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POSITIVE SOLUTIONS FOR FRACTIONAL DIFFERENCE EQUATIONS USING FIXED POINT THEOREMS

LAVANYA A.P., DAPHY LOUIS LOVENIA J.*,

Department of Mathematics, Karunya University, Coimbatore 641114, India

Abstract. Consider a two point boundary value problem for fractional difference equation.

$$\begin{cases} -\Delta^{\mu}x(t) = f(t+\mu-1, x(t+\mu-1)), \\ x(\mu-2) = 0, \\ x(\mu+b+1) = g(x), \end{cases}$$

where $t \in [0,b]_{\mathbb{N}_{0:=\{0,1,\ldots,b\}}}$, $f : [\mu-1,\ldots\mu+b+1]_{\mathbb{N}_{\mu-2}} \times \mathbb{R} \to \mathbb{R}$ is continuous function, $g \in \mathbb{C}\big([\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}, \mathbb{R}\big)$ is a functional and $1 < \mu < 2$. For example, $g(x)$ has the form $g(x) = \sum_{i=1}^{n} c_i x(t_i)$, where $t_i \in [\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}$ and each $c_i \in \mathbb{R}$. Existence and uniqueness of the solutions of (1.1) are established using the contraction mapping theorem and krasnosel'skii theorem. Examples are provided to illustrate the result.

Keywords: Fractional difference equations; Boundary value problems; Existence and uniqueness of solutions; Green's function; Fixed point theorem.

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1. Introduction

*Corresponding author.

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In recent years, fractional difference equations have been of great interest. The researchers started doing research in the field of fractional difference equations motivated by the concepts of fractional differential equations. Fractional difference equations have various applications in the field of science and engneering. They are used in physics, chemistry, mechanics, control theory, signals and electrical circuits.

In particular, fractional differential equations, fractional calculus and fractional difference equations appear in the field of oncology, references there in [1,2,3].

In [4], Goodrich studied the following FBVP with non-local conditions,

$$-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)),$$
$$y(\nu-2) = g(x), \quad y(\nu+b) = 0.$$

In [5], Atici and Eloe solved the two-point boundary value problem for a finite fractional difference equations (FBVP) of the form,

$$-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)),$$
$$y(\nu-2) = 0, \quad y(\nu+b+1) = 0,$$

where $1 < \nu \leq 2$.

In [6], Goodrich also derived first Green's function for the three point nonlinear discrete FBVP

$$-\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)),$$
$$y(\nu-2) = 0, \quad \alpha y(\nu+k) = y(\nu+b),$$

where $\nu \in (1,2], \quad \alpha \in [0,1] \quad k \in [-1, b-1]_{\mathbb{Z}}$.

Motivated by these papers and by refering the papers [7-12], the two-point boundary value problem of fractional difference equations (FBVP) has been presented, which has the form,

$$\begin{cases} -\Delta^{\mu} x(t) = f(t+\mu-1, x(t+\mu-1)), \\ x(\mu-2) = 0, \\ x(\mu+b+1) = g(x), \end{cases} \tag{1.2}$$

where $t \in [0,b]_{\mathbb{N}_{0:=\{0,1,\ldots,b\}}}$, $f : [\mu-2,\ldots\mu+b+1]_{\mathbb{N}_{\mu-2}} \times \mathbb{R} \to \mathbb{R}$ is continuous function, $g \in \mathbb{C}\big([\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}},\mathbb{R}\big)$ is a functional and $1<\mu<2$. For example, $g(x)$ has the form $g(x)=\sum_{i=1}^{n} c_i x(t_i)$, where $t_i \in [\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}$ and each $c_i \in \mathbb{R}$.

Section 2 provids the basic definitions and Lemmas of fractional difference equations which are useful in the following sequel.

Section 3 obtains the sufficient conditions for the uniqueness solutions of two-point boundary value problem (1.1) using contraction mapping theorem.

Section 4 presents the sufficient conditions for the existence of positive solutions of (1.1) using krasnosel'skii theorem.

Section 5 obtains the sufficient conditions for nonexistence of positive solutions of problem (1.1).

In section 6, examples are given to illustrate the results of FBVP (1.1).

2. Preliminaries

Definition 1.1. [13] Define $t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}$ for any t and μ, for which the right-hand side is defined. If $t+1-\mu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{\underline{\mu}}=0$.

Definition 1.2. [7] The μ^{th} fractional sum of a function f, for $\mu>0$, is defined by

$$\Delta^{-\mu} f(t;a) := \frac{1}{\Gamma(\mu)} \sum_{s=a}^{t-\mu} (t-s-1)^{\underline{\mu-1}} f(s)$$

for $t \in \{a+\mu, a+\mu+1,\ldots\} := \mathbb{N}_{a+\mu}$. Also define the μ^{th} fractional difference for $\mu>0$ by $\Delta^{\mu} f(t) := \Delta^{N}\Delta^{\mu-N} f(t)$, where $t \in \mathbb{N}_{a+\mu}$ and $\mu \in \mathbb{N}$ is chosen so that $o \le N-1 < \mu \le N$.

Lemma 1.3. [7] *Let t and μ be any numbers for which t^{μ} and $t^{\mu-1}$ are defined. Then $\Delta t^{\mu} = \mu t^{\mu-1}$.*

Lemma 1.4. [13] *Let $0 \le N-1 < \mu \le N$. Then $\Delta^{-\mu}\Delta^{\mu} x(t) = x(t) + c_1 t^{\mu-1} + c_2 t^{\mu-2} + \ldots + c_N t^{\mu-N}$, for some $c_i \in \mathbb{R}$, with $1 \le i \le N$.*

Lemma 1.5. [4] *For t and s, for which both $(t-s-1)^{\mu}$ and $(t-s-2)^{\mu}$ are defined, find that $\Delta_s[(t-s-1)^{\mu}] = -\mu(t-s-1)^{\mu-1}$.*

Lemma 1.6. [14] *If $t \le r$, then $t^{\alpha} \le r^{\alpha}$ for any $\alpha > 0$.*

