Available online at http://scik.org
Adv. Inequal. Appl. 2015, 2015:12
ISSN: 2050-7461

GENERALIZED HERMITE-HADAMARD INEQUALITY FOR LIPSCHTIZ FUNCTIONS

A. BARANI

Department of Mathematics, Lorestan University, PO Box 465, Khoramabad, Iran

Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for Lipschitz functions defined on invex subsets of real line.

Keywords: Hermite-Hadamard inequality; Invex sets; Lipschitz functions.

2010 AMS Subject Classification: 26D15, 26A51.

1. Introduction

Let $I = [c,d]$ be an interval on the real line $\mathbb{R}$, $f : I \to \mathbb{R}$ be a convex function and $a,b \in [c,d], a < b$. We consider the well-known Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Several refinements and generalizations of Hermite-Hadamard have been found in [1-5, 8-12, 16] and references therein. In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of preinvex functions introduced by Ben-Israel and Mond in [7] (see [6, 14] for more property and generalizations).

E-mail address: barani.a@lu.ac.ir

Received June 13, 2015

Now, we recall some notions in invexity analysis which will be used throughout the paper (see [2, 13, 14, 17] and references therein). A set $S \subseteq \mathbb{R}$ is said to be invex with respect to the map $\eta : S \times S \to S$, if for every $x, y \in S$ and every $t \in [0,1]$, $y + t\eta(x,y) \in S$. Recall that an $\eta-$path for $x, y \in S$ is a subset of S defined by

$$P_{xy} := \{x + t\eta(y,x) |\ 0 \leq t \leq 1\}.$$

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets which are not convex. The mapping $\eta : S \times S \to S$ is said to be satisfies the condition C if for every $x, y \in S$ and $t \in [0,1]$,

$$\eta(y, y + t\eta(x,y)) = -t\eta(x,y),$$
$$\eta(x, y + t\eta(x,y)) = (1-t)\eta(x,y).$$

For every $x, y \in S$ and every $t_1, t_2 \in [0,1]$ from condition C we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y). \tag{1.2}$$

Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to S$. Then, the function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if for every $x, y \in S$ and $t \in [0,1]$,

$$f(y + t\eta(x,y)) \leq tf(x) + (1-t)f(y).$$

Every convex function is a preinvex with respect to the map $\eta(x,y) = x - y$ but the converse does not holds. The following Hermite-Hadamard inequality for preinvex functions is introduced in [15],

$$f(a + \frac{1}{2}\eta(b,a)) \leq \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.3}$$

where $a, b \in S$, (see also [5]).

On the other hand, Dragomir in [9] defined two mapping $H, F : [0,1] \to \mathbb{R}$, as follows and established several important results in connection to Hermite-Hadamard inequality.

$$\begin{aligned} H(t) &:= \frac{1}{b-a} \int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)dx, \\ F(t) &:= \frac{1}{(b-a)^2} \int_a^b \int_a^b f\Big(tx + (1-t)y\Big)dxdy. \end{aligned} \tag{1.4}$$

Since then numerous articles have appeared in the literature reflecting further applications and properties of mappings H, F, (see [3, 8, 10, 11, 16]) and references therein. In [10] Dragomir by relaxing convexity and utilizing two above mapping, introduced some Hermite-Hadamard inequalities for Lipschitz functions defined on intervals as follows;

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $M-$Lipschitz function and $a, b \in I$ with $a < b$. Then, we have the following inequalities*

$$\left| f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{4}(b-a), \tag{1.5}$$

and

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M}{3}(b-a). \tag{1.6}$$

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $M-$Lipschitz function and $a, b \in I$ with $a < b$. Then,*

(i) The mapping H is $\frac{M}{4}(b-a)-$Lipschitz on $[0,1]$.

