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# SOME REMARKS ON NIEZGODA'S EXTENSION OF JENSEN-MERCER INEQUALITY

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**Abstract.** In this article, we would restate the Niezgoda's extension of Jensen-Mercer inequality in an alternate form and we would give refinement of the Niezgoda's result with some applications in terms of weighted power mean.

**Keywords:** Convex functions; Jensen's inequality; Mercer's inequality; Refinement.

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## 1. Introduction

In article [11], A. McD. Mercer proved the following variant of Jensen's inequality, to which we will refer as to the Jensen-Mercer inequality.

**Theorem 1.** *Let $[a,b]$ be an interval in $\mathbb{R}$ and $x_1,\dots,x_n \in [a,b]$. Let $w_1,\dots,w_n$ be nonnegative real numbers such that $\sum_{i=1}^n w_i = 1$. If $\phi$ is a convex function on $[a,b]$, then*

$$\phi\left(a+b-\sum_{i=1}^n w_i x_i\right) \le \phi(a)+\phi(b)-\sum_{i=1}^n w_i\phi(x_i). \tag{1}$$

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**Definition 1.** Given two real row $n$-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{y}$ is said to majorize $\mathbf{x}$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$$

holds for $k \in \{1, \ldots, n-1\}$ and

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $x_{[1]} \geq \ldots \geq x_{[n]}$ and $y_{[1]} \geq \ldots \geq y_{[n]}$ are the entries of $\mathbf{x}$ and $\mathbf{y}$, respectively, in nonincreasing order.

This notion and notation of majorization was introduced by Hardy et al. in [5]. Now, we state the well-known majorization theorem from the same book [5] as follows (see also [9]).

**Theorem 2.** *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. *The inequality*

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$$

*holds for every continuous convex function* $f : \mathbb{R} \to \mathbb{R}$ *if and only if* $\mathbf{x} \prec \mathbf{y}$.

The following extension of (1) was given by Niezgoda in [12], to which we will refer as to the Niezgoda's inequality (see [6, 8, 13, 14] for recent extensions of (1).)

**Theorem 3.** *Let* $\phi : [a,b] \to \mathbb{R}$ *be a continuous convex function. Suppose that* $\boldsymbol{a} = (a_1, \ldots, a_m)$ *with* $a_j \in [a,b]$ *and* $\boldsymbol{X} = (x_{ij})$ *is a real* $n \times m$ *matrix such that* $x_{ij} \in [a,b]$ *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$.

*If* $\boldsymbol{a}$ *majorizes each row of* $\boldsymbol{X}$, *that is,*

$$\boldsymbol{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \boldsymbol{a} \text{ for each } i \in \{1, \ldots, n\},$$

*then we have the inequality*

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \phi(x_{ij}), \tag{2}$$

*where* $\sum_{i=1}^{n} w_i = 1$ *with* $w_i \geq 0$.

In this article, we would restate the Niezgoda's result in an alternate form and we would give refinement of the Niezgoda's inequality with some applications in terms of weighted power mean. This article is based on four main sections. The first section is devoted to introduction and preliminaries. The second section deals with some alternate forms of Niezgoda's inequality. The third section deals with the refinement of the result which is stated in the second section. The last section totally based on some of the applications of the refinement of the Niezgoda's inequality.

## 2. Niezgoda's inequality- An alternate form

Here we need some construction for our next result. Given a function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ and an $n \times m$ matrix $\mathbf{z} = (z_{ij})$ such that $z_{ij} \in I$ for all $i, j$, we define $g(\mathbf{z})$ to be the matrix $(g(z_{ij}))$. The $i$th row and $j$th column of $\mathbf{z}$ are denoted by $\mathbf{z}_{i.}$ and $\mathbf{z}_{.j}$, respectively. For example, $g(\mathbf{z}_{.j}) = (z_{1j}, \ldots, z_{nj})^T$. By $\mathbf{1}_n$ we denote the (column) $n$-tuple of ones.

Now we would restate the Niezgoda's inequality in an alternate form and by using the techniques of article [12] we would prove it.

**Theorem 4.** *Let all the assumptions of Theorem 3 be valid. Then we have the inequality*

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i \phi(x_{ij}) - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i \phi(x_{ij}) \tag{3}$$

*where* $\sum_{i=1}^{n} w_i = 1$ *with* $w_i \geq 0$ *and* $k \in \{1, \ldots, m\}$.