Theorem 1.7. *Let $h : [\mu-1,\dots \mu+b+1]_{\mathbb{N}_{\mu-1}} \to \mathbb{R}$ and $g : \mathbb{R}^{b+3} \to \mathbb{R}$ be given. A function x is a solution of the discrete FBVP*

$$\begin{cases} -\Delta^{\mu} x(t) = h(t+\mu-1), \\ x(\mu-2) = 0, \\ x(\mu+b+1) = g(x), \end{cases} \tag{2.1}$$

where $t \in [0,b]_{\mathbb{N}_0}$, if and only if $x(t), \quad t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}$ has the form,

$$x(t) = \sum_{s=0}^{b} G(t,s)h(s+\mu-1) + \frac{g(x)t^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}},$$

where $G(t,s) = \frac{1}{\Gamma\mu}\begin{cases} \frac{t^{\underline{\mu-1}}(\mu+b-s)^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}} - (t-s-1)^{\underline{\mu-2}} & 0 \le s \le t-\mu+2 \le b, \\ \frac{t^{\underline{\mu-1}}(\mu+b-s)^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}} & 0 \le t-\mu+2 \le s \le b. \end{cases}$

Proof. Consider a function that is the general solution of (1.1),

$$x(t) = -\Delta^{-\mu}h(t+\mu-1) + c_1 t^{\underline{\mu-1}} + c_2 t^{\underline{\mu-2}},$$

where $t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}$. Now applying boundary condition $x(\mu-2) = 0$ implies that

$$x(\mu-2) = -\Delta^{-\mu}h(t)_{t=\mu-2} + c_1(\mu-2)^{\underline{\mu-1}} + c_2(\mu-2)^{\underline{\mu-2}}$$

$$x(\mu-2) = c_2$$

$$c_2 = 0.$$

On the other hand, by applying boundary condition $x(\mu+b+1) = g(x)$, one finds that

$$\begin{aligned} x(\mu+b+1) &= [-\Delta^{-\mu}h(t)]_{t=\mu+b-s} + c_1(\mu+b+1)^{\mu-1} + c_2(\mu+b+1)^{\mu-2} \\ &= -\frac{1}{\Gamma\mu}\sum_{s=0}^{b}(\mu+b-s)h(s+\mu-1) + c_1(\mu+b+1)^{\mu-1} \end{aligned}$$

$$c_1(\mu+b+1)^{\mu-1} = \frac{1}{\Gamma\mu}\sum_{s=0}^{b}(\mu+b-s)^{\mu-1}h(s+\mu-1) + g(x)$$

$$c_1 = \frac{1}{(\mu+b+1)^{\mu-1}\Gamma\mu}\sum_{0}^{b}(\mu+b-s)^{\mu-1}h(s+\mu-1) + \frac{g(x)}{(\mu+b+1)^{\mu-1}}$$

Then we get, $x(t)$ as follows

$$x(t) = -\frac{1}{\Gamma\mu}\sum_{s=0}^{t-\mu}(t-s-1)^{\mu-1}h(s+\mu-1) + \frac{t^{\mu-1}}{(\mu+b+1)^{\mu-1}}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}h(s+\mu-1) + \frac{g(x)t^{\mu-1}}{(\mu+b+1)^{\mu-1}}.$$

Consequently, deduce that $x(t)$ has the form

$$x(t) = \sum_{s=0}^{b} G(t,s)h(s+\mu-1) + \frac{g(x)t^{\mu-1}}{(\mu+b+1)^{\mu-1}}.$$

Lemma 1.8. [8] *The Green's function $G(t,s)$ satisfies the following conditions:*

(i) $G(t,s) > 0 \quad for \quad (t,s) \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}} \times [0,b]$.

(ii) $\max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}} G(t,s) = G(s+\mu-1,s) \quad for \quad s \in [0,b]$.

(iii) $\min_{t\in[\frac{\mu+b}{4},\frac{3(\mu+b)}{4}]} G(t,s) \geq \frac{1}{4}\max_{t\in[\mu-3,\mu+b]_{\mathbb{N}_{\mu-1}}} G(t,s) = \frac{1}{4}G(s+\mu-1,s) \quad for \quad s \in [0,b]$.

3. Existence and uniqueness of solutions

Let us consider, x is a solution of (1.1), if and only if x is a fixed point of the operator $T : \mathbb{R}^{b+3} \to \mathbb{R}^{b+3}$, where

$$\begin{aligned}(Tx)(t) = -\frac{1}{\Gamma\mu}\sum_{0}^{t-\mu}(t-s-1)^{\mu-1}h(s+\mu-1) + \frac{t^{\mu-1}}{\Gamma\mu(\mu+b+1)^{\mu-1}} \\ \sum_{s=0}^{t-\mu}(\mu+b+1)^{\mu-1}h(s+\mu-1) + \frac{g(x)}{(\mu+b+1)^{\mu-1}}\end{aligned}$$

for $t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}$.

Theorem 3.1. *Assume that $f(t,x)$ and $g(x)$ are Lipschitz in x. If the condition*

$$\alpha \prod_{j=1}^{b}\Big(\frac{\mu+j}{j}\Big)\Big[\frac{\mu+2b+2}{(b+1)}\Big] + \beta < 1$$

hold then (1.1) has a unique solution.

Proof. Since $f(t,x)$ and $g(x)$ are Lipschitz in x. That is, there exists $\alpha, \beta > 0$ such that $|f(t,x_1) - f(t,x_2)| \leq \alpha \|x_1 - x_2\|$ whenever $x_1, x_2 \in \mathbb{R}$, and $|g(x_1) - g(x_2)| \leq \beta \|x_1 - x_2\|$ whenever $x_1, x_2 \in \mathbb{C}\big([\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}, \mathbb{R}\big)$.

First we show that T is contraction mapping. Let us define $\|x\| = \max_{t\in[\mu-2,\mu+b+1]} |x(t)|$

$$\begin{aligned}\|Tx_1 - Tx_2\| \leq \alpha \|x_1 - x_2\| & \max_{t\in[\mu-2,\mu+b+1]_{\mu-2}} \left[\frac{1}{\Gamma\mu}\sum_{s=0}^{b-\mu}(t-s-1)^{\underline{\mu-1}}\right] + \alpha \|x_1 - x_2\| \\ & \max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}} \left[\frac{t^{\mu-1}}{\Gamma\mu(\mu+b+1)^{\mu-1}}\sum_{s=0}^{t-\mu}(\mu+b-s)^{\underline{\mu-1}}\right] + \\ & \beta \|x_1 - x_2\| \max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}} \left[\frac{t^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}}\right].\end{aligned} \tag{3.1}$$

