Available online at http://scik.org
Adv. Inequal. Appl. 2016, 2016:6
ISSN: 2050-7461

SOME GENERALIZED OSTROWSKI TYPE INEQUALITIES INVOLVING LOCAL FRACTIONAL INTEGRALS AND APPLICATIONS

MEHMET ZEKI SARIKAYA[1], SAMET ERDEN[2,*], HÜSEYIN BUDAK[1]

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

[2]Department of Mathematics, Faculty of Science, Bartın University, Bartın-TURKEY

Abstract. In this paper, we establish the generalized Ostrowski type inequality involving local fractional integrals on fractal sets R^{α} $(0<\alpha\leq 1)$ of real line numbers. Some applications for special means of fractal sets R^{α} are also given. The results presented here would provide extensions of those given in earlier works.

Keywords: Generalized Ostrowski inequality; Generalized Hölder's inequality; Generalized convex functions.

2010 AMS Subject Classification: 26D07, 26D10, 26A33.

1. Introduction

In 1938, Ostrowski established the following interesting integral inequality for differentiable mappings with bounded derivatives [10].

Theorem 1.1. [Ostrowski inequality] *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) whoose derivative $f':(a,b)\to R$ is bounded on (a,b), i.e. $\|f'\|_{\infty}:=\sup\limits_{t\in(a,b)}|f'(t)|<\infty$. Then,*

*Corresponding author.

Received November 14, 2015

we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty , \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

This inequality is well known in the literature as the Ostrowski inequality. For more information recent development on Ostrowski inequality, please refer to [1]-[5], [7], [8] and [11]-[15].

Definition 1.1. [Convex function] The function $f : [a,b] \subset R \to R$, is said to be convex if the following inequality holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

Theorem 1.2. [Hermite-Hadamard inequality] *Let $f : I \subseteq R \to R$ be a convex function on the interval I of real numbers and $a,b \in I$ with $a < b$. If f is a convex function then the following double inequality, which is well known in the literature as the Hermite–Hadamard inequality, holds* [12]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}. \tag{2}$$

2. Preliminaries

Recall the set R^α of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [17, 18] and so on.

Recently, the theory of Yang's fractional sets [17] was introduced as follows.

For $0 < \alpha \leq 1$, we have the following α-type set of element sets:

Z^α : The α-type set of integer is defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, ..., \pm n^\alpha, ...\}$.

Q^α : The α-type set of the rational numbers is defined as the set $\{m^\alpha = \left(\frac{p}{q}\right)^\alpha : p, q \in Z,$ $q \neq 0\}$.

J^α : The α-type set of the irrational numbers is defined as the set $\{m^\alpha \neq \left(\frac{p}{q}\right)^\alpha : p, q \in Z,$ $q \neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};

(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;

(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;

(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;

(5) $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;

(6) $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;

(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. [17] A non-differentiable function $f:R\to R^{\alpha}$, $x\to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2.2. [17]The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}},$$

where $\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(\alpha+1)(f(x)-f(x_0))$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 2.3. [17] Let $f(x)\in C_{\alpha}[a,b]$. Then the local fractional integral is defined by,

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}[a,b]$.

Definition 2.4. [Generalized convex function] [17] Let $f:I\subseteq R\rightarrow R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in[0,1]$, if the following inequality

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;

(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer function.

Lemma 2.1. [17] *(1) (Local fractional integration is anti-differentiation) Suppose that* $f(x)=g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, *then we have*

$${}_aI_b^{\alpha}f(x)=g(b)-g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x),g(x)\in D_{\alpha}[a,b]$ *and* $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)\in C_{\alpha}[a,b]$, *then we have*

$${}_aI_b^{\alpha}f(x)g^{(\alpha)}(x)=f(x)g(x)|_a^b-{}_aI_b^{\alpha}f^{(\alpha)}(x)g(x).$$

Lemma 2.2. [17] *We have*

i) $\dfrac{d^{\alpha}x^{k\alpha}}{dx^{\alpha}}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)}\int\limits_a^b x^{k\alpha}(dx)^{\alpha}=\dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right)$, $k\in R$.

