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BEST UPPER AND LOWER BOUNDS FOR WALLIS' INEQUALITY OF HIGHER ORDER

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Abstract. In this paper, we obtain conditions for the best refinement of Wallis' inequality of order m, and verify it for $m = 2,3$ and 4. An open problem is proposed.

Keywords: Gamma function; Wallis' inequality; Wallis' formula.

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1. Introduction

The Gamma function $\Gamma : (0,\infty) \longrightarrow \mathbb{R}$ is defined by the relation

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \quad (x > 0)$$

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The Gamma function has the following well known properties (see [1],[2]):

$$\Gamma(1)=1, \quad \Gamma(x+1)=x\Gamma(x) \quad (\Gamma(n+1)=n!, n\in\mathbb{N}) \tag{1.1}$$

$$\prod_{k=0}^{m-1}\Gamma(x+\frac{k}{m})=(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mx}\Gamma(mx) \quad \text{(Gauss multiplication formula)} \tag{1.2}$$

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma+\int_0^{+\infty}\frac{e^{-t}-e^{-xt}}{1-e^{-t}}dt \quad (x>0, \gamma=\text{Euler's canstant}) \tag{1.3}$$

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}=1+\frac{(a-b)(a+b-1)}{2x}+O(\frac{1}{x^2}) \quad \text{(asymptotic formula)} \tag{1.4}$$

$$\lim_{x\to\infty}\left[(b-a)\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]=1 \quad (a,b\in\mathbb{R}) \quad \text{(Wendel's limit)} \tag{1.5}$$

$$\psi(x)=\ln x-\frac{1}{2x}+O(\frac{1}{x^2}) \tag{1.6}$$

We need the following Lemma regarding to Γ and ψ functions.

Lemma 1.1. *(Convolution theorem of Laplace transform) Let f and g be piecewise continuous for $t\geq 0$ on any given finite interval and there exist two constants $M>0$ and $C\geq 0$ such that $|f(t)|\leq Me^{Ct}$, then we have*

$$\int_0^{+\infty}\left[\int_0^s f(u)g(t-u)du\right]e^{-st}dt=\int_0^{+\infty}f(u)e^{-su}du\int_0^{+\infty}g(v)e^{-sv}dv.$$

For proof see [2].

The authors in [3], [4], and [5] proved the following theorem by different methods.

Theorem 1.2. *For all integers $n\geq 1$, we have*

$$\frac{1}{\sqrt{\pi(n+\frac{4}{\pi}-1)}}\leq\frac{(2n)!}{(n!)^2 2^{2n}}<\frac{1}{\sqrt{\pi(n+\frac{1}{4})}} \tag{1.7}$$

The constants $\frac{4}{\pi}-1$ and $\frac{1}{4}$ are best possible.

In a recent paper[7], anong other things, we obtained Wallis' inequality of order m and showed that the Wallis' inequality of order 2 is the classic Wallis' inequality:

$$\frac{mn\prod\limits_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(n!)^{m-1}\sqrt{(2\pi)^{m-1}m}\sqrt[m]{\prod\limits_{k=1}^{m-1}\Gamma(n+\frac{2k+1}{m})}}<\frac{(mn)!}{(n!)^m m^{mn}}<\frac{\sqrt{m}}{\sqrt{(2\pi n)^{m-1}}}$$

In this paper we claim that we can refine the Wallis' inequality of order m by the following

$$L < \frac{(mn)!}{(n!)^m m^{mn}} < U \tag{1.8}$$

where

$$L = \frac{1}{\sqrt{\frac{n^{m-2}}{m^{2-2m}((m-1)!)^2} + \frac{(2\pi)^{m-1}(n^{m-1}-n^{m-2})}{m}}}$$

and

$$U = \sqrt{\frac{6m^2}{(2\pi)^{m-1}n^{m-2}(m^2-1+6mn)}}$$

and we show that it holds if

$$g(x) = \int_0^{+\infty}\left(\int_0^t (2h(s)h(t-s)-h(t))ds\right)e^{-tx}dt > 0$$

where

$$h(t) = \frac{me^{-\frac{t}{m}} + me^{(1-\frac{1}{m})t} - (m-2)e^t - m - 2}{2(1-e^{-\frac{t}{m}})(e^t-1)}$$

In theorem 2.5 we obtain some properties of h. In Corollary 2.7 we show that for $m = 2,3$ and 4, $g(x) > 0$. Finally we propose an open problem.