*Proof.* Fix $k \in \{1, \ldots, m\}$. Taking

$$\Psi\mathbf{z} = \sum_{i=1}^{n} w_i z_i \quad \text{for} \quad \mathbf{z} = (z_1, \ldots, z_n)^T, \tag{4}$$

we have an observation that $\Psi$ is linear and monotonic with $\Psi\mathbf{1}_n = 1$. So by using linearity of $\Psi$ and convexity of $\phi$ we can write

$$\begin{aligned}
&\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \\
&= \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} \Psi \mathbf{x}_{.j} - \sum_{j=k+1}^{m} \Psi \mathbf{x}_{.j}\right) \\
&= \phi\left(\sum_{j=1}^{m} a_j \cdot 1 - \sum_{j=1}^{k-1} \Psi \mathbf{x}_{.j} - \sum_{j=k+1}^{m} \Psi \mathbf{x}_{.j}\right) \\
&= \phi\Psi\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{k-1} \mathbf{x}_{.j} - \sum_{j=k+1}^{m} \mathbf{x}_{.j}\right) \\
&\leq \Psi\phi\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{k-1} \mathbf{x}_{.j} - \sum_{j=k+1}^{m} \mathbf{x}_{.j}\right).
\end{aligned} \tag{5}$$

Now using majorization for each $i \in \{1, \ldots, n\}$ we have

$$\sum_{j=1}^{m} x_{ij} = \sum_{j=1}^{m} a_j,$$

which gives

$$\sum_{j=1}^{m} a_j = \sum_{j=1}^{k-1} x_{ij} + x_{ik} + \sum_{j=k+1}^{m} x_{ij}$$

or we can write

$$\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij} = x_{ik}.$$

Now applying $\phi$ and using majorization theorem we get

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right) = \phi(x_{ik}) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{k-1} \phi(x_{ij}) - \sum_{j=k+1}^{m} \phi(x_{ij})$$

or

$$\phi\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{k-1} \mathbf{x}_{.j} - \sum_{j=k+1}^{m} \mathbf{x}_{.j}\right) \leq \sum_{j=1}^{m} \phi(a_j)\mathbf{1}_n - \sum_{j=1}^{k-1} \phi(\mathbf{x}_{.j}) - \sum_{j=k+1}^{m} \phi(\mathbf{x}_{.j}).$$

Now, using monotonicity property of $\Psi$ we obtain

$$\begin{aligned}
&\Psi\phi\left(\sum_{j=1}^{m} a_j \mathbf{1}_n - \sum_{j=1}^{k-1} \mathbf{x}_{.j} - \sum_{j=k+1}^{m} \mathbf{x}_{.j}\right) \leq \Psi\left(\sum_{j=1}^{m} \phi(a_j)\mathbf{1}_n - \sum_{j=1}^{k-1} \phi(\mathbf{x}_{.j}) - \sum_{j=k+1}^{m} \phi(\mathbf{x}_{.j})\right) \\
&= \sum_{j=1}^{m} \phi(a_j)\Psi\mathbf{1}_n - \sum_{j=1}^{k-1} \Psi\phi(\mathbf{x}_{.j}) - \sum_{j=k+1}^{m} \Psi\phi(\mathbf{x}_{.j}) \\
&= \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{k-1} \Psi\phi(\mathbf{x}_{.j}) - \sum_{j=k+1}^{m} \Psi\phi(\mathbf{x}_{.j}).
\end{aligned}$$

Combining with (5) we have

$$\phi\Psi\left(\sum_{j=1}^{m} a_j\mathbf{1}_n - \sum_{j=1}^{k-1}\mathbf{x}_{.j} - \sum_{j=k+1}^{m}\mathbf{x}_{.j}\right) \le \sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{k-1}\Psi\phi(\mathbf{x}_{.j}) - \sum_{j=k+1}^{m}\Psi\phi(\mathbf{x}_{.j}). \tag{6}$$

Using (5) and (6) we finally obtain our required result

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \le \sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i\phi(x_{ij}) - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i\phi(x_{ij}).$$

**Remark 1.** Now we show that Theorem 3 and Theorem 4 are equivalent and from one we can deduce the other. With $k = m$, inequality (3) gives us inequality (2). On the other hand, the majorization preorder $\prec$ on $\mathbb{R}^n$ is permutation-invariant. Therefore columns of the matrix $\mathbf{X}$ can be permuted, and Theorem 3 remains still true. Thus Theorem 4 follows.

