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MATHEMATICAL PROGRAMMING PROBLEMS HAVING PARAMETERS AS GAMMA RANDOM VARIABLES IN CHANCE CONSTRAINTS

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Abstract. In this paper we have discussed a transformation procedure of the chance constraints, to arrive at the deterministic constraints, for mathematical programming problem having parameters as Gamma random variables in chance constraints. We have used geometric inequality and some other inequalities for this transformation. We solved, the transformed deterministic problem having non-linear constraints, using generic package LINGO 10 and verified the solution using MATLAB 7.6.

Keywords: Stochastic programming; chance constraint programming; gamma random variable; geometric inequality; deterministic implicative reduction; non-linear programming.

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1. Introduction

Stochastic programming [7, 14, 26] is an optimization method which is based on the probability theory and have been developing in various ways [22, 19], including two stage problems [17] and chance constrained problems [6]. One of the difficulties that one faces while dealing

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with a mathematical programming problem having chance constraints [6] is that, all or some of the parameters in the formulation are not constant or deterministic but random in nature.

Charnes and Cooper [6] have first modelled chance constrained programming (CCP). Here, they have developed a new analytical and conceptual method which reffer to a planning of optimal stochastic decision rules under uncertainty. The indefiniteness and risk in planning and managing problems was examined by Kolbin [16] and presented chance-constraint programming models. For the class of chance-constraint programming problem, Symonds [24] has presented deterministic solution procedure. A minimum-risk approach to multi-objective stochastic linear programming problems was suggested by Stancu-Minasian [23]. Chance-constraint programming in stochastic is expanded to fuzzy parameters concept by Liu and Iwamura [18]. They have presented certain equations with chance constraint in some fuzzy concept identical to stochastic programming. Also Liu and Iwamura [18] have suggested a fuzzy simulation method for chance constraints for which it is usually difficult to be changed into certain equations and they have used the concept of genetic algorithm for solving this type of problems and discussed numerical examples. Hulsurkar et al. [12] extended their studies on multi-objective stochastic linear programming problems using the concept of fuzzy programming approach. They first changed the stochastic programming problem into a linear or a nonlinear deterministic problem and then used fuzzy programming approach for finding a solution. In the study of the control of nitrate pollution, Kampas and White [15] have suggested use of probabilistic programming and adopted a comparative study of this model with the approaches of various probabilistic constraints. Mohammed [20] has extended his contribution on chance-constraint fuzzy goal programming containing right-hand side values with uniform random variables. He demonstrated the core idea regarding the stochastic goal programming and chance-constraint linear goal programming. Apak and Gken [2] developed new mathematical models for stochastic traditional and U-type assembly lines with a chance-constrained 0-1 integer programming technique. For portfolio selection with fuzzy returns, Huang [11] presented two types of credibility-based chance-constrained models . In case of transmission system planning in the competitive electricity market environment, Yang and Wen [28] suggested a chance-constrained

programming model. Sarkar et al. [25] studied CCP, where the parameters are lognormal in nature and extended their study in some managerial apllications and Fisher's discriminant function for separation of populations. For study of uncertainty of municipal solid waste management a robust hybrid stochastic chance-constraint programming model was suggested by Xu et al. [27]. Henrion and Strugarek [10] explored the convexity of chance constraints with independent random variables. A stochastic algorithm was suggested by Parpas and Rstem [21] for the global optimization of chance-constrained problems assuming that, the probability measure used to evaluate the constraints is known only through its moments. A chance-constrained approach and a compromise programming approach was recomended by Abdelaziz and Masri [1], to transform the multi-objective stochastic linear program with partial linear information on the probability distribution into its equivalent uni-objective problem. A polynomial time approximation method was suggested by Goyal and Ravi [9], for the chance-constrained knapsack problem when item sizes are normally distributed and independent of other items. The concept of Essen inequality used by Atalay and Apaydin [3], to transform the chance constrained model into a deterministic model where pameters are independent with Gamma distributions.
Although, many authors studied and solved CCP for the parameters having normal distribution, uniform distribution, lognormal distribution, and exponential distribution with the linear chance constraints [22, 4, 5, 19, 25]. But the gamma distribution [13], which has a variety of applications, has been discussed a little so far. In particular, it can be used to model:

(1) the queuing systems;
(2) the flow of items through manufacturing and distribution processes;
(3) the load on web servers;
(4) the probability of ruin & value (risk management), etc..