Now taken the right hand side terms, and from lemma 1.3,

$$\begin{aligned}\alpha \|x_1 - x_2\| \left[\frac{1}{\Gamma\mu}\sum_{s=0}^{t-\mu}(t-s-1)^{\mu-1}\right] &= \frac{\alpha \|x_1 - x_2\|}{\Gamma\mu}\left[-\frac{1}{\mu}(t-s)^{\mu}\right]_{s=0}^{t-\mu+1} \\ &= \alpha \|x_1 - x_2\| \left[\frac{\Gamma(t+1)}{\Gamma(t-\mu+1)\quad\Gamma(\mu+1)}\right] \\ &\leq \alpha \|x_1 - x_2\| \left[\frac{\Gamma(\mu+b+1)}{\Gamma(b+1)\quad\Gamma(\mu+1)}\right] \\ &= \alpha \prod_{j=1}^{b}\left(\frac{\mu+j}{j}\right) \|x_1 - x_2\|\end{aligned} \tag{3.2}$$

Again using the lemma, we get

$$\begin{aligned}\alpha \|x_1 - x_2\| \left[\frac{t^{\underline{\mu-1}}}{\Gamma\mu(\mu+b+1)^{\underline{\mu-1}}}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}\right] &\leq \alpha\frac{\|x_1 - x_2\|}{\Gamma\mu}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}} \\ &= \frac{\alpha \|x_1 - x_2\|}{\Gamma\mu}\left[-\frac{1}{\mu}(\mu+b-s+1)\right]_{s=0}^{b+1} \\ &= \alpha \|x_1 - x_2\| \, . \prod_{j=1}^{b+1}\left(\frac{\mu+j}{j}\right).\end{aligned}$$

Now, taking the third term in (3.1) and applying the lemma, we get

$$\beta \|x_1 - x_2\| \left[\frac{t^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}}\right] \leq \beta \|x_1 - x_2\| \, . \tag{3.3}$$

Substitute the equations (3.2), (3.3) in (3.1), we get

$$\begin{aligned}\|Tx_1-Tx_2\| &\leq \left\{\alpha\prod_{j=1}^{b}\Big(\frac{\mu+j}{j}\Big)+\alpha\prod_{j=1}^{b+1}\Big(\frac{\mu+j}{j}\Big)+\beta\right\}\|x_1-x_2\| \\ &= \left\{\alpha\prod_{j=1}^{b}\Big(\frac{\mu+j}{j}\Big)\Big[1+\frac{\mu+(b+1)}{b+1}\Big]+\beta\right\}\|x_1-x_2\| \\ &= \left\{\alpha\prod_{j=1}^{b}\Big(\frac{\mu+j}{j}\Big)\Big[\frac{\mu+2b+2}{b+1}\Big]+\beta\right\}\|x_1-x_2\|.\end{aligned}$$

Hence equation (1.1) has a unique solution. This completes the proof.

Theorem 3.2. *Suppose that there exists a constant $K>0$ such that $f(t,x)$ satisfies the inequality*

$$\max_{(t,x)\in[\mu-2,\mu+b+2]_{\mathbb{N}_{\mu-3}}\times[-K,K]}|f(t,x)|\leq\frac{K}{\frac{\Gamma\mu+b+1+\Gamma\mu+b}{\mu\Gamma b+1}+1} \tag{3.4}$$

and $g(x)$ satisfies the inequality

$$\max_{x\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-3}}\times[-K,K]}|g(x)|\leq\frac{K}{\frac{\Gamma\mu+b+1+\Gamma\mu+b}{\mu\Gamma b+1}+1}. \tag{3.5}$$

Then (1.1) has at least one solution x_0 satisfying $|x_0|\leq k$ for all $t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}$.

Proof Consider the Banach space

$$\mathbb{B}:=\{x\in\mathbb{R}^{b+3}:\|x\|\leq K\}.$$

Obviously, T is a continuous opeartor, which is defined in (2.1). To prove $T:\mathbb{B}\to\mathbb{B}$, it is enough to prove that $\|Tx\|\leq K$. Assume that inequalities (3.4) & (3.5) hold for given f & g. For convenience, let

$$\Phi=\frac{K}{\frac{\Gamma(\mu+b+1)+\Gamma(\mu+b)}{\mu\Gamma(b+1)}+1} \tag{3.6}$$

$$\begin{aligned}\|Tx\|\leq{}&\max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}}\frac{1}{\Gamma(\mu)}\sum_{s=0}^{t-\mu}(t-s-1)^{\underline{\mu-1}}|f(s+\mu-1,x(s+\mu-1))|+\\ &\max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}}\frac{t^{\underline{\mu-1}}}{\Gamma\mu(\mu+b+1)^{\underline{\mu-1}}}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}(f(s+\mu-1,x(s+\mu-1)))+\\ &\max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}}\frac{t^{\underline{\mu-1}}|g(x)|}{(\mu+b+1)^{\underline{\mu-1}}}\end{aligned} \tag{3.7}$$

$$\leq \Phi \max_{t\in[\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}} \left[\frac{1}{\Gamma\mu}\sum_{s=0}^{t-\mu}(t-s-1)^{\underline{\mu-1}} + \frac{t^{\underline{\mu-1}}}{\Gamma\mu(\mu+b+1)^{\underline{\mu-1}}}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}\right]$$

$$+\Phi\frac{t^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}} \tag{3.8}$$

Now simplify the terms on the right-hand side in inequality (3.1) as follows,

$$\begin{aligned}
&\frac{1}{\Gamma\mu}\sum_{s=0}^{t-\mu}(t-s-1)^{\underline{\mu-1}} + \frac{t^{\underline{\mu-1}}}{\Gamma\mu(\mu+b+1)^{\underline{\mu-1}}}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}\\
&\leq \frac{1}{\Gamma\mu}\sum_{s=0}^{t-\mu}(t-s-1)^{\underline{\mu-1}} + \frac{1}{\Gamma}\mu\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}\\
&\leq \frac{1}{\Gamma\mu}\sum_{s=0}^{b}(\mu+b-s-1)^{\underline{\mu-1}} + \frac{1}{\Gamma\mu}\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}}.
\end{aligned} \tag{3.9}$$

On the other hand, we know $t^{\mu-1}$ is increasing in t. Thus

$$\sum_{n=0}^{b}(\mu+b-s+1)^{\underline{\mu-1}} = \Big[-\frac{1}{\Gamma\mu}(\mu+b-s-1)^{\mu}\Big]_{s=0}^{b+1} = \frac{\Gamma(\mu+b+1)}{\mu\Gamma(b+1)}. \tag{3.10}$$