2. **Main Results**

Theorem 2.1. *Let the sequence*

$$Q_n = \frac{\Gamma^{2m-2}(n+1)}{n^{m-2}\prod_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})} - n \quad (m \in \mathbb{N} \text{ and } m \geq 2)$$

be strictly decreasing. Then the following inequalities hold.

$$L < \frac{(mn)!}{(n!)^m m^{mn}} < U$$

where

$$L = \frac{1}{\sqrt{\frac{n^{m-2}}{m^{2-2m}((m-1)!)^2} + \frac{(2\pi)^{m-1}(n^{m-1}-n^{m-2})}{m}}}$$

and

$$U=\sqrt{\frac{6m^2}{(2\pi)^{m-1}n^{m-2}(m^2-1+6mn)}}$$

Proof. First we show that $\lim_{n\to\infty} Q_n = \frac{m^2-1}{6m}$. We have

$$Q_n = n\left[\frac{\Gamma^{2m-2}(n+1)}{n^{m-1}\prod_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}-1\right] = n\left[n^{1-m}\frac{\Gamma^{2m-2}(n+1)}{\prod_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}-1\right]$$

$$= n\left[n^{\frac{1-m}{2}}\frac{\Gamma^{m-1}(n+1)}{\prod_{k=1}^{m-1}\Gamma(n+\frac{k}{m})}-1\right]\left[n^{\frac{1-m}{2}}\frac{\Gamma^{m-1}(n+1)}{\prod_{k=1}^{m-1}\Gamma(n+\frac{k}{m})}+1\right]$$

$$= n\left[\prod_{k=1}^{m-1} n^{\frac{k}{m}-1}\frac{\Gamma(n+1)}{\Gamma(n+\frac{k}{m})}-1\right]\left[\prod_{k=1}^{m-1} n^{\frac{k}{m}-1}\frac{\Gamma(n+1)}{\Gamma(n+\frac{k}{m})}+1\right] \tag{2.1}$$

Using the asymptotic formula we deduce that

$$Q_n = n\left[\prod_{k=1}^{m-1}\left(1+\frac{(1-\frac{k}{m})\frac{k}{m}}{2n}+o(\frac{1}{n^2})\right)-1\right]\left[\prod_{k=1}^{m-1}\left(1+\frac{(1-\frac{k}{m})\frac{k}{m}}{2n}+o(\frac{1}{n^2})\right)+1\right]$$

Hence

$$\lim_{n\to\infty} Q_n = \frac{1}{2}\sum_{k=1}^{m-1}(1-\frac{k}{m})(\frac{k}{m}).2 = \sum_{k=1}^{m-1}\frac{k}{m}-\sum_{k=1}^{m-1}\frac{k^2}{m^2}$$

$$= \frac{1}{m}.\frac{m(m-1)}{2}-\frac{1}{m^2}.\frac{m(m-1)(2m-1)}{6} = \frac{m^2-1}{6}.$$

Since Q_n is strictly decreasing, we have

$$\lim_{n\to\infty} Q_n < Q_n < Q_1$$

so

$$\frac{m^2-1}{6} < \frac{\Gamma^{2m-2}(n+1)}{n^{m-2}\prod_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}-n < \frac{1}{\prod_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}-1 = \frac{1}{(2\pi)^{m-1}m^{1-2m}(m-1)!}-1$$

multiplying both side of inequalities by $\dfrac{(2\pi)^{m-1}n^{m-2}}{m}$, we get

$$\frac{(2\pi)^{m-1}n^{m-2}(m^2-1)}{6m^2} < \frac{(2\pi)^{m-1}(\Gamma(n+1))^{2m-2}}{m\prod_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}-\frac{(2\pi)^{m-1}n^{m-1}}{m} < \frac{n^{m-2}}{m^{2-2m}((m-1)!)^2}-\frac{(2\pi)^{m-1}n^{m-2}}{m}$$

Hence

$$\frac{(2\pi)^{m-1}n^{m-2}(m^2-1)}{6m^2}+\frac{(2\pi)^{m-1}n^{m-1}}{m}<\frac{(2\pi)^{m-1}(\Gamma(n+1))^{2m-2}}{m\prod\limits_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}$$
$$<\frac{n^{m-2}}{m^{2-2m}((m-1)!)^2}-\frac{(2\pi)^{m-1}n^{m-2}}{m}+\frac{(2\pi)^{m-1}n^{m-1}}{m}$$