The three-parameter gamma distribution is defined in terms of a shape parameter, a scale parameter and a location parameter. However, the two-parameter version of this distribution is more commonly used in practice.

The Probability Density Function (pdf) of a three-parameter gamma distribution [13] is,

$$f(x) = \frac{(x-\gamma)^{\alpha-1}}{\beta^{\alpha}\Gamma(\alpha)} \exp(-(x-\gamma)/\beta),$$

where, the domain is $\gamma \leq x < +\infty$, and the parameter α is the shape parameter ($\alpha > 0$), β is the scale parameter ($\beta > 0$) and γ is the location parameter. In this model $\gamma \equiv 0$ yields the two-parameter Gamma distribution.

Further, for simplicity one may consider $\beta = 1$. But in reality β may not be equal to one. In that case we have to consider scale transformation so that the scale parameter becomes one.

The Essen inequality apprach were used by Atalay and Apaydin [3], to transform the chance constrained model into a deterministic model where pameters are independent with Gamma distributions. In our paper we have reduced the chance constraints with independent Gamma parameters to deterministic constraints.Here geometric inequality and some other inequalities are used, where the objective function may be considered as linear or nonlinear with deterministic cost coefficient.

2. Mathematical model with linear constraints

2.1 Mathematical model with linear probabilistic constraints

Let us consider an optimization problem having linear probabilistic constraints that can be modeled as follows:

To find $X = (x_1, x_2, x_3, ..., x_n)$ so as to,

$$Minimize\, f(X) \tag{1}$$

subject to chance constraints,

$$\Pr[\sum_{j=1}^{n} a_{ij}x_j \leq b_i] \geq 1 - p_i \, , i = 1,2,3, ..., m, \tag{2}$$

$$x_j \geq 0 \, , j = 1,2,3, ..., n. \tag{3}$$

where, $0 < p_i < 1$ and $b_i > 0$, for $i = 1,2,3, ..., m$, are given constants and a_{ij} 's are independently distributed gamma random variable with parameters $\alpha_{ij}(> 0), \beta_{ij}(> 0)$ and $\gamma_{ij}(\geq 0)$, which are respectively shape parameter, scale parameter and location parameter for $i =$

$1,2,3,.........,m$ and $j = 1,2,3,..............,n$. Let us consider the following transformation in respect of location and scale as $a'_{ij} = \frac{a_{ij}-\gamma_{ij}}{\beta_{ij}}$.

Then the event $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$, $i = 1,2,3,.........,m$ is equivalent to the following event:

$$\sum_{j=1}^{n}[a'_{ij}\beta_{ij}x_j + \gamma_{ij}x_j] \leq b_i,\, i = 1,2,3,.........,m$$

i.e.,

$$\sum_{j=1}^{n} a'_{ij}(\beta_{ij}x_j) \leq b_i - \sum_{j=1}^{n}\gamma_{ij}x_j = b'_i \text{ (say)},\, i = 1,2,3,.........,m.$$

So, the original problem (1) – (3) can be equivalently expressed in the following form:

To find $X = (x_1,x_2,x_3,.............,x_n)$ so as to,

$$Minimize\, f(X) \tag{4}$$

subject to chance constraints,

$$\Pr[\sum_{j=1}^{n} a'_{ij}\beta_{ij}x_j \leq b'_i] \geq 1-p_i\,,\, i = 1,2,3,.........,m, \tag{5}$$

$$x_j \geq 0\,,\, j = 1,2,3,..............,n \tag{6}$$

where, $0 < p_i < 1$ and $b'_i = b_i - \sum_{j=1}^{n}\gamma_{ij}x_j$, $i = 1,2,3,.........,m$ are constants and a'_{ij} 's are independently distributed gamma random variable with parameter $\alpha_{ij}(>0)$, i.e., $a'_{ij} \sim G(\alpha_{ij})$ for $i = 1,2,3,.........,m$ and $j = 1,2,3,..............,n$.

2.2 Deterministic reduction of the model

Here we shall reduce the probabilistic liner constraints (5) to deterministic non-linear constraints as follows:

Let us consider the event $b'_i \geq \sum_{i=1}^{n} a'_{ij}\beta_{ij}x_j$.

