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BERNARDI'S INTEGRAL OPERATORS OF JANOWSKI CLASS OF FUNCTIONS

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Abstract. Let $\mathcal{P}(A,B)$ denote the Janowski class of analytic functions subordinate to $\frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$. We determine C so that whenever $\frac{zf'(z)}{f(z)}$ is subordinate to $\frac{1+Cz}{1-Cz}$, $\frac{zF'(z)}{F(z)}$ is subordinate to $\frac{1+Az}{1+Bz}$ where F is the Bernardi's integral operator defined by $F(z) = I_\gamma(f)(z) = \frac{\gamma+1}{z^\gamma}\int\limits_0^z t^{\gamma-1}f(t)dt$. A similar result for $f'(z)$ is also dealt.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{N} = \{f \in A : f(0) = f'(0) - 1 = 0\}$. Let $\mathcal{P}$ denote the class of analytic functions defined on $\mathcal{U}$ satisfying $p(0) = 1$, $\Re\{p(z)\} > 0$. Let Ω designate the class of analytic functions ω in $\mathcal{U}$ such that $\omega(0) = 0$ and $|\omega(z)| < 1$. A function $p \in \mathcal{P}$ has the representation $p(z) = \frac{1+\omega(z)}{1-\omega(z)}$ where $\omega \in \Omega$. This representation for functions with positive real part in terms of analytic functions on $\mathcal{U}$ satisfying the conditions of Schwarz's lemma [6] motivated Janowski [2] to define the class $\mathcal{P}(A,B)$. Let $\mathcal{P}(A,B)$, where $-1 \leq B < A \leq 1$ denote the class of analytic functions p defined on $\mathcal{U}$ with the representation $p(z) = \frac{1+A\omega(z)}{1+B\omega(z)}$, $z \in \mathcal{U}$

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and $\omega \in \Omega$. Special choices for the parameters A and B yield the following:

$$\begin{aligned}
\mathcal{P}(1-2\beta,-1) &= \{p : \Re\{p(z)\} > \beta, z \in \mathcal{U}, 0 \leq \beta < 1\} \\
\mathcal{P}(1,-1+1/M) &= \{p : |p(z)-M| < M, z \in \mathcal{U}, M > 1/2\} \\
\mathcal{P}(\beta,0) &= \{p : |p(z)-1| < \beta, z \in \mathcal{U}, 0 < \beta \leq 1\} \\
\mathcal{P}(\beta,-\beta) &= \left\{p : \left|\tfrac{p(z)-1}{p(z)+1}\right| < \beta, z \in \mathcal{U}, 0 < \beta \leq 1\right\}
\end{aligned}$$

Several results concerning these classes may be found in Janowski [2], McCarty ([3], [4]) and Schaffer [8]. In recent years there have been several papers in literature on $\mathcal{P}(A,B)$ or on classes with different parametrization. Let $S^*(A,B) = \left\{f \in \mathcal{N} : \dfrac{zf'(z)}{f(z)} \in \mathcal{P}(A,B)\right\}$. Since $\mathcal{P}(A,B) \subset \mathcal{P}$, it follows that $S^*(A,B) \subset S^*$ where S^* is the class of starlike functions. Bernardi's integral operator [1] is defined as

$$F(z) = I_\gamma(f)(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt \tag{1.1}$$

with $\gamma \geq 0$ and $f \in \mathcal{N}$. Under the above transform the class of convex functions,the class of starlike functions and the class of close-to-convex functions are closed. In this paper we determine C so that if $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Cz}{1-Cz}$ then $\dfrac{zF'(z)}{F(z)} \prec \dfrac{1+Az}{1+Bz}$. We also discuss a similar problem for f'. We need the following lemma due to Miller and Mocanu [5] to prove our main results .

Lemma 1.1 Suppose that the function ω is regular in $\mathcal{U}$ with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathcal{U}$, we have

(1) $z_0\omega'(z_0) = k\omega(z_0)$ and
(2) $\Re\left\{1 + \dfrac{z_0\omega''(z_0)}{\omega'(z_0)}\right\} \geq k$ where k real and $k \geq 1$.