Thus

(2.2)

$$\frac{1}{\frac{n^{m-2}}{m^{2-2m}((m-1)!)^2}+\frac{(2\pi)^{m-1}(n^{m-1}-n^{m-2})}{m}}<\frac{m\prod\limits_{k=1}^{m-1}\Gamma^2(n+\frac{k}{m})}{(2\pi)^{m-1}(\Gamma(n+1))^{2m-2}}<\frac{6m^2}{(2\pi)^{m-1}n^{m-2}(m^2-1+6mn)}$$

On the other hand by formula (1.2) we have

$$\frac{(mn)!}{(n!)^m m^{mn}}=\frac{(mn)\Gamma(mn)}{(n!)^m m^{mn}}=\frac{(mn)\prod\limits_{k=0}^{m-1}\Gamma(n+\frac{k}{m})}{(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mn}(n!)^m m^{mn}}$$
$$=\frac{(mn)\Gamma(n)\prod\limits_{k=1}^{m-1}\Gamma(n+\frac{k}{m})}{(2\pi)^{\frac{m-1}{2}}m^{\frac{1}{2}-mn}(n!)^m m^{mn}}=\frac{\sqrt{m}\prod\limits_{k=1}^{m-1}\Gamma(n+\frac{k}{m})}{(2\pi)^{\frac{m-1}{2}}(\Gamma(n+1))^{m-1}}$$

So, the inequalities (2.2) are equivalat to

$$\frac{1}{\frac{n^{m-2}}{m^{2-2m}((m-1)!)^2}+\frac{(2\pi)^{m-1}(n^{m-1}-n^{m-2})}{m}}<\left(\frac{(mn)!}{(n!)^m m^{mn}}\right)^2$$
$$<\frac{6m^2}{(2\pi)^{m-1}n^{m-2}(m^2-1+6mn)}$$

Thus

$$L<\frac{(mn)!}{(n!)^m m^{mn}}<U.$$

□

Theorem 2.2. *Let $m\in\mathbb{N}$, $m\geq 2$ and*

$$f(x)=\frac{\Gamma^{2m-2}(x+1)}{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}-x\quad(x>0)$$

Then f is strictly decreasing on $(0,\infty)$, if

$$g(x)=\int_0^{+\infty}\left(\int_0^t (2h(s)h(t-s)-h(t))ds\right)e^{-tx}dt>0$$

where

$$h(s)=\frac{me^{-\frac{s}{m}}+me^{(1-\frac{1}{m})s}-(m-2)e^{-s}-m-2}{2(1-e^{-\frac{s}{m}})(e^s-1)}.$$

Proof. Differentiation $f(x)$ gives us

$$f'(x)=\frac{\Gamma^{2m-2}(x+1)}{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}\left[(2m-2)\psi(x+1)-2\sum_{k=0}^{m-1}\psi(x+\frac{k}{m})-(m-2)\frac{1}{x}\right]-1$$

so

$$\frac{f'(x)+1}{(2m-2)\psi(x+1)-2\sum\limits_{k=1}^{m-1}\psi(x+\frac{k}{m})-(m-2)\frac{1}{x}}=\frac{\Gamma^{2m-2}(x+1)}{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})} \tag{2.3}$$

By differentiating $f'(x)$, we obtain

$$\begin{aligned}f''(x)=&\frac{\Gamma^{2m-2}(x+1)}{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}\left[\left((2m-2)\psi(x+1)-2\sum_{k=1}^{m-1}\psi(x+\frac{k}{m})-\frac{m-2}{x}\right)^2\right]\\&+\frac{\Gamma^{2m-2}(x+1)}{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}\left[(2m-2)\psi'(x+1)-2\sum_{k=1}^{m-1}\psi'(x+\frac{k}{m})+\frac{m-2}{x^2}\right]\end{aligned}$$

so

$$\begin{aligned}f''(x)=&\frac{\Gamma^{2m-2}(x+1)}{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}\left[\left((2m-2)\psi(x+1)-2\sum_{k=1}^{m-1}\psi(x+\frac{k}{m})-\frac{m-2}{x}\right)^2\right.\\&\left.+\left((2m-2)\psi'(x+1)-2\sum_{k=1}^{m-1}\psi'(x+\frac{k}{m})+\frac{m-2}{x^2}\right]\right.\end{aligned}$$