This can be written as,

$$b'_i \geq \sum_{j=1}^{n}(a'_{ij}+1)\beta_{ij}x_j - \sum_{j=1}^{n}\beta_{ij}x_j,\, i = 1,2,3,.........,m$$

i.e.,

$$b_i^{'} \geq n(\frac{1}{n}\sum_{j=1}^{n}(a_{ij}^{'}+1)\beta_{ij}x_j) - \sum_{j=1}^{n}\beta_{ij}x_j, \, i = 1,2,3,..........,m. \tag{7}$$

Now from geometric inequality (G.I.) we have,

$$n(\frac{1}{n}\sum_{j=1}^{n}(a_{ij}^{'}+1)\beta_{ij}x_j) \geq n(\prod_{j=1}^{n}[(a_{ij}^{'}+1)\beta_{ij}x_j]^{\frac{1}{n}}), \, i = 1,2,3,..........,m \tag{8}$$

Thus from (7) and (8) we get,

$$b_i^{'} \geq n(\prod_{j=1}^{n}[(a_{ij}^{'}+1)\beta_{ij}x_j]^{\frac{1}{n}}) - \sum_{j=1}^{n}\beta_{ij}x_j, \, i = 1,2,3,..........,m,$$

i.e.,

$$\prod_{j=1}^{n}(a_{ij}^{'}+1)^{\frac{1}{n}}\prod_{j=1}^{n}(\beta_{ij}x_j)^{\frac{1}{n}} \leq \frac{1}{n}b_i^{'}+\frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j, \, i = 1,2,3,..........,m,$$

i.e.,

$$\prod_{j=1}^{n}(a_{ij}^{'}+1) \leq (\frac{1}{n}b_i^{'}+\frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j)^n/\prod_{j=1}^{n}(\beta_{ij}x_j), \, i = 1,2,3,..........,m. \tag{9}$$

Now, as $1+\sum_{j=1}^{n}a_{ij}^{'} \leq \prod_{j=1}^{n}(1+a_{ij}^{'})$, $i = 1,2,3,...........,m$ for non negative $a_{ij}^{'}$'s, so from the inequality (9) we have,

$$1+\sum_{j=1}^{n}a_{ij}^{'} \leq (\frac{1}{n}b_i^{'}+\frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j)^n/\prod_{j=1}^{n}(\beta_{ij}x_j), \, i = 1,2,3,..........,m,$$

i.e.,

$$\sum_{j=1}^{n}a_{ij}^{'} \leq (\frac{1}{n}b_i^{'}+\frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j)^n/\prod_{j=1}^{n}(\beta_{ij}x_j)-1, \, i = 1,2,3,..........,m. \tag{10}$$

Therefore the probabilistic constraints (5) can be written, using the implicative relation (10) as follows:

$$\Pr[\sum_{j=1}^{n}a_{ij}^{'} \leq (\frac{1}{n}b_i^{'}+\frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j)^n/\prod_{j=1}^{n}(\beta_{ij}x_j)-1] \geq 1-p_i, \, i = 1,2,3,..........,m. \tag{11}$$

Let, $A_i = \sum_{j=1}^{n} a'_{ij}$, $\forall i = 1,2,3,....,m$. Since,$\forall i,j$, a'_{ij} 's follows independent $G(\alpha_{ij})$, so

$$A_i \sim G(\sum_{j=1}^{n} \alpha_{ij}), \forall i = 1,2,3,....,m. \tag{12}$$

Let, $\Pr[A_i \leq t_i] = 1 - p_i$, where, $A_i \sim G(\sum_{j=1}^{n} \alpha_{ij})$ and $0 < p_k < 1$. This gives

$$t_i = G_{p_i}(\sum_{j=1}^{n} \alpha_{ij}). \tag{13}$$

Therefore, from (11) and (13),using above result we have,

$$[(\frac{1}{n}b'_i + \frac{1}{n}\sum_{j=1}^{n} \beta_{ij}x_j)^n / \prod_{j=1}^{n}(\beta_{ij}x_j)] - 1 \geq G_{p_i}(\sum_{j=1}^{n} \alpha_{ij}),\ i = 1,2,3,..........,m \tag{14}$$

where, $b'_i = b_i - \sum_{j=1}^{n} \gamma_{ij}x_j$, $i = 1,2,3,..........,m$.