On the other hand, we have

$$\sum_{s=0}^{b}(\mu+b-s)^{\underline{\mu-1}} = \Big[-\frac{1}{\mu}(\mu+b-s-1)^{\mu}\Big]_{s=0}^{b+1} = \frac{\Gamma(\mu+b)}{\mu\Gamma(b+1)}. \tag{3.11}$$

Substituting (3.9)-(3.11) into (3.8), we get

$$\begin{aligned}
\|Tx\| &\leq \Phi\Big[\frac{\Gamma(\mu+b+1)}{\mu+\Gamma(b+1)} + \frac{\Gamma(\mu+b)}{\mu\Gamma(b+1)}\Big] + \Phi\\
&= \Phi\Big[\frac{\Gamma(\mu+b+1)+\Gamma(\mu+b)}{(\mu\Gamma(b+1))} + 1\Big].
\end{aligned} \tag{3.12}$$

By inserting (3.6) into (3.12), we obtain

$$\begin{aligned}
\|Tx\| &\leq \Phi\Big[\frac{\Gamma(\mu+b+1)+\Gamma(\mu+b)}{\mu\Gamma(b+1)} + 1\Big]\\
&= \frac{K}{\frac{\Gamma(\mu+b+1)+\Gamma(\mu+b)}{\mu\Gamma(b+1)}+1}\Big[\frac{\Gamma(\mu+b+1)+\Gamma(\mu+b)}{\mu\Gamma(b+1)} + 1\Big] = K.
\end{aligned} \tag{3.13}$$

By Browder theorem, $Tx_0 = x_0$ with $x_0 \in \mathbb{B}$. Therefore, the function x_0 is a solution of (1.1) and x_0 satisfies $|x_0(t)| \leq K$ for each $t \in [\mu-2,\mu+b+1]_{\mathbb{B}_{\mu-2}}$. This completes the proof of the theorem.

Lemma 3.3. *If $f : [\mu, \mu - 1, \dots \mu + b + 1]_{\mathbb{N}_{\mu-1}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Then the solution x of the FBVP (1.1) satisfy*

$$\min_{\frac{b+\mu}{4} \leq t \leq \frac{3(b+\mu)}{4}} x(t) \geq \frac{1}{4} \max_{(\mu-1, b+\mu+1)_{\mathbb{N}_{\mu-1}}} |x(t)|.$$

4. Existence of positive solutions

In this section, the existence theorems are solved using Krasnosel'skii theorem and the sufficient conditions for positive solutions are established.

Lemma 4.1 [15] *Let $\mathbb{B}$ be a banach space and let $K \leq \mathbb{B}$ be a cone. Assume that $\Omega_1 \quad \& \quad \Omega_2$ are bounded open sets contained in $\mathbb{B}$ such that $0 \in \Omega_1 \quad \& \quad \overline{\Omega_1} \leq \Omega_2$. Further, assume that $T : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ is a completely continuous operator. If either*

$$(i) \, \|Tx\| \leq \|x\| \quad for \quad x \in K \cap \partial\Omega_1 \quad \& \quad \|Tx\| \geq \|x\| \, for \quad x \in K \cap \partial\Omega_2;$$

or

$$(ii) \, \|Tx\| \geq \|x\| \quad for \quad x \in K \cap \partial\Omega_1 \quad \& \quad \|Tx\| \leq \|x\| \quad for \quad x \in K \cap \partial\Omega_2.$$

Then the operator T has atleast one fixed point in $K \in (\overline{\Omega_2} \backslash \Omega_1)$.

Let

$$f_0 = \liminf_{x \to 0} \min_{t \in [\mu-1, \mu+b+1]_{\mathbb{N}_{\mu-1}}} f(t,x)$$

$$f^0 = \limsup_{x \to 0} \max_{t \in [\mu-1, \mu+b+1]_{\mathbb{N}_{\mu-1}}} f(t,x)$$

$$f_\infty = \liminf_{x \to \infty} \min_{t \in [\mu-1, \mu+b+1]_{\mathbb{N}_{\mu-1}}} f(t,x)$$

$$f^\infty = \limsup_{x \to \infty} \max_{t \in [\mu-1, \mu+b+1]_{\mathbb{N}_{\mu-1}}} f(t,x)$$

$$\frac{1}{A} = \sum_{s=0}^{b+1} G(b+\mu+1, s), \quad \frac{1}{B} = \frac{1}{4} \sum_{s=[\frac{b+\mu+1}{4} - \mu + 2]}^{[\frac{3(b+\mu+1)}{4}] - \mu + 2} G\Big([\frac{b-\mu-1}{2}] + (\mu+1), s\Big)$$

The following conditions are required to prove the existence theorems of positive solutions.
$(H) f : [\mu, \mu-1, \dots \mu+b+1]_{\mathbb{N}_\mu} \times \mathbb{R} \to \mathbb{R}$ is continuous, and $g \in \mathbb{C}\Big([\mu-1, \mu+b+1]_{\mathbb{N}_{\mu-2}}, \mathbb{R}\Big)$ is a function.
(H_1) There is a number $p > 0$, such that $g(x) < p$ for $0 \leq x < p$.

(H_2) There is a number $p > 0$, such that $f(t,x) < Ap$ for $0 \le x \le p \quad \& \quad \mu - 1 \le t \le b + \mu + 1$.

(H_3) There is a number $p > 0$, such that $f(t,x) > Bp$ for $\frac{1}{4}p \le x \le p \quad \& \quad \frac{b+\mu+1}{4} \le t \le \frac{3(b+\mu+1)}{4}$.

(H_4) There is a number $s > 0$ such that $\frac{g(x)}{2^{\underline{\mu-1}}} > s$ for $o < s \le x < p$.