By (2.3) we have

$$\begin{aligned}
&\frac{(2m-2)\psi(x+1)-2\sum\limits_{k=1}^{m-1}\psi(x+\frac{k}{m})-(m-2)\frac{1}{x}}{1+f'(x)}f''(x)\\
&=\left[2\sum_{k=1}^{m-1}\left(\psi'(x+1)-\psi'(x+\frac{k}{m})\right)-\frac{m-2}{x^2}\right]\\
&+4\left[\sum_{k=1}^{m-1}\left(\psi(x+1)-\psi(x+\frac{k}{m}\right)-\frac{m-2}{2x}\right]^2
\end{aligned}$$

or

$$\frac{(m-1)\psi(x+1)-\sum\limits_{k=1}^{m-1}\psi(x+\frac{k}{m})-\frac{m-2}{2x}}{2(1+f'(x))}f''(x)=\left[\sum_{k=1}^{m-1}(\psi'(x+1)-\psi'(x+\frac{k}{m}))+\frac{m-2}{2x^2}\right]$$

(2.4)

$$+2\left[\sum_{k=1}^{m-1}(\psi(x+1)-\psi(x+\frac{k}{m}))-\frac{m-2}{2x}\right]^2$$

set

$$g(x)=\sum_{k=1}^{m-1}\left(\psi'(x+1)-\psi'(x+\frac{k}{m})\right)+\frac{m-2}{2x^2}+2\left[\sum_{k=1}^{m-1}(\psi(x+1)-\psi(x+\frac{k}{m}))-\frac{m-2}{2x}\right]^2$$

Thus

$$\frac{(m-1)\psi(x+1)-\sum\limits_{k=1}^{m-1}\psi(x+\frac{k}{m})-\frac{m-2}{2x}}{1+f'(x)}f''(x)=g(x) \tag{2.5}$$

since

$$\frac{(m-1)\psi(x+1)-\sum\limits_{k=1}^{m-1}\psi(x+\frac{k}{m})-\frac{m-2}{2x}}{1+f'(x)}=\frac{x^{m-2}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}{\Gamma^{2m-2}(x+1)}>0$$

By (2.5) we see that $f''(x)$ and $g(x)$ have the same sign. If $g(x)>0$ then $f''(x)>0$. So $f'(x)$ is strictly increasing on $(0,+\infty)$. We have

$$\begin{aligned}
f'(x)&=\frac{\Gamma^{2m-2}(x+1)}{x^{m-1}\prod\limits_{k=1}^{m-1}\Gamma^2(x+\frac{k}{m})}\left[2\sum_{k=1}^{m-1}\left((\psi(x+1)-\psi(x+\frac{k}{m})\right)-(m-2)\right]-1\\
&=\left(\prod_{k=1}^{m-1}\frac{\Gamma(x+1)}{\Gamma(x+\frac{k}{m})}x^{\frac{k}{m}-1}\right)^2\left[2\sum_{k=1}^{m-1}x\left((\psi(x+1)-\psi(x+\frac{k}{m})\right)-(m-2)\right]-1
\end{aligned}$$

since by (1.6) and (1.4),

$$x\left(\psi(x+1)-\psi(x+\frac{k}{m})\right)=x\left[\ln(x+1)-\frac{1}{2(x+1)}+o((x+1)^{-2})\right.$$
$$\left.-\ln(x+\frac{k}{m})+\frac{1}{2(x+\frac{k}{m})}-o((x+\frac{k}{m})^{-2})\right]$$
$$=x\left[\ln\frac{x+1}{x+\frac{k}{m}}\right]-\frac{x}{2(x+1)}+\frac{x}{2(x+\frac{k}{m})}+o((x+1)^{-2})-o((x+\frac{k}{m})^{-2})$$

and

$$\lim_{x\to\infty}x^{\frac{k}{m}-1}\frac{\Gamma(x+1)}{\Gamma(x+\frac{k}{m})}=1$$

we conclude that

$$\lim_{x\to\infty}f'(x)=1\cdot\left[2\sum_{k=1}^{m-1}(1-\frac{k}{m})-(m-2)\right]-1=0$$

so

$$f'(x)<\lim_{x\to\infty}f'(x)=0\Longrightarrow f'(x)<0$$

Hence f is strictly decreasing on $(0,\infty)$.