(ii) For every $t \in [0,1]$ we have the following inequalities

$$\left| H(t) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{M(1-t)}{4}(b-a), \tag{1.7}$$

$$\left| f\Big(\frac{a+b}{2}\Big) - H(t) \right| \leq \frac{Mt}{4}(b-a), \tag{1.8}$$

and

$$\left| H(t) - t\frac{1}{b-a}\int_a^b f(x)dx - (1-t)f\Big(\frac{a+b}{2}\Big) \right| \leq \frac{t(1-t)M}{2}(b-a). \tag{1.9}$$

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $M-$Lipschitz function and $a, b \in I$ with $a < b$. Then,*

(i) $F(t) = F(1-t)$, for all $t \in [0,1]$

(ii) The mapping F is a $\frac{M(b-a)}{3}-$Lipsschitz function on $[0,1]$.

(iii) We have the following inequalities

$$\left| F(t) - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\Big(\frac{x+y}{2}\Big)dxdy \right| \leq \frac{M|2t-1|}{6}(b-a), \tag{1.10}$$

$$\left| F(t) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{Mt}{3}(b-a), \tag{1.11}$$

and

$$|F(t)-H(t)| \leq \frac{M(1-t)}{4}(b-a). \tag{1.12}$$

The main purpose of this paper is to establishing some new inequalities involving generalizations of two above mappings for Lipschitz functions on invex subsets of $\mathbb{R}$.

2. Main results

At first we start with the following theorem connecting two inequalities of Hermite-Hadamard type for Lipschitz functions defined on invex sets.

Theorem 2.1. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to S$. Suppose that η satisfies condition C. Assume that $f : S \to \mathbb{R}$ is a $M-$Lipschitz function and $a, b \in S$ with $\eta(a,b) \neq 0$. Then,*

(i)

$$\left| f\big(a+\frac{1}{2}\eta(a,b)\big) - \frac{1}{\eta(a,b)} \int_b^c f(x)dx \right| \leq \frac{1}{4} M |\eta(a,b)|, \tag{2.1}$$

where $c := b+\eta(a,b)$.

(ii) If $a > b$ and $\eta(a,b) \geq a-b$, then,

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{1}{\eta(a,b)} \int_b^c f(x)dx \right| \\ &\leq \frac{M(a-b)^3}{3\eta(a,b)^2} + \frac{M\eta(a,b)}{2} + \frac{M(b-a)}{2}, \end{aligned} \tag{2.2}$$

where $c := b+\eta(a,b)$.

Proof. (i) Let $a, b \in S$ and $t \in [0,1]$. Then,

$$\begin{aligned} &\Big| tf(a) + (1-t)f(b) - f(b+t\eta(a,b)) \Big| \\ &= \Big| t\big(f(a) - f(b+t\eta(a,b))\big) + (1-t)\big(f(b) - f(b+t\eta(a,b))\Big| \\ &\leq t\Big| f(a) - f(b+t\eta(a,b))\Big| + (1-t)\Big| f(b) - f(b+t\eta(a,b))\Big| \\ &\leq tM|a-b-t\eta(a,b)| + t(1-t)M|\eta(a,b)|. \end{aligned} \tag{2.3}$$

For $t=\frac{1}{2}$, we get

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2}-f(b+\frac{1}{2}\eta(a,b))\Big|\\&\leq\frac{1}{2}M\left|a-b-\frac{1}{2}\eta(a,b)\right|+\frac{1}{2}M\left|\eta(a,b)\right|.\end{aligned}\tag{2.4}$$

If in (2.4) we put $b+t\eta(a,b)$ and $b+(1-t)\eta(a,b)$ instate of a and b, respectively then we obtain

$$\begin{aligned}&\Big|\frac{f(b+t\eta(a,b))+f(b+(1-t)\eta(a,b))}{2}-f(b+\frac{1}{2}\eta(a,b))\Big|\\&\leq\frac{1}{2}M|2t-1|\left|\eta(a,b)\right|.\end{aligned}\tag{2.5}$$