Lemma 2.3. [Generalized Hölder's inequality] [17] *Let* $f,g\in C_{\alpha}[a,b]$, $p,q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b|f(x)g(x)|(dx)^{\alpha}\leq\left(\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b|f(x)|^p(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int\limits_a^b|g(x)|^q(dx)^{\alpha}\right)^{\frac{1}{q}}.$$

In [9], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.1. *Let $f(x) \in I_x^{\alpha}[a,b]$ be generalized convex function on $[a,b]$ with $a<b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{\alpha} f(x) \leq \frac{f(a)+f(b)}{2^{\alpha}}. \tag{3}$$

The interested reader is refer to [6],[9],[16]-[22] for local freactional theory.

In this paper, we establish the generalized Ostrowski type inequalities and we obtain some inequalities using generalized convex function. The results presented here would provide extensions of those given in earlier works.

3. Main results

Theorem 3.1. *Let $I \subseteq \mathbb{R}$ be an interval, $f: I^0 \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ (I^0 is the interior of I) such that $f \in D_{\alpha}(I^0)$ and $f^{(\alpha)} \in C_{\alpha}[a,b]$ for $a,b \in I^0$ with $a<b$. Then, for all $x \in [a,b]$, we have the identity*

$$\begin{aligned} &(1-h)^{\alpha} f(x) + h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_aI_b^{\alpha} f(t) \\ = \; &\frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}} \int_a^b p(x,t) f^{(\alpha)}(t)\,(dt)^{\alpha} \end{aligned}$$

where

$$p(x,t) = \begin{cases} \left(t-\left(a+h\frac{b-a}{2}\right)\right)^{\alpha}, & t \in [a,x] \\ \\ \left(t-\left(b-h\frac{b-a}{2}\right)\right)^{\alpha}, & t \in (x,b] \end{cases}$$

for all $h \in [0,1]$ and $a+h\frac{b-a}{2} \leq x \leq b-h\frac{b-a}{2}$.

Proof. Using the local fractional integration by parts, we have

$$\begin{aligned}
& \frac{1}{\Gamma(1+\alpha)}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha \\
= & \frac{1}{\Gamma(1+\alpha)}\int_a^x \left(t-\left(a+h\frac{b-a}{2}\right)\right)^\alpha f^{(\alpha)}(t)\,(dt)^\alpha \\
& +\frac{1}{\Gamma(1+\alpha)}\int_x^b \left(t-\left(b-h\frac{b-a}{2}\right)\right)^\alpha f^{(\alpha)}(t)\,(dt)^\alpha \\
= & \left(t-\left(a+h\frac{b-a}{2}\right)\right)^\alpha f(t)\Big|_a^x-\frac{1}{\Gamma(1+\alpha)}\int_a^x \Gamma(1+\alpha)\,f(t)\,(dt)^\alpha \\
& +\left(t-\left(b-h\frac{b-a}{2}\right)\right)^\alpha f(t)\Big|_x^b-\frac{1}{\Gamma(1+\alpha)}\int_x^b \Gamma(1+\alpha)\,f(t)\,(dt)^\alpha \\
= & (b-a)^\alpha(1-h)^\alpha f(x)+\left(h\frac{b-a}{2}\right)^\alpha[f(a)+f(b)]-\Gamma(1+\alpha)\,{}_aI_b^\alpha f(t).
\end{aligned}$$

If we devide the rsulting equality with $(b-a)^\alpha$, then we complete the proof. □

Corollary 3.1. *Under the same assumptions of Theorem with $h=1$, then the following equality holds:*

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha=\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t).$$

Remark 3.1. If we choose $h=0$ in Theorem , then we have the following equality

$$\frac{1}{\Gamma(1+\alpha)(b-a)^\alpha}\int_a^b p(x,t)f^{(\alpha)}(t)\,(dt)^\alpha=f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)$$

which is given by Sarikaya and Budak in [16].

Theorem 3.2. *Suppose that the assumptions of Theorem are satisfied, then we have the inequality*

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)\right| \tag{4}$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[2^{\alpha}\left(x-\frac{a+b}{2}\right)^{2\alpha}+\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[(h-1)^{2\alpha}+h^{2\alpha}\right]\right]$$

for all $h\in[0,1]$ *and* $a+h\frac{b-a}{2}\leq x\leq b-h\frac{b-a}{2}$.