So if $g(x)>0$, then f is striclly decreasing.

Now we show that

$$g(x)=\int_0^{+\infty}\left(\int_0^t(2h(s)h(t-s)-h(t))ds\right)e^{-tx}dt$$

since

$$\psi(x)=\int_0^{+\infty}\frac{e^{-t}-e^{-tx}}{1-e^{-t}}dt-\gamma \quad \text{and} \quad \psi'(x)=\int_0^{+\infty}\frac{te^{-tx}}{1-e^{-t}}dt$$

we have

$$\psi(x+1)-\psi(x+\frac{k}{x})=\int_0^{+\infty}\frac{e^{-t(x+\frac{k}{m})}-e^{-t(x+1)}}{1-e^{-t}}dt$$

and

$$\psi'(x+1)-\psi'(x+\frac{k}{m})=\int_0^{+\infty}\frac{te^{-t(x+1)}-te^{-t(x+\frac{k}{m})}}{1-e^{-t}}dt=\int_0^{+\infty}\frac{te^{-tx}(e^{-t}-e^{-t\frac{k}{m}})}{1-e^{-t}}dt$$

so

$$
\begin{aligned}
g(x) &= \sum_{k=1}^{m-1}\left(\psi'(x+1)-\psi'(x+\frac{k}{m})\right)+\frac{m-2}{2x^2}+2\left[\sum_{k=1}^{m-1}\left(\psi(x+1)-\psi(x+\frac{k}{m})\right)-\frac{m-2}{2x}\right]^2 \\
&= \left[-\sum_{k=1}^{m-1}\int_0^{+\infty}\frac{e^{-t\frac{k}{m}}-e^{-t}}{1-e^{-t}}te^{-tx}dt+\frac{m-2}{2x^2}\right]+2\left[\sum_{k=1}^{m-1}\frac{e^{-t\frac{k}{m}}-e^{-t}}{1-e^{-t}}e^{-tx}dt-\frac{m-2}{2x}\right]^2
\end{aligned}
$$

since

$$
\begin{aligned}
&\sum_{k=1}^{m-1}\int_0^{+\infty}\frac{e^{-t\frac{k}{m}}-e^{-t}}{1-e^{-t}}e^{-tx}dt \\
&\quad=\int_0^{+\infty}\frac{e^{-tx}}{1-e^{-t}}\left(\sum_{k=1}^{m-1}\left(e^{-t\frac{k}{m}}-e^{-t}\right)\right)dt \\
&\quad=\int_0^{+\infty}\frac{e^{-tx}}{1-e^{-t}}\left(\sum_{k=1}^{m-1}e^{-t\frac{k}{m}}-(m-1)e^{-t}\right)dt \\
&\quad=\int_0^{+\infty}\frac{e^{-tx}}{1-e^{-t}}\left(e^{-\frac{t}{m}}\frac{1-e^{-\frac{m-1}{m}t}}{1-e^{-\frac{t}{m}}}-(m-1)e^{-t}\right)dt \\
&\quad=\int_0^{+\infty}\frac{e^{-tx}}{1-e^{-t}}\left(\frac{e^{-\frac{t}{m}}-e^{-t}}{1-e^{-\frac{t}{m}}}-(m-1)e^{-t}\right)dt \\
&\quad=\int_0^{+\infty}\frac{e^{-tx}}{1-e^{-t}}.\frac{(m-1)e^{-(1+\frac{1}{m})t}+e^{-\frac{t}{m}}-me^{-t}}{1-e^{-\frac{t}{m}}}dt \\
&\quad=\int_0^{+\infty}\frac{(m-1)e^{-\frac{t}{m}}+e^{(1-\frac{1}{m})t}-m}{(1-e^{-\frac{t}{m}})(e^t-1)}e^{-tx}dt
\end{aligned}
$$