## 3. Refinements

Let a convex function $\phi : [a,b] \to \mathbb{R}$ . Suppose that $\mathbf{a} = (a_1,\dots,a_m)$ with $a_j \in [a,b]$, and $\mathbf{X} = (x_{ij})$ is a real $n \times m$ matrix such that $x_{ij} \in [a,b]$ and $\mathbf{a}$ majorizes each row of $\mathbf{X}$ for all $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,m\}$ with $\sum_{i=1}^{n} w_i = 1$. Then for any non-empty subset $I$ of $\{1,2,\dots,n\}$ we take $\bar{I} := \{1,2,\dots,n\} \setminus I \neq \emptyset$ and define $W_I = \sum_{i\in I} w_i$ and $W_{\bar{I}} = 1 - \sum_{i\in I} w_i$. For the convex function $\phi$ and the matrix $\mathbf{X} = (x_{ij})$ for $i \in I$, $j \in \{1,\dots,m\}$ and $\mathbf{w} = (w_1,\dots,w_n)$ as above, we can define the following functional

$$\begin{aligned} D(\mathbf{w},\mathbf{X},\phi;I) := \quad & W_I\phi\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}\right) \\ + \quad & W_{\bar{I}}\phi\left(\sum_{j=1}^{m} a_j - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{k-1}\sum_{i\in \bar{I}} w_i x_{ij} - \frac{1}{W_{\bar{I}}}\sum_{j=k+1}^{m}\sum_{i\in \bar{I}} w_i x_{ij}\right). \end{aligned} \tag{7}$$

Then the following refinement holds.

**Theorem 5.** *Let all the assumptions of Theorem 4 be valid. Then for any non empty subset I of* $\{1,\dots,n\}$ *we have*

$$\begin{aligned} \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) &\le D(\mathbf{w},\mathbf{X},\phi;I) \\ &\le \sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i\phi(x_{ij}) - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i\phi(x_{ij}) \end{aligned} \tag{8}$$

*Proof.* Fix $k \in \{1,\dots,m\}$. By the property of convex function we have

$$
\begin{aligned}
&\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right)\\
&= \phi\left[\sum_{i=1}^{n} w_i \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right)\right]\\
&= \phi\left[W_I\left(\frac{1}{W_I}\sum_{i\in I} w_i \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right)\right)\right.\\
&\left. + W_{\bar{I}}\left(\frac{1}{W_{\bar{I}}}\sum_{i\in I} w_i \left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} x_{ij} - \sum_{j=k+1}^{m} x_{ij}\right)\right)\right]\\
&\le W_I \phi\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}\right)\\
&+ W_{\bar{I}} \phi\left(\sum_{j=1}^{m} a_j - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{k-1}\sum_{i\in \bar{I}} w_i x_{ij} - \frac{1}{W_{\bar{I}}}\sum_{j=k+1}^{m}\sum_{i\in \bar{I}} w_i x_{ij}\right) = D(\mathbf{w},\mathbf{X},\phi;I).
\end{aligned}
$$

Now using generalized Niezgoda's inequality (2) in the following functional

$$
\begin{aligned}
D(\mathbf{w},\mathbf{X},\phi;I) &= W_I \phi\left(\sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}\right)\\
&+ W_{\bar{I}} \phi\left(\sum_{j=1}^{m} a_j - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{k-1}\sum_{i\in \bar{I}} w_i x_{ij} - \frac{1}{W_{\bar{I}}}\sum_{j=k+1}^{m}\sum_{i\in \bar{I}} w_i x_{ij}\right)\\
&\le W_I \left(\sum_{j=1}^{m} \phi(a_j) - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i \phi(x_{ij}) - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i \phi(x_{ij})\right)\\
&+ W_{\bar{I}} \left(\sum_{j=1}^{m} \phi(a_j) - \frac{1}{W_{\bar{I}}}\sum_{j=1}^{k-1}\sum_{i\in \bar{I}} w_i \phi(x_{ij}) - \frac{1}{W_{\bar{I}}}\sum_{j=k+1}^{m}\sum_{i\in \bar{I}} w_i \phi(x_{ij})\right)\\
&= \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i \phi(x_{ij}) - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i \phi(x_{ij})
\end{aligned}
$$

for any $I$, which proves the second inequality in (8).