Proof. Taking modulus in Theorem , we get

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)\right| \tag{5}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}|p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}\\
&\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}|p(x,t)|(dt)^{\alpha}\\
&= \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{a}^{x}\left|t-\left(a+h\frac{b-a}{2}\right)\right|^{\alpha}(dt)^{\alpha}\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\alpha)}\int_{x}^{b}\left|t-\left(b-h\frac{b-a}{2}\right)\right|^{\alpha}(dt)^{\alpha}\right]\\
&= \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\left[K_1+K_2\right].
\end{aligned}$$

Using Lemma , we have

$$\begin{aligned} K_1 &= \frac{1}{\Gamma(1+\alpha)} \int_a^x \left| t - \left(a + h\frac{b-a}{2} \right) \right|^\alpha (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \int_a^{a+h\frac{b-a}{2}} \left(a + h\frac{b-a}{2} - t \right)^\alpha (dt)^\alpha \\ &\quad + \frac{1}{\Gamma(1+\alpha)} \int_{a+h\frac{b-a}{2}}^x \left(t - \left(a + h\frac{b-a}{2} \right) \right)^\alpha (dt)^\alpha \\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\left(h\frac{b-a}{2} \right)^{2\alpha} + \left(x - \left(a + h\frac{b-a}{2} \right) \right)^{2\alpha} \right] \end{aligned}$$

and similarly

$$\begin{aligned} K_2 &= \frac{1}{\Gamma(1+\alpha)} \int_x^b \left| t - \left(b - h\frac{b-a}{2} \right) \right|^\alpha (dt)^\alpha \\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\left(h\frac{b-a}{2} \right)^{2\alpha} + \left(b - h\frac{b-a}{2} - x \right)^{2\alpha} \right]. \end{aligned}$$

Substituting the calculated intagrals K_1 and K_2 in (5), then we have

$$\left| (1-h)^\alpha f(x) + h^\alpha \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^\alpha f(t) \right|$$

$$\begin{aligned} \leq & \frac{\left\| f^{(\alpha)} \right\|_\infty}{(b-a)^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\left(h\frac{b-a}{2} \right)^{2\alpha} + \left(x - \left(a + h\frac{b-a}{2} \right) \right)^{2\alpha} \right. \\ & \left. + \left(h\frac{b-a}{2} \right)^{2\alpha} + \left(b - h\frac{b-a}{2} - x \right)^{2\alpha} \right] \end{aligned}$$

$$= \frac{\left\|f^{(\alpha)}\right\|_\infty}{(b-a)^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[2^\alpha\left(h\frac{b-a}{2}\right)^{2\alpha}+2^\alpha\left(x-\frac{a+b}{2}\right)^{2\alpha}\right.$$

$$\left.+\frac{(b-a)^{2\alpha}}{2^\alpha}-h^\alpha(b-a)^{2\alpha}+h^{2\alpha}(b-a)^{2\alpha}\right]$$

$$= \frac{\left\|f^{(\alpha)}\right\|_\infty}{(b-a)^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[2^\alpha\left(x-\frac{a+b}{2}\right)^{2\alpha}+\frac{(b-a)^{2\alpha}}{2^\alpha}\left[(h-1)^{2\alpha}+h^{2\alpha}\right]\right]$$

which completes the proof. □

Remark 3.2. If we choose $h=0$ in Theorem , then we have the following inequality

$$\left|f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right|\le 2^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\frac{1}{4^\alpha}+\frac{\left(x-\frac{a+b}{2}\right)^{2\alpha}}{(b-a)^{2\alpha}}\right](b-a)^\alpha\left\|f^{(\alpha)}\right\|_\infty,$$

which is given by Sarikaya and Budak in [16].

Remark 3.3. If we choose $\alpha=1$ in Theorem , then we have the following inequality

$$\left|(1-h)f(x)+h\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(t)dt\right|$$

$$\le \frac{\|f'\|_\infty}{b-a}\left[\frac{1}{4}(b-a)^2\left[(h-1)^2+h^2\right]+\left(x-\frac{a+b}{2}\right)^2\right]$$

which is proved by Dragomir et al. in [8].