Thus, in (14), we have the implicatively reduced deterministic form of the probabilistic constraints (5). We can note that the reduced deterministic constraints (14) are nonlinear and hence the reduced problem is a non-linear optimization problem.

Thus, the implicative reduction to deterministic form of the probabilistic model (1) – (3) is as follows:

To find $X = (x_1, x_2, x_3, ..., x_n)$ so as to,

$$Minimize\, f(X) \tag{15}$$

subject to the constraints,

$$[(\frac{1}{n}b'_i + \frac{1}{n}\sum_{j=1}^{n} \beta_{ij}x_j)^n / \prod_{j=1}^{n}(\beta_{ij}x_j)] - 1 \geq G_{p_i}(\sum_{j=1}^{n} \alpha_{ij}),\ i = 1,2,3,..........,m, \tag{16}$$

$$x_j \geq 0\,,\ j = 1,2,3,..............,n. \tag{17}$$

2.2.1 Spacial Case:

In case, when $2\sum_{j=1}^{n} \alpha_{ij}$is a positive integer, then from (12) we have,

$$2A_i \sim \chi^2_{2\sum_{j=1}^{n} \alpha_{ij}}, \forall i = 1,2,3,........,m. \tag{18}$$

Let, $\Pr[2A_i \leq t_i] = 1 - p_i$, where, $2A_i \sim \chi^2_{2\sum_{j=1}^{n}\alpha_{ij}}, \forall i = 1,2,3,........,m$.

This implies,

$$t_i = \chi^2_{2\sum_{j=1}^{n}\alpha_{ij}}(1-p_i), i = 1,2,3,............,m. \tag{19}$$

Therefore, from (11) and (19) we have,

$$[(\frac{1}{n}b_i^{'} + \frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j)^n / \prod_{j=1}^{n}(\beta_{ij}x_j)] - 1 \geq G_{p_i}(\sum_{j=1}^{n}\alpha_{ij}),\, i = 1,2,3,..........,m \tag{20}$$

where, $b_i^{'} = b_i - \sum_{j=1}^{n}\gamma_{ij}x_j$, $i = 1,2,3,..........,m$.

Thus, we have the implicative reduction to deterministic form of the probabilistic model (1) – (3) as follows:

To find $X = (x_1, x_2, x_3, ..., x_n)$so as to,

$$Minimize\, f(X) \tag{21}$$

subject to the constraints,

$$[(\frac{1}{n}b_i^{'} + \frac{1}{n}\sum_{j=1}^{n}\beta_{ij}x_j)^n / \prod_{j=1}^{n}(\beta_{ij}x_j)] - 1 \geq \frac{1}{2}\chi^2_{2\sum_{j=1}^{n}\alpha_{ij}}(1-p_i),\, i = 1,2,3,..........,m. \tag{22}$$

$$x_j \geq 0,\, j = 1,2,3,..............,n. \tag{23}$$

3. Mathematical model with non-linear constraints

3.1 Mathematical model with non-linear probabilistic constraints

Let us consider an optimization problem having non-linear probabilistic constraints that can be modeled as follows:

To find $X = (x_1, x_2, x_3, ..., x_n)$so as to,

$$Minimize\, f(X) \tag{24}$$

subject to chance constraints,

$$\Pr[\sum_{i=1}^{n}\sum_{j=1}^{n} a_{kij}x_ix_j \leq b_k] \geq 1-p_k\,, k=1,2,3,..........,m. \tag{25}$$

$$x_i \geq 0\,, i=1,2,3,..............,n. \tag{26}$$

where, $0 < p_k < 1$ and $b_k > 0$, for $k = 1,2,3,..........,m$, are given constants. Let us consider that a_{kij}'s are independent one parameter gamma random variable for all or some i, j, k i.e., $a_{kij} \sim G(\xi_{kij})$ for all or some i, j, k or a_{kij}'s are identically zero for some (not all) i, j, k.