Integrating on $[0,1]$ implies that

$$\begin{aligned}&\left|\frac{\int_0^1 f(b+t\eta(a,b))dt+\int_0^1 f(b+(1-t)\eta(a,b))dt}{2}-f(b+\frac{1}{2}\eta(a,b))\right|\\&\leq\frac{1}{4}M\left|\eta(a,b)\right|.\end{aligned}\tag{2.6}$$

(ii) For every $t\in[0,1]$, from (2.3) we have

$$\begin{aligned}&\left|tf(a)+(1-t)f(b)-f(b+t\eta(a,b))\right|\\&\leq tM\left|a-b-t\eta(a,b)\right|+t(1-t)M\eta(a,b).\end{aligned}\tag{2.7}$$

By integrating on $[0,1]$ we get

$$\begin{aligned}&\left|f(a)\int_0^1 tdt+f(b)\int_0^1(1-t)dt-\int_0^1 f(a+t\eta(a,b))dt\right|\\&\left(=\left|\frac{f(a)+f(b)}{2}-\frac{1}{\eta(a,b)}\int_a^c f(x)dx\right|\right)\\&\leq\left[\int_0^1 tM\left|a-b-t\eta(a,b)\right|dt\right]+\int_0^1 t(1-t)M\eta(a,b)dt\\&=\left[\frac{M(a-b)^3}{3\eta(a,b)^2}+\frac{M\eta(a,b)}{3}+\frac{M(b-a)}{2}\right]+\frac{M\eta(a,b)}{6}\\&=\frac{M(a-b)^3}{3\eta(a,b)^2}+\frac{M\eta(a,b)}{2}+\frac{M(b-a)}{2}.\end{aligned}\tag{2.8}$$

Note that, by simple computation we have

$$\begin{aligned}
&\int_0^1 t\,|a-b-t\eta(a,b)|\,dt\\
&=\int_0^{\lambda} t\big(a-b-t\eta(a,b)\big)dt+\int_{\lambda}^1 t\big(b-a+t\eta(a,b)\big)dt\\
&=\frac{(a-b)^3}{3\eta(a,b)^2}+\frac{\eta(a,b)}{3}+\frac{b-a}{2},
\end{aligned}$$

where $\lambda := \frac{a-b}{\eta(a,b)}$.

Remark 2.1. *If in Theorem 2.1, $\eta(x,y)=x-y$, for every $x,y\in S$, then we have the results in Theorem 1.1, as a special case.*

Motivated by [9] for a real valued function f defined on an invex set $S\subseteq\mathbb{R}$ with respect to $\eta : S\times S\to S$, we consider two mappings $\mathsf{H},\mathsf{F} : [0,1]\to R$, as follows;

$$\mathsf{H}(t) := \frac{1}{\eta(b,a)}\int_a^c f\big(a+\frac{1}{2}\eta(b,a)+t\eta(y,a+\frac{1}{2}\eta(b,a))\big)dy,$$

and

$$\mathsf{F}(t) := \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f(x+t\eta(y,x))dxdy,$$

where $a,b\in S$ and $c := a+\eta(b,a)$. Note that in the special case, if $\eta(x,y)=x-y$ for every $x,y\in S$ then, the mappings H and F reduce to mappings H and F defined in (1.4), respectively. The following theorem is a generalization of theorem 1.2 in invexity setting.

Theorem 2.2. *Let $S\subseteq\mathbb{R}$ be an invex set with respect to $\eta : S\times S\to S$. Suppose that η satisfies condition C and for every $x\neq y\in S$, $\eta(y,x)\neq 0$. Assume that $f : S\to\mathbb{R}$ is a $M-$Lipschitz function. Then, for every $a,b\in S$ one has*

(i) The mapping H is $\frac{M}{4}|\eta(b,a)|-$Lipschitz on $[0,1]$.