2. Main results

Theorem 1.1. *Let* $C = (A-B)\left\{\dfrac{(\gamma+2)+(A+B\gamma)}{(2+A+B)(\gamma+1+A+B\gamma)+(A-B)}\right\}$ *and* $F(z) = \dfrac{\gamma+1}{z^\gamma}\displaystyle\int_0^z t^{\gamma-1}f(t)dt$. *If* $\dfrac{zf'(z)}{f(z)} \prec \dfrac{1+Cz}{1-Cz}$ *,then* $\dfrac{zF'(z)}{F(z)} \prec \dfrac{1+Az}{1+Bz}$ *for all* $\gamma \geq 0$, $-1 \leq B < A \leq 1$.

Proof. Let a function ω be defined by

$$\omega(z) = \frac{\frac{zF'(z)}{F(z)} - 1}{A - B\frac{zF'(z)}{F(z)}} \tag{1.2}$$

for $-1 \leq B < A \leq 1$ and $\omega(z) \neq 1$ for $z \in \mathcal{U}$.

Now $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$ we need only to show that $|w(z)| < 1$ in $\mathcal{U}$.
From (2.1) we have $\frac{zF'(z)}{F(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}$
Logarithmic differentiation yields

$$1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)} = \frac{(A - B)z\omega'(z)}{\{1 + Aw(z)\}\{1 + Bw(z)\}}$$

Using the definition of Bernardi integral operator $(\gamma + 1)f(z) = zF'(z) + \gamma F(z)$. By taking logarithmic derivative we get

$$\frac{zf'(z)}{f(z)} = \frac{zF'(z)}{F(z)}\left\{\frac{1 + \frac{zF''(z)}{F'(z)} - \frac{zF'(z)}{F(z)}}{\frac{zF'(z)}{F(z)} + \gamma} + 1\right\}$$

$$= \frac{(A - B)z\omega'(z)}{\{1 + B\omega(z)\}\{(\gamma + 1) + (A + B\gamma)\omega(z)\}} + \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1$, then by Miller Mocanu's lemma we have $z_0\omega'(z_0) = k\omega(z_0), k \geq 1$. Thus we have

$$\left\{\frac{\frac{z_0f'(z_0)}{f(z_0)} - 1}{\frac{z_0f'(z_0)}{f(z_0)} + 1}\right\} = \frac{(A - B)z\omega(z_0)\{k + 1 + \gamma + (A + B\gamma)\omega(z_0)\}}{\{2 + (A + B\gamma)\omega(z_0)\}\{\gamma + 1 + \omega(z_0)(A + B\gamma)\} + (A - B)k\omega(z_0)}$$

and

$$\left|\frac{\frac{z_0f'(z_0)}{f(z_0)} - 1}{\frac{z_0f'(z_0)}{f(z_0)} + 1}\right| = \frac{(A - B)\left|k + 1 + \gamma + (A + B\gamma)e^{i\theta}\right|}{|\{2 + (A + Be^{i\theta}\}\{\gamma + 1 + e^{i\theta}(A + B\gamma)\} + (A - B)ke^{i\theta}|}$$

$$= \varphi(\cos\theta).$$

Now $\varphi(t)$ is a decreasing function of $t = \cos\theta$ in $[-1, 1]$ for $\gamma \geq 0$. Hence we get

$$\left|\frac{\frac{z_0 f'(z_0)}{f(z_0)} - 1}{\frac{z_0 f'(z_0)}{f(z_0)} + 1}\right| \geq (A - B)\left\{\frac{(\gamma + 2) + A + B\gamma}{(2 + A + B) + (\gamma + 1 + A + B\gamma) + (A - B)}\right\}$$
$$= C$$

a contradiction to the hypothesis that $\frac{zf'(z)}{f(z)} \prec \frac{1 + Cz}{1 - Cz}$.

Hence we have

$$|\omega(z)| = \left|\frac{\frac{zF'(z)}{F(z)} - 1}{A - B\frac{zF'(z)}{F(z)}}\right| < 1$$

Equivalently $\frac{zF'(z)}{F(z)} \prec \frac{1 + Az}{1 + Bz}$ which completes the proof. □

Corollary 1.2. *For the parametric values $A = \alpha, B = -\alpha$ we get Theorem 1 in [7] which reads:*

Let $\beta = \alpha\left\{\frac{2 + \alpha + \gamma(1 - \alpha)}{1 + 2\alpha + \gamma(1 - \alpha)}\right\}$ and $F(z) = \frac{\gamma + 1}{z^\gamma}\int_0^z t^{\gamma - 1} f(t)dt$.