$(H_5) f^0 < Ax, f^\infty < Ax$

$(H_6) f_0 > Bx, f_\infty > Bx$

$(H_7) f^\infty < Ax, f_0 > Bx$

$(H_8) f_0 > Bx, f^\infty < Ax$

$(H_5^*) f^0 = 0, f^\infty = 0$

$(H_6^*) f_0 = \infty, f_\infty = \infty$

$(H_7^*) f^\infty = 0, f_0 = \infty$

$(H_8^*) f_0 = \infty, f^\infty = 0.$

Let $\mathbb{B} = \left\{ x : [\mu - 2, \mu + b + 1]_{\mathbb{N}_{\mu-2}} \to \mathbb{R}, x(\mu - 2) = 0, x(\mu + b + 1) = g(x) \right\}$

Then $\mathbb{B}$ is a Banach space with respect to the norm

$$\|x\| = \max_{t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}} |x(t)|.$$

Define a cone in $\mathbb{B}$ by

$$K = \left\{ x \in \mathbb{B} : x(t) \ge 0, \min_{\frac{b+\mu+1}{4} \le t \le \frac{3(\mu+b+1)}{4}} x(t) \ge \frac{1}{4} \|x\| \right\}$$

$$(Tx)(t) = \sum_{s=0}^{b+1} G(t,s) + \frac{g(x)}{(\mu + b + 1)^{\underline{\mu-1}}} t^{\underline{\mu-1}} \tag{4.1}$$

From the lemma, we get

$$\min_{\frac{b+\mu+1}{4} \le t \le \frac{3(b+\mu+1)}{4}} (Tx)(t) \ge \frac{1}{4} \sum_{s=0}^{b+1} G(\mu + b + 1, s) f(s + \mu - 1, x(s + \mu - 1)) + g(x) \frac{(b + \mu + 1)^{\mu-1}}{(\mu + b + 1)^{\mu-1}}.$$

$$\begin{aligned} \min_{\frac{b+\mu+1}{4} \le t \le \frac{3(b+\mu+1)}{4}} (Tx)(t) &\ge \frac{1}{4} \max_{t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}} \sum_{s=0}^{b+1} G(t,s) f(s + \mu - 1, x(s + \mu - 1)) + \|x\| \quad (4.2) \\ &= \tfrac{1}{4} \|Tx\|. \end{aligned}$$

hence $TK < K$. In the sequel, let

$$\Omega_{\lambda+\nu} = \{ x \in K : \|x\| < \lambda + \nu \}$$

$$\partial\Omega_{\lambda+\nu} = \{x \in K, \|x\| = \lambda + \nu\}.$$

Theorem 4.2. *Assume that there exists three different positive numbers q, r & R such that g satisfies conditions (H_1) and (H_3) at q, condition (H_2) at r and conditions (H_4) at R. Then FBVP (1.1) has atleast one positive solution $x_0 \in K$ satisfying* $\min\{q,r,R\} \leq \|x_0\| \leq \max\{q,r,R\}$.

Proof We know that $T : K \to K$ & T is completely continuous. Without loss of generality suppose that $q < r < R$. Note that for $x \in \partial\Omega_{r+q}$. We've $\|x\| = r+q$, so that condition (H_1) & (H_2) holds for all $x \in \partial\Omega_{r+q}$.

Then,

$$\begin{aligned}
(Tx)(t) &\leq \sum_{s=0}^{b+1} G(b+\mu+1,s) f(s+\mu-1, x(s+\mu-1)) + g(x). \\
&\leq Ar \sum_{s=0}^{b+1} G(b+\mu+1,s) + q \\
&\leq r+q. \\
&= \|x\|.
\end{aligned}$$

(ie) $\|T\| \leq \|x\|$ for $x \in \partial\Omega_{r+q}$

Note that for $y \in \partial\Omega_{R+q}$, and $\|x\| = R+q$, so condition (H_3) & (H_4) holds for all $x \in \partial\Omega_{R+q}$, since $[\frac{b-\mu-1}{2}] + (\mu+1) \in \left[\frac{\mu+b+1}{4}, \frac{3(\mu+b+1)}{4}\right]$. $(Tx)\Big([\frac{b-\mu-1}{2}] + (\mu+1)\Big)$

$$\begin{aligned}
&= \sum_{s=0}^{b+1} G\Big([\frac{b-\mu-1}{2}] + (\mu+1), s\Big) f\big(s+\mu-1, x(s+\mu-1)\big) + g(x)\frac{(\mu+b-1/2)^{\underline{\mu-1}}}{(\mu+b-1)^{\mu-1}} \\
&\geq \frac{1}{4} \sum_{s=\frac{b+\mu+1}{4}-\mu+2}^{\frac{3(b+\mu+1)}{4}-\mu+2} G\Big([\frac{b-\mu-1}{2}] + (\mu+1), s\Big) f\big(s+\mu-1, x(s+\mu-1)\big) + \frac{g(x)}{2^{(\mu-1)}} \\
&\geq \frac{BR}{4} \sum_{s=\frac{b+\mu+1}{4}-\mu+2}^{\frac{3(b+\mu+1)}{4}-\mu+2} G\Big([\frac{b-\mu-1}{2}] + (\mu+1), s\Big) f(s+\mu-1, x(s+\mu-1)) + q \\
&= R+q.
\end{aligned}$$

(ie) $\|Tx\| \geq \|x\|$ for $x \in K \cap \partial\Omega_{R+q}$

By lemma 4.1, T has atleast one fixed point say $x_0 \in \underline{\Omega}_{R+q}/\Omega_{r+q}$. This function $x_0(t)$ is a positive solution of (1.1) and satisfies $r+q \leq \|x_0\| \leq R+q$. This completes the proof.

Theorem 4.3. *Suppose that conditions $(H),(H_1),(H_4) \quad \& \quad (H_5)$ hold, $f>0$ for $t \in [\mu-2,\mu+b+1]_{\mu-2}$. Then the FBVP (1.1) has atleast two positive solutions x_1 & x_2 with $0<\|x_1\|<p<\|x_2\|$.*

Proof. Suppose that (H_5) holds. Since $f^0 < A$, one can find $\varepsilon > 0 (\varepsilon < A) \quad \& \quad 0 < r_0 < p$ such that

$$f(t,x) \leq (A-\varepsilon)x, \quad 0 \leq x \leq r_0, \quad t \in [\mu-2,\mu+b+1]_{\mathbb{N}_{\underline{\mu-2}}}.$$

Letting $r_1 \in (0, r_0+q)$, for $x \in \partial\Omega_{r_1+q}$, we get

$$\begin{aligned}
(Tx)(t) &= \sum_{s=0}^{b+1} G(t,s) f(s+\mu-1, x(s+\mu-1)) + \frac{g(x)t^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}} \\
&\leq \sum_{s=0}^{b+1} G(b+\mu+1,s)(A-\varepsilon)r_1 + g(x) \\
&< A r_1 \sum_{s=0}^{b+1} G(b+\mu+1,s) + g(x) \\
&= r_1 + q,
\end{aligned}$$

from that we get $\|Tx\| < \|x\|$ for $x \in \partial\Omega_{r_1+q}$.