Hence

$$
\begin{aligned}
g(x)=&-\int_0^{+\infty}\frac{(m-1)e^{-\frac{t}{m}}+e^{(1-\frac{1}{m})t}-m}{(1-e^{-\frac{t}{m}})(e^t-1)}te^{-tx}dt+\frac{m-2}{2x^2} \\
&+2\left[\int_0^{+\infty}\frac{(m-1)e^{-\frac{t}{m}}+e^{(1-\frac{1}{m})t}-m}{(1-e^{-\frac{t}{m}})(e^t-1)}e^{-tx}dt-\frac{m-2}{2x}\right]^2
\end{aligned}
$$

By easy calculation we see that

$$
\frac{1}{x}=\int_0^{+\infty}e^{-tx}dt(x>0)\quad\text{and}\quad\frac{1}{x^2}=\int_0^{+\infty}te^{-tx}dt\quad(x>0)
$$

so

$$g(x)=\int_0^{+\infty}\left[-t\frac{(m-1)e^{-\frac{t}{m}}+e^{(1-\frac{1}{m})t}-m}{(1-e^{-\frac{t}{m}})(e^t-1)}+\frac{m-2}{2}t\right]e^{-tx}dt$$
$$+2\left[\int_0^{+\infty}\left(\frac{(m-1)e^{-\frac{t}{m}}+e^{(1-\frac{1}{m})t}-m}{(1-e^{-\frac{t}{m}})(e^t-1)}-\frac{m-2}{2}\right)e^{-tx}dt\right]^2$$

since

$$\frac{(m-1)e^{-\frac{t}{m}}+e^{(1-\frac{1}{m})t}-m}{(1-e^{-\frac{t}{m}})(e^t-1)}-\frac{m-2}{2}=\frac{me^{-\frac{t}{m}}+me^{(1-\frac{1}{m})t}-(m-2)e^t-m-2}{2(1-e^{-\frac{t}{m}})(e^t-1)}$$

It follows that

$$g(x)=-\int_0^{+\infty}t\frac{me^{-\frac{t}{m}}+me^{(1-\frac{1}{m})t}-(m-2)e^t-m-2}{2(1-e^{-\frac{t}{m}})(e^t-1)}e^{-tx}dt$$
$$+2\left[\int_0^{+\infty}\frac{me^{-\frac{t}{m}}+me^{(1-\frac{1}{m})t}-(m-2)e^t-m-2}{2(1-e^{-\frac{1}{m}})(e^t-1)}e^{-tx}dt\right]^2$$

set

$$h(t)=\frac{me^{-\frac{t}{m}}+me^{(1-\frac{1}{m})t}-(m-2)e^t-m-2}{2(1-e^{-\frac{t}{m}})(e^t-1)}$$

Then

$$g(x)=-\int_0^{+\infty}t\,h(t)e^{-tx}dt+2\left[\int_0^{+\infty}h(t)e^{-tx}dt\right]^2$$
$$=-\int_0^{+\infty}\left(\int_0^t h(t)e^{-tx}ds\right)dt+2\left[\int_0^{+\infty}h(t)e^{-tx}dt\right]^2$$

Finally by using the convolution theorem (Lemma 1.1), We obtain

$$g(x)=-\int_0^{+\infty}\left(\int_0^t h(t)e^{-tx}ds\right)dt+2\int_0^{+\infty}\int_0^t(h(s)h(t-s)ds)\,e^{-tx}dt$$
$$=\int_0^{+\infty}\left(\int_0^t(2h(s)h(t-s)-h(t))ds\right)e^{-tx}dt$$

Notice that the function h holds in conditions of lemma 1.1, by theorem 2.5. □

Corollary 2.3. *Let* $g(x)=\int_0^{+\infty}\left(\int_0^t(2h(s)h(t-s)-h(t)ds\right)e^{-tx}dt>0\ (x>0)$. *Then the following inequalities hold*

$$L<\frac{(mn)!}{(n!)^m m^{mn}}<U$$

The proof is obvious by theorems 2.1 and 2.2

Definition 2.4. Let

$$h(t) = \frac{me^{-\frac{t}{m}} + me^{(1-\frac{1}{m})t} - (m-2)e^t - m - 2}{2(1-e^{-\frac{t}{m}})(e^t-1)}$$
$$= \frac{m + me^t - (m-2)e^{(1+\frac{1}{m})t} - (m+2)e^{\frac{t}{m}}}{2(e^{\frac{t}{m}}-1)(e^t-1)} \quad (0 < t < +\infty)$$

we define

$$H(b) = h(m\ln b) = \frac{m + mb^m - (m-2)b^{m+1} - (m+2)b}{2(b-1)(b^m-1)} \quad (1 < b < \infty)$$