(ii) For every $t\in[0,1]$ we have the following inequalities

$$\left|H(t)-\frac{1}{\eta(b,a)}\int_a^c f(x)dx\right| \le \frac{M(1-t)}{4}\,|\eta(b,a)|\,, \tag{2.9}$$

$$\left|f\big(a+\frac{1}{2}\eta(b,a)\big)-H(t)\right| \le \frac{Mt}{4}|\eta(b,a)|, \tag{2.10}$$

and

$$\begin{aligned}&\Big|H(t)-t\frac{1}{\eta(b,a)}\int_a^c f(x)dx-(1-t)f\big(a+\frac{1}{2}\eta(b,a)\big)\Big|\\&\le\frac{t(1-t)M}{2}|\eta(b,a)|,\end{aligned}\tag{2.11}$$

where $c := a+\eta(b,a)$.

Proof. Let $t_1,t_2\in[0,1]$. Then,

$$\begin{aligned}&\Big|\mathsf{H}(t_2)-\mathsf{H}(t_1)\Big|\\&=\frac{1}{|\eta(b,a)|}\Big|\int_a^c f\big(a+\frac{1}{2}\eta(b,a)+t_2\eta(x,a+\frac{1}{2}\eta(b,a))\big)\\&-\int_a^c f\big(a+\frac{1}{2}\eta(b,a)+t_1\eta(x,a+\frac{1}{2}\eta(b,a))\big)dx\Big|\\&\le\frac{1}{|\eta(b,a)|}\int_a^c\Big|f\big(a+\frac{1}{2}\eta(b,a)+t_2\eta(x,a+\frac{1}{2}\eta(b,a))\big)\\&-f\big(a+\frac{1}{2}\eta(b,a)+t_1\eta(x,a+\frac{1}{2}\eta(b,a))\big)\Big|dx\\&\le\frac{M}{|\eta(b,a)|}\int_a^c\Big|t_2\eta(x,a+\frac{1}{2}\eta(b,a))-t_1\eta(x,a+\frac{1}{2}\eta(b,a))\Big|dx\\&=\frac{M|t_2-t_1|}{|\eta(b,a)|}\int_a^c\big|\eta(x,a+\frac{1}{2}\eta(b,a))\big|dx\\&=\frac{M|\eta(b,a)|}{4}|t_2-t_1|.\end{aligned}\tag{2.12}$$

Indeed, if we choose the change of variable $x:=a+s\eta(b,a), s\in[0,1]$, and using (1.2) then we have

$$\begin{aligned}&\int_a^c\big|\eta(x,a+\frac{1}{2}\eta(b,a))\big|dx\\&\int_0^1\big|\eta(a+s\eta(b,a),a+\frac{1}{2}\eta(b,a))\big||\eta(b,a)|ds\\&=|\eta(b,a)|^2\int_0^1\big|s-\frac{1}{2}\big|ds=\frac{1}{4}|\eta(b,a)|^2,\end{aligned}\tag{2.13}$$

this completes the proof of (i).

(ii) It is easy to see that

$$\mathsf{H}(0)=f\big(a+\frac{1}{2}\eta(b,a)\big),$$

and

$$\mathsf{H}(1)=\frac{1}{\eta(b,a)}\int_a^c f(x)dx.$$

Now, the inequalities (2.9) and (2.10) follow from (2.12) by choosing $t_1 = t, t_2 = 1$ and $t_1 = 0, t_2 = t$, respectively. Inequality (2.11) follow by adding t times (2.9) and $(1-t)$ times (2.10). This completes the proof.

Theorem 2.2. *Let $S \subseteq \mathbb{R}$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}$. Suppose that for every $x \neq y \in S$, $\eta(y,x) \neq 0$. If $f : S \to \mathbb{R}$ is a $M-$Lipsschitz function then; for every $a,b \in S$,*

(i) $\mathsf{F}(t) = \mathsf{F}(1-t)$, for all $t \in [0,1]$

(ii) If η satisfies condition C then, F is a $\frac{M|\eta(b,a)|}{3}-$Lipsschitz function on $[0,1]$.