On the other hand, since $f^\infty < A$, there exists $0 < \sigma A \quad \& \quad R_0 + q > 0$ such that $f(t,x) \leq \sigma_x, \quad x \geq R_0 + q, \quad t \in [\mu-2,\mu+b+1]_{\mathbb{N}_{\mu-2}}$. Let $M = \max_{(t,x)\in[\mu-1,\mu+b+1]\times[0,\mathbb{R}_0]} f(t,x)$, then $0 \leq f(t,x) \leq \sigma x + M, \quad o < x < \infty$. Let $R_1 + q > \max\{p, \frac{M}{A-\sigma}\}$, for $x \in \partial\Omega_{R_1+q}$ we have

$$\begin{aligned}
\|Tx\| \leq \sum_{s=0}^{b+1} G(b+\mu,s) f(s+\mu-1, x(s+\mu-1)) + g(x) &\leq (\sigma\|x\|) \sum_{s=0}^{b+1} G(b+\mu,s) + g(x) \\
&= (\sigma R_1 + M)\frac{1}{A} + q \\
&\leq R_1 + q.
\end{aligned}$$

Therefore $\|Tx\| \leq \|x\|$ for $x \in \partial\Omega_{R_1+q}$.

Finally, for any $x \in \partial\Omega_{p+q}$, since $\frac{1}{4}p \leq x(t) \leq p$, for $\frac{b+\mu}{4} \leq t \leq \frac{3(b+\mu)}{4}$, then we get $(Tx)\big([\frac{b-\mu-1}{2}] + (\mu+1)\big)$

$$
\begin{aligned}
&= \sum_{s=0}^{b+1} G\big([\frac{b-\mu-1}{2}] + (\mu+1), s\big) f(s+\mu-1, x(s+\mu-1)) + g(x)\Big(\frac{(\mu+b+1/2)^{\underline{\mu-1}}}{(\mu+b+1)^{\underline{\mu-1}}}\Big) \\
&> B.\frac{1}{4}p \sum_{s=0}^{b+1} G\big([\frac{b-\mu-1}{2}] + (\mu+1), s\big) + q \\
&= p + q = \|x\|,
\end{aligned}
$$

from that we obtain $\|Tx\| > \|x\|$ for $x \in K \cap \partial\Omega_{p+q}$. By the lemma, the proof is complete.

Corollary 4.4. *Suppose that conditions $(H), (H_1), \quad (H_4)$ and (H_5^*) hold. Then the FBVP (1.1) has at least two positive solutions.*

Theorem 4.5. *Suppose that conditions $(H), \quad (H_1) \quad (H_4)$ and (H_6) hold, $f > 0$ for $t \in [\mu - 2, b+\mu]_{\mathbb{N}_{\mu-2}}$. Then the FBVP (1.1) has at least two positive solutions x_1 and x_2 with $0 < \|x_1\| < p < \|x_2\|$.*

Proof. Suppose that (H_6) holds. Since $f_0 > B$, there exists $\varepsilon > 0$ and $0 < r_0 < p$ such that $f(t,x) \geq (B+\varepsilon)x, o \leq x \leq r_0, \qquad t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}$. Let $r_2 \in (0, r_0)$. Thus for $x \in \partial\Omega_{r_2}$, then we get

$$
\begin{aligned}
(Tx)\Big([\frac{b-\mu-1}{2}] + (\mu+1)\Big) &= \sum_{s=0}^{b+1} G\big([\frac{b-\mu-1}{2}] + (\mu+1), s\big)(B+\varepsilon)x + g(x)\frac{(\mu+b-1/2)^{\underline{\mu-1}}}{(\mu+b-1)^{\mu-1}} \\
&\geq (B+\varepsilon)\frac{1}{4}\|x\| \sum_{s=\frac{b+\mu+1}{4}-\mu+2}^{\frac{3(b+\mu+1)}{4}-\mu+2} G\big([\frac{b-\mu-1}{2}] + (\mu+1), s\big) + \frac{g(x)}{2^{\mu-1}} \\
&> B\frac{1}{4}\|x\| \sum_{s=\frac{b+\mu+1}{4}-\mu+2}^{\frac{3(b+\mu+1)}{4}-\mu+2} G\big([\frac{b-\mu-1}{2}] + (\mu+1), s\big) + q \\
&> r_2 + q,
\end{aligned}
$$

from that we see that $\|Tx\| > \|x\|$ for $x \in K \cap \partial\Omega_{r_2+q}$.

On the other hand, since $f_\infty > B$, there exists $\eta > 0$, and $R_0 + q > 0$ such that $f(t,x) \geq (B+\eta)x$, $x \geq R_0 + q$, $t \in [\mu-2, \mu+b+1]_{\mathbb{N}_{\mu-2}}$. Choose $R_2 + q > \max\{4R_2 + q, p\}$. For $x \in \partial\Omega_{R_2+q}$, since $x(t) \geq \frac{1}{4}\|x\| > R_0$ for $\frac{b+\mu}{4} \leq t \leq \frac{3(b+\mu)}{4}$, then get

$$
\begin{aligned}
(Tx)\Big([\frac{b-\mu-1}{2}]+(\mu+1)\Big) &= \sum_{s=0}^{b+1} G\big([\frac{b-\mu-1}{2}]+(\mu+1),s\big)(B+\eta)x+g(x)\frac{(\mu+b-1/2)^{\underline{\mu-1}}}{(\mu+b-1)^{\mu-1}} \\
&\geq (b+\eta)\frac{1}{4}\|x\| \sum_{s=\frac{b+\mu+1}{4}-\mu+2}^{\frac{3(b+\mu+1)}{4}-\mu+2} G\big([\frac{b-\mu-1}{2}]+(\mu+1),s\big)+\frac{g(x)}{2^{\mu-1}} \\
&> B\frac{1}{4}\|x\| \sum_{s=\frac{b+\mu+1}{4}-\mu+2}^{\frac{3(b+\mu+1)}{4}-\mu+2} G\big([\frac{b-\mu-1}{2}]+(\mu+1),s\big)+q \\
&> R_2+q.
\end{aligned}
$$

Therefore $\|Tx\| > \|x\|$ for $x \in K \cap \partial\Omega_{R_2+q}$. For any $x \in \partial\Omega_{p+q}$, from (H_2), we get $f(t,x) < Ap, \quad t \in [\mu-1, \mu+b+1]_{\mathbb{N}_{\mu-1}}$, then

$$
\begin{aligned}
(Tx)(t) &= \sum_{s=0}^{b+1} G(t,s) f\big(s+\mu-1, x(s+\mu-1)\big)+g(x) \\
&\leq \sum_{s=0}^{b+1} G(b+\mu,s)Ap+q \\
&= p+q.
\end{aligned}
$$

Hence $\|Tx\| \leq \|x\|$ for $x \in K \cap \partial\Omega_{p+q}$. Therefore, by the Lemma the proof is complete.