Theorem 2.5. *Let*

$$H(b) = \frac{m + mb^m - (m-2)b^{m+1} - (m+2)b}{2(b-1)(b^m-1)} \quad (1 < b < \infty)$$

Then

(i) $\lim_{b\to 1} H(b) = \frac{1}{2}$ *and* $\lim_{b\to\infty} H(b) = 1 - \frac{m}{2}$,
(ii) H is strictly decreasing on $(1,\infty)$ *and* $1 - \frac{m}{2} < H(b) < \frac{1}{2}$,
(iii) h is striclly decreasing on $(0,\infty)$ *and* $1 - \frac{m}{2} < h(t) < \frac{1}{2}$.

Proof. (i) It is obvious that $\lim_{b\to\infty} H(b) = 1 - \frac{m}{2}$. By L'Hospital's rule we have

$$\lim_{b\to 1} H(b) = \frac{1}{2}\lim_{b\to 1} \frac{m + mb^m - (m-2)b^{m+1} - (m+2)b}{b^{m+1} - b - b^m + 1}$$
$$= \frac{1}{2}\lim_{b\to 1} \frac{m^2b^{m-1} - (m-2)(m+1)b^m - (m+2)}{(m+1)b^m - 1 - mb^{m-1}}$$
$$= \frac{1}{2}\lim_{b\to 1} \frac{m^2(m-1)b^{m-2} - m(m-2)(m+1)b^{m+1}}{m(m+1)b^{m-1} - m(m-1)b^{m-2}} = \frac{1}{2}.1 = \frac{1}{2}$$

(ii) By easy calculations we obtain

$$H(b) = \frac{-mb^m(b-1) - m(b-1) + 2b(b^m-1)}{2(b-1)(b^m-1)} = \frac{-m(b-1)(b^m+1) + 2b(b^m-1)}{2(b-1)(b^m-1)}$$
$$H(b) = -\frac{m}{2}.\frac{b^m+1}{b^m-1} + \frac{b}{b-1}$$

So

$$H'(b) = \frac{-m}{2}.\frac{-2mb^{m-1}}{(b^m-1)^2} - \frac{1}{(b-1)^2} = \frac{m^2b^{m-1} - (1+b+b^2+\cdots+b^{m-1})^2}{(b^m-1)^2} < 0$$

Because the geometric mean of positive numbers $1, b, \ldots, b^{m-1}$ is not greater than the arithmetic mean of its, that is

$$\frac{1+b+\cdots+b^{m-1}}{m} > \sqrt[m]{b^1.b^2.\ldots.b^{m-1}} = \sqrt[m]{b^{\frac{m(m-1)}{2}}} = b^{\frac{m-1}{2}}$$

or, $(1+b+\cdots+b^{m-1})^2 > m^2 b^{m-1}$.

Thus H is strictly decreasing on $(1,\infty)$.

(iii) By the relation between h and H the assertion is clear. □

Lemma 2.6. *Let* $2H(a)H(\frac{b}{a}) - H(b) > 0$ $(b > a > 1)$. *Then*

$$g(x) = \int_0^{+\infty} \left(\int_0^t (2h(s)h(t-s) - h(t))ds \right) e^{-tx} dt > 0$$

Proof. By change of variable $s = m\ln a$ and setting $t = m\ln b$ we get

$$\begin{aligned}
\int_0^t (2h(s)h(t-s) - h(t))ds &= \int_1^b [2h(m\ln a)h(m\ln b - m\ln a) - h(m\ln b)]\frac{mda}{a} \\
&= \int_1^b [2h(m\ln a)h(m\ln \frac{b}{a}) - h(m\ln b)]\frac{mda}{a} \\
&= m\int_1^b [2H(a)H(\frac{b}{a}) - H(b)]\frac{da}{a}.
\end{aligned}$$

The rest of proof is obvious. □

Now we want show that for $m = 2, 3$ and 4, $g(x) > 0$. There are several ways for $m = 2$ (see [3],[4] and [5]). If we use in a similar way for $m = 3$ and 4 we would have very long and complicated calculations. Notice that by theorem 2.5, H may be negative for $m \geq 3$. So we prefer straight forwardly calculaling.