(iii) If η satisfies condition C then, for every $t \in [0,1]$ we have the following inequalities

$$\begin{aligned} &\left|\mathsf{F}(t) - \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f\big(x+\frac{1}{2}\eta(y,x)\big)dxdy\right| \\ &\leq \frac{M|2t-1|}{6}|\eta(b,a)|, \end{aligned} \tag{2.14}$$

$$\left|\mathsf{F}(t) - \frac{1}{\eta(b,a)}\int_a^c f(x)dx\right| \leq \frac{Mt}{3}|\eta(b,a)|, \tag{2.15}$$

and

$$|\mathsf{F}(t) - \mathsf{H}(t)| \leq \frac{M(1-t)}{4}|\eta(b,a)|. \tag{2.16}$$

Proof. (i) It is obvious.

(ii) Let $t_1, t_2 \in [0,1]$. Then,

$$\begin{aligned} &|\mathsf{F}(t_2) - \mathsf{F}(t_1)| \\ &= \frac{1}{\eta(b,a)^2}\left|\int_a^c\int_a^c \big[f\big(x+t_2\eta(y,x)\big) - f\big(x+t_1\eta(y,x)\big)\big]dxdy\right| \\ &\leq \frac{1}{\eta(b,a)^2}\int_a^c\int_a^c \Big|f\big(x+t_2\eta(y,x)\big) - f\big(x+t_1\eta(y,x)\big)\Big|dxdy \\ &\leq \frac{M|t_2-t_1|}{\eta(b,a)^2}\int_a^c\int_a^c \big|\eta(y,x)\big|dxdy. \end{aligned} \tag{2.17}$$

On the other hand, if we use the change of variables $x := a + r\eta(b,a), y := a + s\eta(b,a),\ r,s \in [0,1]$ then by simple computation we get

$$\frac{\partial(x,y)}{\partial(r,s)} = \begin{pmatrix} \eta(y,x) & 0 \\ 0 & \eta(y,x) \end{pmatrix},$$

hence $\det \frac{\partial(x,y)}{\partial(r,s)} = \eta(b,a)^2$. Now, by using (1.2) we obtain

$$\begin{aligned}
&\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c \big|\eta(y,x)\big|dxdy \\
&= \frac{1}{\eta(b,a)^2}\int_0^1\int_0^1 \big|\eta(a+s\eta(b,a),a+r\eta(b,a))\big|\eta(b,a)^2 drds \\
&= |\eta(b,a)|\int_0^1\int_0^1 \big|s-r\big|drds = \frac{|\eta(b,a)|}{3}.
\end{aligned} \tag{2.18}$$

By combining (2.17) and (2.18) it follows that

$$|\mathsf{F}(t_2)-\mathsf{F}(t_1)| \le \frac{M|\eta(b,a)|}{3}|t_2-t_1|. \tag{2.19}$$

(iii) The inequalities (2.14) and (2.15) follows from (2.19) if we choose $t_1=\frac{1}{2}, t_2=t$ and $t_1=0, t_2=t$, respectively.

Now, we prove the inequality (2.16). Since f is $M-$Lipschitz if we set

$$y := a+s\eta(b,a),\ x := a+r\eta(b,a),\ r,s\in[0,1]$$

then, by using (1.2) we have

$$\begin{aligned}
&\left|f(y+t\eta(x,y)) - f\Big(a+\frac{1}{2}\eta(b,a)+t\eta(x,a+\frac{1}{2}\eta(b,a))\Big)\right| \\
&\le M\left|y+t\eta(x,y)-a-\frac{1}{2}\eta(b,a)-t\eta(x,a+\frac{1}{2}\eta(b,a))\right| \\
&= M\left|s\eta(b,a)+t(r-s)\eta(b,a)-\frac{1}{2}\eta(b,a))-t(r-\frac{1}{2})\eta(b,a))\right| \\
&= M\left|s\eta(b,a)-ts\eta(b,a)-\frac{1}{2}\eta(b,a))+\frac{1}{2}t\eta(b,a))\right| \\
&= M\left|(1-t)s\eta(b,a)-(1-t)\frac{1}{2}\eta(b,a))\right| \\
&= (1-t)M\left|y-a-\frac{1}{2}\eta(b,a)\right|, \text{ for all } t\in[0,1].
\end{aligned} \tag{2.20}$$