**Remark 2.** It can be observed that the inequality (8) can also be written as

$$
\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1}\sum_{i=1}^{n} w_i x_{ij} - \sum_{j=k+1}^{m}\sum_{i=1}^{n} w_i x_{ij}\right) \le \min_{I} D(\mathbf{w},\mathbf{X},\phi;I) \tag{9}
$$

and

$$\max_I D(\mathbf{w}, \mathbf{X}, \phi; I) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{k-1} \sum_{i=1}^{n} w_i \phi(x_{ij}) - \sum_{j=k+1}^{m} \sum_{i=1}^{n} w_i \phi(x_{ij}). \tag{10}$$

**Remark 3.** Similar remarks as given in Remarks 1 hold for Theorem 5 as well.

For our next corollary we need the following definition.

**Definition 2.** [9, p. 10] An $m \times m$ matrix $\mathbf{A} = (a_{jk})$ is said to be doubly stochastic, if $a_{jk} \geq 0$ and $\sum_{j=1}^{m} a_{jk} = \sum_{k=1}^{m} a_{jk} = 1$ for all $j, k \in \{1, \ldots, m\}$.

It is well known [9, p. 31] that if $\mathbf{A}$ is an $m \times m$ doubly stochastic matrix, then

$$\mathbf{aA} \prec \mathbf{a} \text{ for each real } m\text{−tuple } \mathbf{a} = (a_1, \ldots, a_m). \tag{11}$$

By applying Theorem 4 and (11), one obtains:

**Corollary 1.** Let $\phi : [a,b] \to \mathbb{R}$ be a continuous convex function on $[a,b]$. Suppose that $\mathbf{a} = (a_1, \ldots, a_m)$ with $a_j \in [a,b]$ $j \in \{1, \ldots, m\}$ and $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ are $m \times m$ doubly stochastic matrices. Set

$$X = (x_{ij}) = \begin{pmatrix} \mathbf{aA}_1 \\ \vdots \\ \mathbf{aA}_n \end{pmatrix}.$$

Then inequality (8) holds.

## 4. Applications

For $\emptyset \neq I \subseteq \{1, \ldots, n\}$ $A_I, G_I, H_I$ and $M_I^{[r]}$ are the arithmetic, geometric, harmonic means, and power mean of the order $r \in \mathbb{R}$, respectively of $x_{ij} \in [a,b]$, where $0 < a < b$, formed along with the positive weights $w_i$, $i \in I$. For $I = \{1, \ldots, n\}$ denoting the arithmetic, geometric, harmonic and power means as $A_n, G_n, H_n$ and $M_n^{[r]}$ respectively.

Define

$$\tilde{A}_I := \sum_{j=1}^{m} a_j - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}$$
$$= \sum_{j=1}^{m} a_j - \sum_{j=1}^{k-1} A_I(\mathbf{x}_j,\mathbf{w}) - \sum_{j=k+1}^{m} A_I(\mathbf{x}_j,\mathbf{w})$$

$$\tilde{G}_I := \frac{\prod_{i\in I} a_j}{\left(\prod_{j=1}^{k-1}\prod_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}\left(\prod_{j=k+1}^{m}\prod_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}}$$

$$\tilde{H}_I := \left(\sum_{j=1}^{m} a_j^{-1} - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^{-1} - \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^{-1}\right)^{-1}$$

$$\tilde{M}_I^{[r]} := \begin{cases} \left(\sum_{j=1}^{m}(a_j)^r - M_I^{[r]}\right)^{\frac{1}{r}}, & r\neq 0,\\ \tilde{G}_I, & r=0,\end{cases}$$

where

$$M_I^{[r]} := \begin{cases} \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i\in I} w_i x_{ij}^r + \frac{1}{W_I}\sum_{j=k+1}^{m}\sum_{i\in I} w_i x_{ij}^r, & r\neq 0,\\ \left(\prod_{i\in I} x_{ij}^{w_i}\right)^{\frac{1}{W_I}}, & r=0.\end{cases}$$

Then the following inequalities hold.