3.2 Reduction of the probabilistic model to deterministic form

Let us let us consider the event, $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{kij}x_ix_j \leq b_k\,, k=1,2,3,..........,m.$

This can be written as,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{kij}+1)x_ix_j - \sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j \leq b_k\,, k=1,2,3,..........,m. \tag{27}$$

Applying G.I. on the first term of LHS of (27) we get,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{kij}+1)x_ix_j \geq n^2[\prod_{i=1}^{n} x_i\{\prod_{j=1}^{n}(a_{kij}+1)x_j\}^{\frac{1}{n}}]^{\frac{1}{n}}\,, k=1,2,3,..........,m \tag{28}$$

Thus from (27) and (28) we get,

$$n^2[\prod_{i=1}^{n} x_i\{\prod_{j=1}^{n}(a_{kij}+1)x_j\}^{\frac{1}{n}}]^{\frac{1}{n}} - \sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j \leq b_k, k=1,2,3,..........,m$$

i.e.,

$$\prod_{i=1}^{n} x_i^n\{\prod_{j=1}^{n}(a_{kij}+1)x_j\} \leq \{\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j}{n^2}\}^{n^2}, k=1,2,3,..........,m$$

i.e.,

$$\prod_{i=1}^{n}\prod_{j=1}^{n}(a_{kij}+1) \leq \{\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_j}{n^2}\}^{n^2}/\{(\prod_{i=1}^{n} x_i^n)(\prod_{j=1}^{n} x_j)\},$$
$$k=1,2,3,..........,m. \tag{29}$$

Now for all non-negative a_{kij} 's we have,

$$1+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{kij}\leq\prod_{i=1}^{n}\prod_{j=1}^{n}(1+a_{kij}),k=1,2,3,..........,m. \tag{30}$$

Thus from (29) and (30) we get,

$$\begin{gathered}\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}a_{kij}\leq\{\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j}{n^2}\}^{n^2}/\{(\prod_{i=1}^{n}x_i^n)(\prod_{j=1}^{n}x_j)\}-1,\\ k=1,2,3,..........,m.\end{gathered} \tag{31}$$

Thus, the probabilistic constraints (25) can be written, using the implicative relation (30), as follows:

$$\begin{gathered}\textstyle\Pr[\sum_{i=1}^{n}\sum_{j=1}^{n}a_{kij}\leq\{(\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j}{n^2})^{n^2}/(\prod_{i=1}^{n}x_i^n\prod_{j=1}^{n}x_j)\}-1]\geq 1-p_k,\\ k=1,2,3,..........,m.\end{gathered} \tag{32}$$

Let, $A_k=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{kij},k=1,2,3,.........,m.$

Since, a_{kij} is zero for some (not all) i,j,k and for the rest $a_{kij}\sim G(\xi_{kij})$. So,

$$A_k\sim G(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}),k=1,2,3,.......,m. \tag{33}$$

Let, $\Pr[A_k\leq t_k]=1-p_k,k=1,2,3,.......,m.$

It implies that,

$$t_k=G_{p_k}(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}),k=1,2,3,.......,m. \tag{34}$$

Thus from (32) using (34) we get,

$$\begin{gathered}\textstyle[\{\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n}x_ix_j}{n^2})^{n^2}/(\prod_{i=1}^{n}x_i^n\prod_{j=1}^{n}x_j)\}-1]\geq G_{p_k}(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}),\\ k=1,2,3,..........,m.\end{gathered} \tag{35}$$

Therefore, in (35) we have implicatively reduced deterministic form of the probabilistic constraint (25). Also here the reduced constraints are non-linear and the reduced problem is a non-linear optimization problem.

Thus, the implicative reduction to deterministic form of the probabilistic model (24) - (26) is as follows:

To find $X = (x_1, x_2, x_3, ..., x_n)$ so as to,

$$Minimize\, f(X) \tag{36}$$

subject to the constraints,

$$[\{\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j}{n^2})^{n^2}/(\prod_{i=1}^{n} x_i^n \prod_{j=1}^{n} x_j)\} - 1] \geq G_{p_k}(\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}), \\ k = 1,2,3,.........., m. \tag{37}$$

$$x_i \geq 0\,,\, i = 1,2,3,..............., n. \tag{38}$$

3.2.1 Special Case

In particular from (33) we can write,

$$2A_k \sim \chi^2_{2\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}}, k = 1,2,3,......., m, \tag{39}$$

if $2\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}$ is positive integers.