Corollary 3.2. *Under the same assumptions of Theorem with $h=1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(t)\right|\le\frac{(b-a)^\alpha}{2^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left\|f^{(\alpha)}\right\|_\infty.$$

Theorem 3.3. *Suppose that the assumptions of Theorem are satisfied, then we have the inequality*

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right|$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{p}}{(b-a)^{\alpha}}\left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}}$$

$$\times\left[2\left(h\frac{b-a}{2}\right)^{(q+1)}+\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{(q+1)}+\left(b-h\frac{b-a}{2}-x\right)^{(q+1)}\right]^{\frac{\alpha}{q}}$$

for all $h\in[0,1]$ *and* $a+h\frac{b-a}{2}\leq x\leq b-h\frac{b-a}{2}$, *where* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and* $\left\|f^{(\alpha)}\right\|_{p}$ *is defined by*

$$\left\|f^{(\alpha)}\right\|_{p}=\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(\alpha)}(t)\right|^{p}(dt)^{\alpha}\right)^{\frac{1}{p}}$$

.

Proof. Taking madulus in Theorem and using generalized Hölder's inequality, we have

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(t)\right|$$

$$\leq \frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_{a}^{b}|p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}$$

$$\leq \frac{1}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}\left|f^{(\alpha)}(t)\right|^{p}(dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}|p(x,t)|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{p}}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}|p(x,t)|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}=\frac{\left\|f^{(\alpha)}\right\|_{p}}{(b-a)^{\alpha}}(K_{3})^{\frac{1}{q}}.$$

Using Lemma , we have

$$K_3 = \frac{1}{\Gamma(1+\alpha)} \int_a^x \left| t - \left(a + h\frac{b-a}{2} \right) \right|^{\alpha q} (dt)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_x^b \left| t - \left(b - h\frac{b-a}{2} \right) \right|^{\alpha q} (dt)^{\alpha}$$
$$= \frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \left[2^{\alpha} \left(h\frac{b-a}{2} \right)^{(q+1)\alpha} + \left(x - \left(a + h\frac{b-a}{2} \right) \right)^{(q+1)\alpha} + \left(b - h\frac{b-a}{2} - x \right)^{(q+1)\alpha} \right]. \tag{6}$$

Hence, the proof is completed. □

Remark 3.4. If we choose $h = 0$ in Theorem , then we have the following inequality

$$\left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_b^{\alpha} f(t) \right| \leq \frac{\left\| f^{(\alpha)} \right\|_p}{(b-a)^{\alpha}} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \left[(x-a)^{(q+1)\alpha} + (b-x)^{(q+1)\alpha} \right]^{\frac{1}{q}},$$

which is given by Sarikaya and Budak in [16].

Corollary 3.3. *Under the same assumptions of Theorem with $h = 1$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} \, {}_aI_b^{\alpha} f(t) \right| \leq \frac{(b-a)^{\frac{\alpha}{q}}}{2^{\alpha}} \left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)} \right)^{\frac{1}{q}} \left\| f^{(\alpha)} \right\|_p.$$

Theorem 3.4. *Suppose that the assumptions of Theorem are satisfied. If $\left|f^{(\alpha)}\right|^p$ is generalized convex, then we have the inequality*

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right|$$

$$\leq \frac{1}{(b-a)^{\frac{\alpha}{q}}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}}\left(\left[\left|f^{(\alpha)}(b)\right|^p+\left|f^{(\alpha)}(a)\right|^p\right]\right)^{\frac{1}{p}}$$

$$\times\left[2\left(h\frac{b-a}{2}\right)^{(q+1)}+\left(x-\left(a+h\frac{b-a}{2}\right)\right)^{(q+1)}+\left(b-h\frac{b-a}{2}-x\right)^{(q+1)}\right]^{\frac{\alpha}{q}}$$

for all $h\in[0,1]$ and $a+h\frac{b-a}{2}\leq x\leq b-h\frac{b-a}{2}$, where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking madulus in Theorem and using generalized Hölder's inequality, we have