**Theorem 6.**

$$(i)\tilde{G}_n \le \min_I \tilde{A}_I^{W_I}\tilde{A}_{\bar{I}}^{W_{\bar{I}}} \text{ and } \tilde{A}_n \ge \max_I \tilde{A}_I^{W_I}\tilde{A}_{\bar{I}}^{W_{\bar{I}}}. \tag{12}$$

$$(ii)\tilde{G}_n \le \min_I\left[W_I\tilde{G}_I + W_{\bar{I}}\tilde{G}_{\bar{I}}\right] \text{ and } \tilde{A}_n \ge \max_I\left[W_I\tilde{G}_I + W_{\bar{I}}\tilde{G}_{\bar{I}}\right]. \tag{13}$$

*Proof.* (i) Applying Theorem 5 to the convex function $\phi(x) = -\ln x$, we obtain

$$-\ln\tilde{A}_n \le -W_I\ln\tilde{A}_I - W_{\bar{I}}\ln\tilde{A}_{\bar{I}} \le -\ln\tilde{G}_n. \tag{14}$$

Now (12) follows from Remark 2 and (14).

(ii) Taking convex function $\phi(x) = \exp x$, and replacing $a_j$ and $x_{ij}$ with $\ln a_j$ and $\ln x_{ij}$, respectively, then applying Theorem 5 we get

$$\tilde{G}_n \le W_I\tilde{G}_I + W_{\bar{I}}\tilde{G}_{\bar{I}} \le \tilde{A}_n. \tag{15}$$

now using Remark 2, we get (13).

Theorem 6 gives following interesting case.

**Corollary 2.**

$$(i)\ \frac{1}{\tilde{G}_n} \le \min_I \frac{1}{\tilde{H}_I^{W_I} \tilde{H}_{\bar{I}}^{W_{\bar{I}}}} \quad \text{and} \quad \frac{1}{\tilde{H}_n} \ge \max_I \frac{1}{\tilde{H}_I^{W_I} \tilde{H}_{\bar{I}}^{W_{\bar{I}}}}.$$

$$(ii)\ \frac{1}{\tilde{G}_n} \le \min_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right] \quad \text{and} \quad \frac{1}{\tilde{H}_n} \ge \max_I \left[\frac{W_I}{\tilde{G}_I} + \frac{W_{\bar{I}}}{\tilde{G}_{\bar{I}}}\right].$$

*Proof.* The substitutions $a_j \to \frac{1}{a_j}$ and $x_{ij} \to \frac{1}{x_{ij}}$ gives the proof directly from Theorem 6.

**Theorem 7.** For $r \le 1$, we get the following inequalities

$$\begin{aligned} \tilde{M}_n^{[r]} &\le \min_I \left[W_I \tilde{M}_I^{[r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]}\right], \\ \tilde{A}_n &\ge \max_I \left[W_I \tilde{M}_I^{[r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]}\right]. \end{aligned} \tag{16}$$

For $r \ge 1$, the inequalities in (16) are reversed.

*Proof.* For $r \le 1$, $r \ne 0$, using Theorem 5 for the convex function $\phi(x) = x^{\frac{1}{r}}$ and replacing $a_j$ and $x_{ij}$ by $a_j^r$ and $x_{ij}^r$ respectively we get

$$\tilde{M}_n^{[r]} \le W_I \tilde{M}_I^{[r]} + W_{\bar{I}} \tilde{M}_{\bar{I}}^{[r]} \le \tilde{A}_n. \tag{17}$$

Using Remark 2 we get (16).

For $r = 0$, using Theorem 5 for the convex function $\phi(x) = \exp x$, replacing $a_j$ and $x_{ij}$ with $\ln a_j$ and $\ln x_{ij}$, respectively, Remark 2 verify (16) by getting

$$\tilde{M}_n^{[r]} = \tilde{G}_n \quad \text{and} \quad \tilde{M}_{\bar{I}} = \tilde{G}_{\bar{I}}. \tag{18}$$

If $r \ge 1$, then the function $\phi(x) = x^{\frac{1}{r}}$ is concave, so the inequalities in (16) are reversed.