Let, $\Pr[2A_k \leq t_k] = 1 - p_k, k = 1,2,3,......., m.$

It implies that,

$$t_k = \chi^2_{2\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}}(1-p_k), k = 1,2,3,......., m. \tag{40}$$

Then, the implicative reduction to deterministic form of the probabilistic model (24) - (26) is as follows:

To find $X = (x_1, x_2, x_3, ..., x_n)$so as to,

$$Minimize\, f(X) \tag{41}$$

subject to the constraints,

$$[\{\frac{b_k+\sum_{i=1}^{n}\sum_{j=1}^{n}x_i x_j}{n^2})^{n^2}/(\prod_{i=1}^{n}x_i^n\prod_{j=1}^{n}x_j)\}-1]\geq\frac{1}{2}\chi^2_{2\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{kij}}(1-p_k),\quad k=1,2,3,.........,m, \tag{42}$$

$$x_i\geq 0\,,\,i=1,2,3,..............,n. \tag{43}$$

4. Solution

We can solve, the non-linear deterministic problems (15) - (17) and (21) - (23) corresponding to the probabilistic problem (1) - (3) and the non-linear deterministic problems (36) - (38) and (41) - (43) corresponding to the probabilistic problem (24) - (26), using mathematical programming software LINGO 10 [17]. Since in both the cases the reductions are through implicative relationship, so we need to verify whether the solution so obtained satisfy the respective probabilistic constraints and we use programming code in MATLAB 7.6 for this verification. If the solutions satisfy the respective probabilistic constraints, then we can consider those solutions as the optimal solutions for the probabilistic problems (1) - (3) and (24) - (26) respectively.

5. Numerical Example

5.1 Numerical example for the model having linear probabilistic constraints

Let us consider the following example for illustration.

To find x_1,x_2,x_3 so as to

$$Minimize\, z=2x_1^2+x_2^2+3x_3^2 \tag{44}$$

subject to the chance constraints,

$$\begin{aligned}&Pr(a_{11}x_1+a_{12}x_2+a_{13}x_3\leq b_1)\geq 1-p_1\\&Pr(a_{21}x_1+a_{22}x_2+a_{23}x_3\leq b_2)\geq 1-p_2\\&Pr(a_{31}x_1+a_{32}x_2+a_{33}x_3\leq b_3)\geq 1-p_3\end{aligned} \tag{45}$$

$$x_i\geq 0\,,\,i=1,2,3. \tag{46}$$

where, $a_{ij}\sim G(\alpha_{ij})$; $i,j=1,2,3$ and α_{ij}'s are given in the following matrix:

i \ j	1	2	3
1	0.2	0.1	0.2
2	0.3	0.1	0.6
3	0.2	0.77	0.03

Also given that, $b_1 = 2, b_2 = 1, b_3 = 4$ and $p_1 = 0.01, p_2 = 0.05, p_3 = 0.1$.

5.1.1 Solution of the above problem

Using (21) – (23), the reduced implicative deterministic form of the given problem (44) – (46) can be written as follows:

To find x_1, x_2, x_3 so as to

$$Minimize z = 2x_1^2 + x_2^2 + 3x_3^2 \tag{47}$$

subject to the constraints,

$$\begin{aligned} &[(\tfrac{1}{3}(2+x_1+x_2+x_3))^3/(x_1x_2x_3)] - 1 \geq \tfrac{1}{2}\chi_1^2(0.99) \\ &[(\tfrac{1}{3}(1+x_1+x_2+x_3))^3/(x_1x_2x_3)] - 1 \geq \tfrac{1}{2}\chi_2^2(0.95) \\ &[(\tfrac{1}{3}(4+x_1+x_2+x_3))^3/(x_1x_2x_3)] - 1 \geq \tfrac{1}{2}\chi_2^2(0.90) \end{aligned} \tag{48}$$

$$x_i \geq 0\,,\, i = 1,2,3. \tag{49}$$

And from the statistical table we have, $\chi_1^2(0.99) = 6.63\,,\, \chi_2^2(0.95) = 5.99\,,\, \chi_2^2(0.90) = 4.61$. Now using LINGO10 [17] we can get the global optimal solution, for the problem (47) – (49), which are

$x_1 = 0.52945 * 10^{-4}\,,\, x_2 = 0.1323762 * 10^{-3}\,,\, x_3 = 0.8797592 * 10^{-6}$ and $z_{\min} = 0.2313211 * 10^{-7}$.

