n \, , Bx_n \, , t_2) \end{cases} \\ Sup_{t_{3+t_4}} = \frac{2}{k} t\ max\begin{cases} M(Su, Bx_n, t_3) \\ M(Tx_n \, , Au, t_4) \end{cases} \end{cases} \right\} \right)$$

Taking limit n →∞ we have

$$\text{M } (Au, z \, , t) \geq \varphi\left(\min\left\{ \begin{matrix} \text{M } (z, z, \text{t}) \\ min\begin{cases} \text{M}(z, Au, \frac{2}{k}\ t_0 - \in \\ \text{M}(z, z, \in\ ) \end{cases} \\ max\begin{cases} M(z, z, \in) \\ \text{M}(z, Au, \frac{2}{k}\ t_0 - \in) \end{cases} \end{matrix} \right\} \right)$$

$$\text{M } (Au, z \, , t) \geq \varphi(\text{M } (z, Au, \tfrac{2}{k}\ t_0 - \in))$$

$$> \text{M } (Au, z, \tfrac{2}{k}\ t_0)$$

This is contradiction. Thus Au=z

Hence Au = Su= z

Suppose that T(X) is a closed subset of X so there exist $v \in$ X such that Tv = z

Let Bv = z, if not then

Put $x = y_n$ and $y= v$ in equation (3.1.1)

$$M(Ay_n \ , Bv, t) \geq \varphi\left(\min\left\{ \begin{cases} M(Sy_n \, , Tv, t) \\ Sup_{t_{1+t_2}} = \frac{2}{k}\ t\ min\begin{cases} M(Sy_n \, , Ay_n \ , t_1) \\ M(Tv, Bv, t_2) \end{cases} \\ Sup_{t_{3+t_4}} = \frac{2}{k} t\ max\begin{cases} M(Sy_n \, , Bv, t_3) \\ M(Tv, Ay_n \ , t_4) \end{cases} \end{cases} \right\} \right)$$

$$M(Ay_n\ ,Bv,t) \geq \varphi\left(\min\left\{\begin{matrix} M(Sy_n\ ,Tv,t) \\ \min\{\mathrm{M}\left(Sy_{n\ ,}Ay_n,\frac{2}{k}\ t_0 - \in\right), \mathrm{M}(Tv,Bv,\in\} \\ max\{\mathrm{M}(Sy_n\ ,Bv,\in), \mathrm{M}(,Ay_n,Tv,\frac{2}{k}\ t_0 - \in\} \end{matrix}\right\}\right)$$

$$M(Ay_{n\ ,}\mathrm{Bv}\ ,\mathrm{t}) \geq \varphi\left(\min\left\{\begin{matrix} M(Sy_{n\ ,}\mathrm{Tv}\ ,\mathrm{t}) \\ \min\{\mathrm{M}\left(Sy_{n\ ,}Ay_n,\frac{2}{k}\ t_0 - \in\right), \mathrm{M}(Tv,Bv,\in\} \\ max\{\mathrm{M}(Sy_n\ ,Bv,\in), \mathrm{M}(,Ay_n,Tv,\frac{2}{k}\ t_0 - \in\} \end{matrix}\right\}\right)$$

$\forall \in \varepsilon\ (0,\ \frac{2}{k}\ t_0)$ as $n \to \infty$ it follows that

$$\mathrm{M}\ (\ z\ ,Bv,t_0\ ) \geq \varphi\left(\min\left\{\begin{matrix} \mathrm{M}\ \ (z,z,t_0\ ) \\ min\left\{\begin{matrix} \mathrm{M}(z,z,\frac{2}{k}\ t_0 - \in \\ \mathrm{M}(z,Bv,\in\ ) \end{matrix}\right. \\ max\left\{\begin{matrix} M(z,Bv\ ,\in) \\ \mathrm{M}(z,z,\frac{2}{k}\ t_0 - \in) \end{matrix}\right. \end{matrix}\right\}\right)$$

$\mathrm{M}\ (\ z\ ,Bv,t_0\ ) \geq \phi\ M(z,Bv,\frac{2}{k}\ t_0 - \in) > \mathrm{M}\ (z,Bz,\ \frac{2}{k}\ t_0\ )$

This gives a contradiction. Therefore Bv=z.