If $f \in S^*(\beta)$ then $F \in S^*(\alpha)$ for all $\gamma \geq 0$, $0 < \alpha \leq 1$.

Now we define the class $R(A, B)$ to be the class of all $f' \in \mathcal{P}(A, B)$ and derive a similar result.

Theorem 1.3. *Let $C = \frac{(A - B)\{(\gamma + 1)(1 + B) + 1\}}{(\gamma + 1)(1 + B)\{2 + A + B\} + A - B}$ and $F(z) = \frac{\gamma + 1}{z^\gamma}\int_0^z t^{\gamma - 1} f(t)dt$.*

If $f'(z) \prec \frac{1 + Cz}{1 - Cz}$ then $F'(z) \prec \frac{1 + Az}{1 + Bz}$ for all $\gamma \geq 0$, $-1 \leq B < A \leq 1$.

Proof. Let us define a function

$$\omega(z) = \frac{F'(z) - 1}{A - BF'(z)}, \quad -1 \leq B < A \leq 1.$$

$$F'(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}$$

Differentiating (1.1) we get,

$$f'(z) = F'(z) + \frac{zF'(z)}{\gamma + 1}$$

$$\frac{f'(z)-1}{f'(z)+1} = \frac{F'(z)-1+\dfrac{zF''(z)}{\gamma+1}}{F'(z)+1+\dfrac{zF''(z)}{\gamma+1}}$$

$$= \frac{\dfrac{1+A\omega(z)}{1+B\omega(z)}-1+\dfrac{(A-B)z\omega'(z)}{(\gamma+1)\left\{1+B\omega(z)\right\}^2}}{\dfrac{1+A\omega(z)}{1+B\omega(z)}+1+\dfrac{(A-B)z\omega'(z)}{(\gamma+1)\left\{1+B\omega(z)\right\}^2}}$$

$$= \frac{(\gamma+1)(1+B\omega(z))\left\{(A-B)\omega(z)\right\}+(A-B)z\omega'(z)}{(\gamma+1)(1+B\omega(z))\left\{2+(A+B)\omega(z)\right\}+(A-B)z\omega'(z)}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that $\max\limits_{|z|\leq|z_0|}|\omega(z)| = |\omega(z_0)| = 1$.

Hence by lemma 1.1 we have $z_0\omega'(z_0) = k\omega'(z)$, $k \geq 1$. Thus we obtain

$$\frac{f'(z_0)-1}{f'(z_0)+1} = \frac{(\gamma+1)(1+B\omega(z_0))\left\{(A-B)\omega(z_0)\right\}+(A-B)k\omega(z_0)}{(\gamma+1)(1+B\omega(z_0))\left\{2+(A+B)\omega(z_0)\right\}+(A-B)k\omega(z_0)}$$

$$\left|\frac{f'(z_0)-1}{f'(z_0)+1}\right| = \frac{(A-B)|(\gamma+1)(1+Be^{i\theta}+k)|}{|(\gamma+1)(1+(1+Be^{i\theta})\left\{2+(A+B)e^{i\theta}\right\}+(A-B)ke^{i\theta}|}$$

$$=\varphi(\cos\theta).$$

Now $\varphi(t)$ is a decreasing function of $t=\cos\theta$ in $[-1,1]$ for $\gamma \geq 0$. Hence we get

$$\left|\frac{f'(z_0)-1}{f'(z_0)+1}\right| \geq \frac{(A-B)\left\{1+(\gamma+1)(B+1)\right\}}{(\gamma+1)(1+B)(2+A+B)+(A-B)}$$

$$=C$$

a contradiction to the hypothesis that

$$f'(z) \prec \frac{1+Cz}{1-Cz}$$

Hence we must have

$$|\omega(z)| < \left|\frac{F'(z)-1}{A-BF'(z)}\right| < 1 \text{ or } F'(z) \prec \frac{1+Az}{1+Bz}$$

which proves the theorem. □

Corollary 1.4. *For $A=\alpha, B=-\alpha$ we get Theorem 2 in [7] which reads:*

Let $\beta = \dfrac{2-\alpha+\gamma(1-\alpha)}{1+\gamma(1-\alpha)}$ *and* $F(z) = \dfrac{\gamma+1}{z^\gamma}\displaystyle\int_0^z t^{\gamma-1}f(t)dt.$

If $f \in R(\beta)$ then $F(z) \in R(\alpha)$ for all $C \geq 0$, $0 < \alpha \leq 1$.

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