Now the above solutions satisfy the constraints (45), which can be verified using MATLAB 7.6 [8]. Thus the above solutions are optimal for the given problem (44) – (46).

5.2 Numerical example for the model having non-linear probabilistic constraints

Let us consider the following example for illustration.

To find x_1, x_2, x_3 so as to

$$Minimize\, z = 2x_1^2 + x_2^2 + 3x_3^2 \tag{50}$$

subject to the chance constraints,

$$\begin{aligned} & Pr(\textstyle\sum_{i=1}^{3}\sum_{j=1}^{3} a_{1ij}x_ix_j \leq b_1) \geq 1 - p_1 \\ & Pr(\textstyle\sum_{i=1}^{3}\sum_{j=1}^{3} a_{2ij}x_ix_j \leq b_2) \geq 1 - p_2 \end{aligned} \tag{51}$$

$$x_i \geq 0\,,\, i = 1,2,3. \tag{52}$$

where, $a_{kij} \sim G(\xi_{kij})$; $i, j = 1,2,3$ and $k = 1,2$ and ξ_{kij}'s are given in the following matrices:

For k = 1, ξ_{1ij}'s are

i \ j	1	2	3
1	0.2	0.1	0.2
2	0.3	0.1	0.6
3	0.2	0.77	0.03

For k = 2, ξ_{2ij}'s are

i \ j	1	2	3
1	0.1	0.2	0.1
2	0.3	0.3	0.1
3	0.1	0.03	0.77

Also given that, $b_1 = 2, b_2 = 3$ and $p_1 = 0.01, p_2 = 0.05$.

5.2.1 Solution of the above problem

Using (41)-(43), the reduced implicative deterministic form of the given problem (50)-(52) can be written as follows:

To find x_1, x_2, x_3 so as to

$$Minimize\, z = 2x_1^2 + x_2^2 + 3x_3^2 \tag{53}$$

subject to the constraints,

$$(54)\qquad \begin{array}{l} [\{\frac{(2+\sum_{i=1}^{3}\sum_{j=1}^{3}x_ix_j)}{3*3}\}^{3*3}/(x_1x_2x_3)^4]-1\geq\frac{1}{2}\chi_5^2(0.99) \\ [\{\frac{(3+\sum_{i=1}^{3}\sum_{j=1}^{3}x_ix_j)}{3*3}\}^{3*3}/(x_1x_2x_3)^4]-1\geq\frac{1}{2}\chi_4^2(0.95) \end{array}$$

$$(55)\qquad x_i\geq 0\,,\, i=1,2,3.$$

And from the statistical table we have, $\chi_5^2(0.99)=15.1\,,\,\chi_4^2(0.95)=9.49$

Again we can find global optimal solution for the problem (53)-(55), using LINGO 10 [17], which is as follows:

$x_1=0.06279646\,,\,x_2=0.08667882\,,\,x_3=0.04885335$ and $z_{\min}=0.02255996$.

Now the above solutions satisfy the constraints (51), which can be verified using MATLAB [8]. Thus the above solutions are optimal for the given problem (50) - (52).

6. Conclusion

In this paper we have suggested a reduction procedure of probabilistic constraints to deterministic constraints through implicative relationship. During this reduction procedure the event space under consideration has been enlarged and the implicative reduction calls for verification. We have to verify that, whether the optimal solution of the transformed deterministic problems under extended region satisfies the original probabilistic constraints or not. Once this verification gives a positive response, the obtained optimal solution of the extended deterministic problem becomes the optimal solution of the original CCP. In case of transformation using sharp inequalities including separation of coefficient parameters, if needed under distributional assumption, the positive response during verification becomes a likely one.

For a gamma setup separation of coefficients from the gamma variables is a must, because often wise evaluation of the resultant distribution becomes a tedious task. For a normal setup this separation is redundant.

It may finally be noted that the case exponential distribution can also be studied as a spatial case of gamma distribution in the step parameters equal to unity.

Conflict of Interests

The authors declare that there is no conflict of interests.

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