**Corollary 3.**

$$\begin{aligned} \tilde{H}_n &\le \min_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right], \\ \tilde{A}_n &\ge \max_I \left[W_I \tilde{H}_I + W_{\bar{I}} \tilde{H}_{\bar{I}}\right]. \end{aligned}$$

*Proof.* Taking the convex function $\phi(x) = \frac{1}{x}$ in the Theorem 5 by replacing $a_j \to \frac{1}{a_j}$ and $x_{ij} \to \frac{1}{x_{ij}}$ and using Remark 2 we get the required result.

**Theorem 8.** Let $r, s \in \mathbb{R}$, $r \le s$.

(i) If $s \ge 0$, then

$$\begin{aligned} \left(\tilde{M}_n^{[r]}\right)^s &\le \min_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right], \\ \left(\tilde{M}_n^{[s]}\right)^s &\ge \max_I \left[W_I \left(\tilde{M}_I^{[r]}\right)^s + W_{\bar{I}} \left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right]. \end{aligned} \tag{19}$$

(ii) If $s < 0$, then inequalities in (19) are reversed

*Proof.* Let $s \geq 0$. Taking convex function $\phi(x) = x^{\frac{s}{r}}$, replacing $a_j$ and $x_{ij}$ with $(a_j)^r$ and $(x_{ij})^r$ respectively, and then using Theorem 5 and Remark 2, we obtain (19).
If $s < 0$, then the function $\phi(x) = x^{\frac{s}{r}}$, is concave so inequalities in (19) are reversed.

Let $\phi : [a,b] \to \mathbb{R}$ be a strictly monotonic and continuous function. Then for a given $n$- tuple $\mathbf{x} = (x_1, \ldots, x_n) \in [a,b]^n$ and positive $n$- tuple $\mathbf{w} = (w_1, \ldots, w_n)$ with $\sum_{i=1}^n w_i = 1$, the value

$$M_\phi^{[n]} = \phi^{-1}\left(\sum_{i=1}^n w_i \phi(x_i)\right)$$

is well defined and is called *quasi − arithmetic mean* of $\mathbf{x}$ with weight $\mathbf{w}$ (see for example [2, p. 215]). If we define

$$\tilde{M}_\phi^{[n]} = \phi^{-1}\left(\sum_{j=1}^m \phi(a_j) - \sum_{j=1}^{k-1}\sum_{i=1}^n w_i \phi(x_{ij}) - \sum_{j=k+1}^m \sum_{i=1}^n w_i \phi(x_{ij})\right),$$

then we have the following results.

**Theorem 9.** Let $\phi, \psi : [a,b] \to \mathbb{R}$ be strictly monotonic and continuous functions. If $\psi \circ \phi^{-1}$ is convex on $[a,b]$, then

$$\begin{aligned} \psi\left(\tilde{M}_\phi^{[n]}\right) &\leq \min_I \left[W_I \psi\left(\tilde{M}_\phi^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_\phi^{[\bar{I}]}\right)\right], \\ \psi\left(\tilde{M}_\psi^{[n]}\right) &\geq \max_I \left[W_I \psi\left(\tilde{M}_\phi^{[I]}\right) + W_{\bar{I}} \psi\left(\tilde{M}_\phi^{[\bar{I}]}\right)\right] \end{aligned} \tag{20}$$

where $\tilde{M}_\phi^{[I]}$ is defined as

$$\tilde{M}_\phi^{[I]} = \phi^{-1}\left(\sum_{j=1}^m \phi(a_j) - \frac{1}{W_I}\sum_{j=1}^{k-1}\sum_{i \in I} w_i \phi(x_{ij}) - \frac{1}{W_I}\sum_{j=k+1}^m \sum_{i \in I} w_i \phi(x_{ij})\right).$$

*Proof.* Applying Theorem 5 to the convex function $f = \psi \circ \phi^{-1}$ and replacing $a, b$, and $x_i$ with $\phi(a)$, $\phi(b)$ and $\phi(x_i)$ respectively and then using Remark 2, we obtain (20).

**Remark 4.** Theorems 6, 7 and 8 follow from Theorem 9, by choosing adequate functions $\phi$ and $\psi$, and appropriate substitutions.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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