Hence $Tv = Bv = z$

Suppose u, v are the coincidence point of (A,S) and (B,T) respectively. Since (A,S) and (B,T) are weakly compatible then ASu=SAu and BTv=TBv

This gives $Az = Sz$ and $Bz = Tz$.

Now we show that z is common fixed point of A and S if $Az \neq z$ using (3.1.1) we obtain

$$\mathrm{M}\ (Az,z\ ,t) \geq \varphi\left(\min\left\{\left\{\begin{matrix} \mathrm{M}\ \ (z,z,t_0\ \ ) \\ Sup_{t_1+t_2} = \frac{2}{k}t min\left\{\begin{matrix} \mathrm{M}(z,Az\ \ ,t_1) \\ \mathrm{M}(z,z,t_2) \end{matrix}\right. \\ Sup_{t_3+t_4} = \frac{2}{k}t\ max\left\{\begin{matrix} M(z,z,t_3\ ) \\ \mathrm{M}(z,Az,t_4) \end{matrix}\right. \end{matrix}\right.\right\}\right)$$

$$\geq\varphi\left(\min\left\{\left\{\begin{matrix} \mathrm{M}\ \ (z,z,t\ ) \\ min\left\{\begin{matrix} \mathrm{M}(z,Az,\frac{2}{k}t\ - \in \\ \mathrm{M}(z,\mathrm{z},\in\ \ ) \end{matrix}\right. \\ max\left\{\begin{matrix} M(z,z,\frac{2}{k}\ \ t - \in) \\ \mathrm{M}(z,\ \ Az,\in) \end{matrix}\right. \end{matrix}\right.\right\}\right)$$

Taking $\in \to 0$ we have

$$M\,(Az, z, t) \geq \varphi\,(\min\{M\,(z, z, t), M(z, Az, \tfrac{2}{k} t)\}$$

$$M\,(Az, z, t) > M\,(z, Az, \tfrac{2}{k} t)$$

Hence Az = S$z = z$

Similarly we can show that Bz=z

Thus Az=Bz=Sz=Tz=z

Hence z is a common fixed point of A, B,S and T.

Uniqueness-

Let w be any other fixed point of A,B,S and T such that w ≠ z.

$$M(w, z, t) \geq \varphi\left(\min\left\{ \left\{ \begin{array}{c} M(Sw, Tz, t) \\ Sup_{t_1+t_2} = \frac{2}{k}\, t\, min \begin{cases} M(Sw, Aw, t_1) \\ M(Tz, Bz, t_2) \end{cases} \\ Sup_{t_3+t_4} = \frac{2}{k}\, t\, max \begin{cases} M(Sw, Bz, t_3) \\ M(Tz, Aw, t_4) \end{cases} \end{array} \right. \right\} \right)$$

Taking $limit\ n \to \infty$ we have

$$M\,(w, z, t) \geq \varphi\left(\min\left\{ \left\{ \begin{array}{c} M\,(w, z, t_0) \\ min \begin{cases} M(w, w, t) \\ M\left(z, z, \frac{2}{k}\, t_0 - \in\right) \end{cases} \\ max \begin{cases} M(w, z, \frac{2}{k} t_0 - \in) \\ M(z, w, \in) \end{cases} \end{array} \right. \right\} \right)$$

$$M\,(w, z, t) \geq \varphi(\,M\,(w, z, \tfrac{2}{k}\, t)\,) > M\,(w, z, \tfrac{2}{k}\, t)$$

This is contradiction. Therefore w$= z$

Thus z is a uniqe common fixed point of A, B, S and T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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