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right| \tag{7}$$

$$\leq \frac{1}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^b |p(x,t)|\left|f^{(\alpha)}(t)\right|(dt)^{\alpha}$$

$$\leq \frac{1}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f^{(\alpha)}(t)\right|^p(dt)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b|p(x,t)|^q(dt)^{\alpha}\right)^{\frac{1}{q}}$$

Because of the generalized convexity of $\left|f^{(\alpha)}\right|^p$, we find that

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f^{(\alpha)}(t)\right|^p(dt)^{\alpha} \tag{8}$$

$$\leq \frac{\left|f^{(\alpha)}(b)\right|^p}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^b(t-a)^{\alpha}(dt)^{\alpha}+\frac{\left|f^{(\alpha)}(a)\right|^p}{\Gamma(1+\alpha)(b-a)^{\alpha}}\int_a^b(b-t)^{\alpha}(dt)^{\alpha}$$

$$= \frac{\left|f^{(\alpha)}(b)\right|^p}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b-a)^{2\alpha}+\frac{\left|f^{(\alpha)}(a)\right|^p}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(b-a)^{2\alpha}$$

$$= (b-a)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[\left|f^{(\alpha)}(b)\right|^p+\left|f^{(\alpha)}(a)\right|^p\right]$$

If we substitute the inequalities (8) and (6) in (7), then we obtain required inequality, which completes the proof. □

Remark 3.5. If we choose $h=0$ in Theorem , then we have the following inequality

$$\left| f(x)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right|$$

$$\leq \frac{1}{(b-a)^{\frac{\alpha}{q}}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}}$$

$$\times\left[(x-a)^{(q+1)}+(b-x)^{(q+1)}\right]^{\frac{\alpha}{q}}\left(\left[\left|f^{(\alpha)}(b)\right|^{p}+\left|f^{(\alpha)}(a)\right|^{p}\right]\right)^{\frac{1}{p}}$$

which is given by Sarikaya and Budak in [16].

Corollary 3.4. *Under the same assumptions of Theorem with $h=1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(t)\right|$$

$$\leq \frac{(b-a)^{\alpha}}{2^{\alpha}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+q\alpha)}{\Gamma(1+(q+1)\alpha)}\right)^{\frac{1}{q}}\left(\left[\left|f^{(\alpha)}(b)\right|^{p}+\left|f^{(\alpha)}(a)\right|^{p}\right]\right)^{\frac{1}{p}}.$$

4. Applications to some special means

Let us recall some generalized means:

$$A(a,b)=\frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}};$$

$$L_n(a,b)=\left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\left[\frac{b^{(n+1)\alpha}-a^{(n+1)\alpha}}{(b-a)^{\alpha}}\right]\right]^{\frac{1}{n}},\ n\in Z\backslash\{-1,0\},\ a,b\in R,\ a\neq b.$$

Now, let us reconsider the inequality (4):

$$\left|(1-h)^{\alpha} f(x)+h^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)\right|$$

$$\leq \frac{\left\|f^{(\alpha)}\right\|_{\infty}}{(b-a)^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[2^{\alpha}\left(x-\frac{a+b}{2}\right)^{2\alpha}+\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[(h-1)^{2\alpha}+h^{2\alpha}\right]\right]$$

for all $h\in[0,1]$ and $a+h\frac{b-a}{2}\leq x\leq b-h\frac{b-a}{2}$.

Consider the mapping $f:(0,\infty)\to R^{\alpha},\ f(x)=x^{n\alpha},\ n\in Z\backslash\{-1,0\}$. Then, $0<a<b$, we have

$$\frac{f(a)+f(b)}{2^{\alpha}}=A(a^{n},b^{n})\text{ and }\frac{1}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha} f(t)=[L_{n}(a,b)]^{n},$$

$$\left\|f^{(\alpha)}\right\|_{\infty}=\begin{cases}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\right|b^{(n-1)\alpha}, & n>1\\ \\ \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\right|a^{(n-1)\alpha}, & n\in(-\infty,1]\backslash\{-1,0\},\end{cases}$$