Corollary 4.6. *Assume that $(H) \quad (H_1) \quad$ and $\quad (H_3)$ hold, (H_6) is replaced by (H_6^*). Then the FBVP (1.1) has at least two positive solutions.*

Theorem 4.7. *Suppose that conditions (H) and (H_7) hold. Then (1.1) has atleast one positive solution.*

We remark here that the proof of the theorem is analogous to above theorem.

Corollary 4.8. *Suppose that conditions (H) and (H_7^*) hold. Then (1.1) has at least one positive solution.*

Theorem 4.9. *Suppose that conditions (H) and (H_8) hold. Then the FBVP (1.1) has at least one positive solution.*

We remark here that the proof of the theorem is analogous to above theorem.

Corollary 4.10. *Suppose that conditions (H) and* (H_8^*) *hold. Then (1.1) has at least one positive solution.*

5. Non-existence of positive solutions

In this section sufficient conditions for the non-existence of positive solutions are obtained.

$$(H_9) \quad g^o = \lim_{\|x\|\to 0} \sup g(x) \quad and \quad g^\infty = \lim_{\|x\|\to\infty} \sup g(x)$$

$$(H_{10}) \quad g_o = \lim_{\|x\|\to 0} \inf g(x) \quad and \quad g_\infty = \lim_{\|x\|\to\infty} \inf g(x).$$

Theorem 5.1. *Assume the conditions (H) and* (H_9) *hold. If* $g_o < \infty, \quad g_\infty < \infty$ *and* $cA^{-1}[(b+1)+A] < 1$ *then (1.1) has no positive solutions.*

Proof. Assume that (1.1) has positive solutions. Since $g_0 < \infty$ and $g_\infty < \infty$ then there exists a positive number such that $g(y_1) \le c\,\|x_1\|$. Since $\|x_1\| = \|Tx_1\|$ where $t \in [\mu-1, \mu+b+1]$

$$\begin{aligned}
\|x_1\| = \|Tx_1\| &= \sum_{s=0}^{b} G(t,s)h(s+\mu-1) + \frac{g(x_1)t^{\mu-1}}{(\mu+b+1)^{\mu-1}} \\
&\le \sum_{s=0}^{b} G(t,s)h(s+\mu-1) + g(x_1) \\
&\le \frac{1}{A}\sum_{s=0}^{b+1} g(x_0)\big(\frac{b+1}{A}+1\big) \\
&\le c\big(\frac{b+1}{A}+1\big)\,\|X_1\| \\
&\le \|x_1\|,
\end{aligned}$$

which is a contradiction. Hence the theorem is complete.

Theorem 5.2. *Assume the conditions* (H) *and* (H_{10}) *hold. If* $g^0 > 0, \quad g^\infty > 0$ *and* $lB^{-1}\big[(\mu+b+1)+2B\big] > 2$ *then (1.1) has no positive solutions.*

proof. Suppose assume $x_1(t)$ is a positive solution of (1.1). Since $g^0 > 0, \quad g^\infty > 0$. There exists a positive number such that

$$g(x_1) \ge l\,\|x_1\|.$$

Since $\|x_1\| = \|Tx_1\|$ where $t \in \left[\frac{\mu+b+1}{4}, \frac{3(\mu+b+1)}{4}\right]$

$$\begin{aligned}
\|x_1\| = \|Tx_1\| &\geq \sum_{s=[\frac{\mu+b+1}{4},-\mu+2]}^{[\frac{3(\mu+b+1)}{4}-\mu+2]} G\big([\frac{b-\mu-1}{2}]+(\mu+1)+s)h(s+\mu-1)+\frac{g(x_1)t^{\mu-1}}{(\mu+b+1)^{\mu-1}} \\
&\geq \frac{4}{B}\sum_{s=\left[\frac{\mu+b+1}{4},-\mu+2\right]}^{\left[\frac{3(\mu+b+1)}{4}-\mu+2\right]} g\big(x_1(s+\mu-1)\big)+g(x_1) \\
&\geq g(x_1)\big[\frac{\mu+b+1}{2B}+1\big] \\
&\geq \big[(\mu+b+1)+2B\big]\frac{\|x_1\|}{2B} \\
&\geq \|x_1\|,
\end{aligned}$$

which is a contradiction. Hence (1.1) has no positive solution.

Corollary 5.3. *Suppose the condition (H) and $g_0 < \infty, \quad g^\infty > 0$ hold then (1.1) has no positive solution.*

Corollary 5.4. *Assume the conditions (H) and $g^\infty < \infty, \quad g^0 > 0$ hold then (1.1) has no positive solution.*

6. Examples

Example 6.1. Suppose that $\mu = \frac{4}{3}, \quad b = 6$. Let $f(t,x(t)) = \frac{|cosecx(t)|}{250} + t$, and $g(x) = \frac{\|\sec x(t)\|}{80}$. Then (1.1) becomes,

$$\begin{cases} -\Delta^{4/3}x(t) = \frac{cosecx(t+\frac{1}{3})}{250} + (t+\frac{1}{3}), \\ x(\mu-2) = 0, \\ x(\mu+b+1) = \frac{\|\sec x(t)\|}{80}. \end{cases} \tag{6.1}$$

Here $\alpha = \frac{1}{250}, \quad \beta = \frac{1}{40}$,

$$\prod_{j=1}^{b}\left(\frac{\mu+j}{j}\right) < 12, \quad \prod_{j=1}^{b+1}\left(\frac{\mu+j}{j}\right) < 14.$$

Therefore, the inequality in theorem 3.1 is,

$$\alpha\prod_{j=1}^{b}(\frac{\mu+j}{j})+\alpha\prod_{j=1}^{b+1}(\frac{\mu+j}{j})+\beta<1.$$

Hence, from Theorem 3.1, (6.1) has a unique solution.