Corollary 2.7. *For* $m = 2, 3$, *and* 4, $2H(a)H(\frac{b}{a}) - H(b) > 0$, $(b > a > 1)$. *In the other words the inequality (1.8) hold for* $m = 2, 3$ *and* 4.

Proof. Since $H(a) = \dfrac{m + ma^m - (m-2)a^{m+1} - (m+2)a}{2(a-1)(a^m - 1)}$. For $m = 2$, $H(a) = \frac{1}{a+1}$. Hence

$$2H(a)H(\frac{b}{a}) - H(b) = \frac{2}{a+1}.\frac{a}{a+b} - \frac{1}{b+1} = \frac{(a-1)(b-a)}{(a+1)(a+b)(b+1)} > 0$$

For $m=3$, $H(a)=\dfrac{3+a-a^2}{2(a^2+a+1)}$. Hence

$$2H(a)H(\frac{b}{a})-H(b)=\frac{3+a-a^2}{a^2+a+1}.\frac{3a^2+ab-b^2}{2(b^2+ab+a^2)}-\frac{3+b-b^2}{2(b^2+b+1)}>0.$$

Because

$$(3+a-a^2)(3a^2+ab-b^2)(b^2+b+1)-(3+b-b^2)(a^2+a+1)(b^2+ab+a^2)$$
$$=-2(a-1)(a-b)(a^2b^2+ab^3+2a^2b+2ab^2+b^3+3a^2+6ab+2b^2+3a+3b)>0$$

For $m=4$, $H(a)=\dfrac{2+a-a^3}{a^3+a^2+a+1}$. So

$$2H(a)H(\frac{b}{a})-H(b)=2\frac{2+a-a^3}{a^3+a^2+a+1}.\frac{2a^3+ba^2-b^3}{b^3+b^2a+ba^2+a^3}-\frac{2+b-b^3}{b^3+b^2+b+1}>0.$$

Because

$$2(2+a-a^3)(2a^3+ba^2-b^3)(b^3+b^2+b+1)-(2+b-b^3)(a^3+a^2+a+1)$$
$$(b^3+b^2a+ba^2+a^3)=-(a-1)(a-b)(3a^4b^3+4a^3b^4+3a^2b^5+4a^4b^2$$
$$+8a^3b^3+8a^2b^4+4ab^5+5a^4b+12a^3b^2+12a^2b^3+8ab^4+3b^5$$
$$+6a^4+16a^3b+22a^2b^2+12ab^3+4b^4+8a^3+17a^2b+16ab^2+5b^3+6a^2+8ab+6b^2)>0.$$

$\square$

Remark 2.8. In fact we have proved the following inequalities:

$$\frac{1}{\sqrt{\pi(n+\frac{4}{\pi}-1}}<\frac{(2n)!}{(n!)^2 2^{2n}}<\frac{1}{\sqrt{\pi(n+\frac{1}{4})}}$$
$$\frac{1}{\sqrt{\frac{4\pi^2}{3}n^2+n(\frac{81}{4}-\frac{4\pi^2}{3})}}<\frac{(3n)!}{(n!)^3 3^{3n}}<\frac{1}{\sqrt{\frac{4\pi^2}{3}n^2+\frac{16\pi^2}{27}n}}$$
$$\frac{1}{\sqrt{2\pi^3n^3+n^2(\frac{256}{9}-2\pi^3)}}<\frac{(4n)!}{(n!)^4 4^{4n}}<\frac{1}{\sqrt{2\pi^3n^3+\frac{5}{4}\pi^3n^2}}$$

These inequalities are the best refinement of Wallis' inequality of order $m=2,3$ and 4.

Lastly we propose the following open problem:

open problem: Let

$$h(t)=\frac{m+me^{t}-(m-2)e^{(1+\frac{1}{m})t}-(m+2)e^{\frac{t}{m}}}{2(e^{\frac{t}{m}}-1)(e^{t}-1)}\quad (t>0, m\in\mathbb{N}, m\geq 2).$$

Prove or disprove the following inequality

$$g(x)=\int_{0}^{+\infty}\left(\int_{0}^{t}(2h(s)h(t-s)-h(t))ds\right)e^{-tx}dt>0\quad (x>0).$$

Conflict of Interests

The author declares that there is no conflict of interests.

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