By integrating the inequality (2.20) on $P_{ab} \times P_{ab}$ we have

$$\begin{aligned}
&\Bigg|\frac{1}{\eta(b,a)^2}\int_a^c\int_a^c f\big(x+\frac{1}{2}\eta(y,x)\big)dxdy \\
&-\frac{1}{\eta(b,a)}\int_a^c f\Big(a+\frac{1}{2}\eta(b,a)+t\eta(x,a+\frac{1}{2}\eta(b,a))\Big)dx\Bigg| \\
&\leq (1-t)M\frac{1}{\eta(b,a)}\int_a^c \Big|y-a-\frac{1}{2}\eta(b,a)\Big|dy \\
&=\frac{(1-t)M}{4}|\eta(b,a)|.
\end{aligned}$$

This completes the proof.

Corollary 2.2. *If in Theorem 2.2, $\eta(x,y) = x - y$, for every $x, y \in S$, then we have the results in Theorem 1.3, as a special case.*

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] M. Akkouchi, A result on the mapping H of S.S. Dragomir with applications, Facta Universitatis, Ser. Math. Inform. 17 (2002), 5-12.

[2] T. Antczak, Mean value in invexity analysis, Nonlinear Anal. 60 (2005), 1471-1484.

[3] S. Banić, Mapping connected with Hermite-Hadamard inequalities for superquadratic functions, J. Math. Inequal. 3 (2009), 557-589.

[4] A. Barani, S. Barani, Hermite-Hadamard inequalities for functions when a power of the absolute value of the first derivative is P-convex, Bull. Aust. Math. Soc. 86 (2012), 126-134.

[5] A. Barani, A.G. Ghazanfari and S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, J. Ineque. Appl. 2012 (2012) Article ID 247.

[6] A. Barani and M.R. Pouryayevali, Invex sets and preinvex functions on Reimannian manifolds, J. Math. Anal. Appl. 328 (2007), 767-779.

[7] A. Ben-Israel, B. Mond, What is invexity?, J. Aust. Math. Soc. 28 (1986), 1-9.

[8] S.S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and applications, (RGMIA Monographs http:// rgmia.vu.edu.au/ monographs/ hermite hadamard.html), Victoria University, 2000.

[9] S.S. Dragomir, Two mappings in connection to Hadamard's inequality, J. Math. Anal. Appl. 167 (1992), 49-56.

[10] S.S. Dragomir , Inequalities on Hadamard's type for Lipschitzian mapping and their applications, J. Math. Anal. Appl. 245 (2000), 489-501.

[11] S.S. Dragomir, A sequence of mapping associated with the Hermite-Hadamard inequalities and applications, Applications of Mathematics, 49 (2004), 123-140.

[12] A. G. Ghazanfari, A. Barani, Some Hermite-Hadamard type inequalities for the product of two operator preinvex functions, Banach J. Math. Anal. 9 (2015), 9-20.

[13] M.A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981), 545-550.

[14] S. R. Mohan and S. K. Neogy, On invex sets and preinvex function, J. Math. Anal. Appl. 189 (1995), 901-908.

[15] M.A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory 2 (2007), no. 2, 126-131.

[16] K.L. Tesing, S.R. Hwang and S.S. Dragomir, New Hermite-Hadamard-type inequalities for convex functions (II), Comput. Math. Appl. 62 (2011), 401-418.

[17] T. Weir, and B. Mond, Preinvex Functions in multiple Objective Optimization, J. Math. Anal. Appl. 136 (1998), 29-38.

[18] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.