Example 6.2. Suppose that $\mu=\frac{5}{4}, \quad b=5, \quad \& \quad K=100$. Also suppose that $f(t,x)=\frac{tx^2}{15}\log 10$ and that $g(x)=\frac{\exp^x}{10}$. Thus, Problem (1.1) becomes

$$\begin{cases} -\Delta^{5/4}x(t)=\frac{(t+\frac{1}{4})(x+\frac{1}{4})^2}{35}\log 10 \\ x(\mu-2)=0 \\ x(\mu+b+1)=\exp^{(t+\frac{1}{4})(x+\frac{1}{4})} \end{cases} \tag{6.2}$$

and the banach space

$$\mathbb{B}=\left\{x\in\mathbb{R}^8:\|x\|<100,\right\}.$$

From the conditions (3.4) and (3.5), we get

$$\frac{K}{\frac{2\Gamma\mu+b+1}{\Gamma\mu+1\Gamma b+1}+1}=\frac{100}{\frac{2\Gamma(\frac{5}{4}+5+1)}{\Gamma\frac{5}{4}+1\Gamma 5+1}+1} \tag{6.3}$$

$$=52.2794$$

$$f(t,x)\leq\frac{(6.25)(25)}{15}log10$$

$$10.416667\leq 52.28$$

$$g(x)\leq\frac{e^{(x)}}{10}\approx 14.8413159\leq 52.28.$$

Hence, f and g satisfies the required conditions. From Theorm 3.2, Problem (6.2) has at least one solution. Take $\mu=\frac{13}{7}, \quad b=14$ after computation get the values $A\approx 0.01592088, \quad B\approx 0.138507089$. The following examples are derived using these values.

Example 6.3. Suppose that $f(t,x) = \frac{\exp^{2x}}{1200x^2}$, $\quad g(x) = \frac{\exp^x}{10\cos x}$. Then problem (1.1) becomes

$$\begin{cases} -\Delta^{\frac{13}{7}} x(t) = \frac{\exp^{2x(t+\frac{6}{7})}}{1200\left[x(t+\frac{6}{7})\right]^2}, \\ x(-\frac{1}{7}) = 0, \\ x(\frac{118}{7}) = \frac{\exp^x}{10\cos x}. \end{cases} \tag{6.4}$$

Let $p = 3 \quad \& \quad s = \frac{9}{5}$. Then

$$f(t,x) = \frac{\exp^{2x}}{1200 \times 9} = 0.0373546 < Ap,$$

$$g(x) = \frac{\exp^x}{10cosx} = 2.0113107 < p \quad \& \quad g(x) > s,$$

where $0 < s \le x < p$. Then $f^0 = f^\infty = \infty$. Hence, by the corollary, problem (6.4) has at least two solutions $x_1 \& x_2$ suchthat $0 \le \|x_1\| < 3 < \|x_2\|$.

Example 6.4. Suppose that $f(t,x) = \frac{10x^3\exp^{-x}}{9+\sin t}$, $\quad g(x) = \frac{x^2\cos 2x}{8}$. Then problem (1.1) has the form,

$$\begin{cases} -\Delta^{\frac{13}{7}} x(t) = \frac{10\left[x(t+\frac{6}{7})\right]^3 \exp^{-x(t+\frac{6}{7})}}{9+\sin(t+\frac{6}{7})}, \\ x(-\frac{1}{7}) = 0, \\ x(\frac{118}{7}) = \frac{x^2\cos 2x}{8}. \end{cases} \tag{6.5}$$

Let $p = 5 \quad \& \quad s = 3$. Then

$$f(t,x) = \frac{1250\exp^{-x}}{9+\sin t} = 0.9082515 > Bp$$

and $f^0 = f^\infty = 0$. Then, the functional $g(x) = \frac{25\cos 2x}{8} = 3.0775243 < p$ and $g(x) > s$. Hence, all conditions of corollary 4.1 are satisfied. Therefore the FBVP (6.5) has atleast two solutions $x_1 \quad \& \quad x_2$ such that $0 < \|x_1\| < 5 < \|x_2\|$.

Example 6.5. Suppose that $f(t,x) = \frac{1}{500}\left(25 + \frac{800}{1+x^3}\right) \quad g(x) = \frac{1+cost}{x^2} \quad \& \quad p = 4$, then Problem (1.1) becomes,

$$\begin{cases} -\Delta^{\frac{13}{7}} x(t) = \frac{1}{500}\left(25 + \frac{20[t+\frac{6}{7}]}{1+\left[x(t+\frac{6}{7})\right]}\right), \\ x(-\frac{1}{7}) = 0, \\ x(\frac{118}{7}) = \frac{1+\cos t}{x^2}. \end{cases} \tag{6.6}$$

We get $f^{\infty} = 0.05 < Ax \quad f_0 = 0.6842857 > Bx$. Then applying the conditions in theorem 5.1, the above problem (6.6) has atleast two solutions $x_1 \quad \& \quad x_2$ such that $0 \leq \|x_1\| < 4 < \|x_2\|$.

Example 6.6. Suppose that $f(t,x) = \exp^{x}, \quad g(x) = 50t + \frac{1}{1+x^2}, \quad \mu = \frac{13}{7}, \quad b = 14$ then problem (1.1) becomes

$$\begin{cases} -\Delta^{\frac{13}{7}} x(t) = \exp^{(t+\frac{6}{7})}, \\ x\left(\frac{-1}{7}\right) = 0, \\ x\left(\frac{118}{7}\right) = 50t + \frac{1}{1+x^2}. \end{cases} \tag{6.7}$$

$g_0 = 843.85714 < \infty \quad g_{\infty} = 842.85714 < \infty$

Let $c = \frac{1}{1000}$. Then the condition $cA^{-1}\left[(b+1) + A\right] = 0.9431589 < 1$

By the theorem, the problem (6.7) has no positive solution.

Example 6.7. suppose that $f(t,x) = logx, \quad g(x) = 4x^2 + \frac{70t}{25x}, \quad \frac{13}{7}$ and $b = 14$, then (1.1) becomes,

$$\begin{cases} -\Delta^{\frac{13}{7}} x(t) = logx\left(t + \frac{6}{7}\right), \\ x\left(-\frac{1}{7}\right) = 0, \\ x\left(\frac{118}{7}\right) = 4x^2 + \frac{70t}{25x}. \end{cases} \tag{6.8}$$

$g_0 = g_{\infty} = \infty > 0$. Let $l = \frac{1}{50}$. Then the condition $lB^{-1}\left[(\mu + b + 1) + 2B\right] = 2.5370613 > 2$.

By the theorem, the problem (6.8) has no positive solution.

Conflict of Interests

The authors declare that there is no conflict of interests.

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