and then, we find that

$$\left|(1-h)^{\alpha} x^{\alpha n}+h^{\alpha}A(a^{n},b^{n})-\Gamma(1+\alpha)[L_{n}(a,b)]^{n}\right|$$

$$\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)(b-a)^{\alpha}}\left[2^{\alpha}\left(x-\frac{a+b}{2}\right)^{2\alpha}+\frac{(b-a)^{2\alpha}}{2^{\alpha}}\left[(h-1)^{2\alpha}+h^{2\alpha}\right]\right]\delta_{n}(a,b)$$

where

$$\delta_{n}(a,b)=\begin{cases}\left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\right|b^{(n-1)\alpha}, & n>1\\ \\ \left|\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\right|a^{(n-1)\alpha}, & n\in(-\infty,1]\backslash\{-1,0\},\end{cases}$$

and $h\in[0,1]$, $x\in\left[a+h\frac{b-a}{2},b-h\frac{b-a}{2}\right]$.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] M. W. Alomari, M. Darus, S. S. Dragomir and P. Cerone, Ostrowski type inequalities for functions whose derivatives are convex in the second sense, Appl. Math. Lett. 23 (2010), 1071–1076.

[2] M. W. Alomari, M. E. Özdemir and H. Kavurmacı,On companion of Ostrowski inequality for mappings whose first derivatives absolute value are convex with applications, Miskolc Math. Notes 13 (2012), 233–248.

[3] M. W. Alomari and M. Darus Some Ostrowski's type inequalities for convex functions with application, RGMIA Res. Rep. Coll. 13 (2010), Art. 3.

[4] N. S. Barnett and S. S. Dragomir, An Ostrowski type inequality for double integrals and applications for cubature formulae, Soochow J. Math. 27 (2001), 109-114.

[5] P. Cerone and S.S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, Demonstratio Math. 37 (2004), 299–308.

[6] G-S. Chen, Generalizations of Hölder's and some related integral inequalities on fractal space, J. Funct. Spaces Appl. 2013 (2013), Article ID 198405.

[7] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11 (1998), 91-95.

[8] S. S. Dragomir, P. Cerone and J. Roumeliotis, A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, RGMIA Research Report Collection, 1999.

[9] H. Mo, X Sui and D Yu, Generalized convex functions on fractal sets and two related inequalities, Abst. Appl. Anal. 2014 (2014), Article ID 636751.

[10] A. M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert, Comment. Math. Helv. 10 (1938), 226-227.

[11] M. E. Ozdemir, H. Kavurmacı and M. Avcı, Ostrowski type inequalities for convex functions, Tamkang J. Math. 45 (2014), 335-340.

[12] J.E. Pečarić, F. Proschan and Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Boston, 1992.

[13] M. Z. Sarikaya, On the Ostrowski type integral inequality, Acta Math. Univ. Comenianae, Vol. LXXIX, 1(2010), pp. 129-134.

[14] M. Z. Sarikaya, On the Ostrowski type integral inequality for double integrals, Demonstratio Mathematica, Vol. XLV No 3 2012.

[15] M. Z. Sarikaya and H. Ogunmez, On the weighted Ostrowski type integral inequality for double integrals, The Arabian J. Sci. Eng. Math. 36 (2011), 1153-1160.

[16] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals, Submited, 2015.

[17] X. J. Yang, Advanced Local Fractional Calculus and Its Applications, World Science Publisher, New York, 2012.

[18] J. Yang, D. Baleanu and X. J. Yang, Analysis of fractal wave equations by local fractional Fourier series method, Adv. Math. Phys. 2013 (2013), Article ID 632309.

[19] X. J. Yang, Local fractional integral equations and their applications, Advances in Computer Science and its Applications (ACSA) 1(4), 2012.

[20] X. J. Yang, Generalized local fractional Taylor's formula with local fractional derivative, J. Expert Sys. 1 (2012), 26-30.

[21] X. J. Yang, D. Baleanu and H. M. Srivastava, Local fractional similarity solution for the diffusion equation defined on Cantor sets, Appl. Math. Lett. 47 (2015), 54-60.

[22] X. J. Yang, Local fractional Fourier analysis, Adv. Mech. Eng. Appl. 1 (2012), 12-16.