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JENSEN'S INEQUALITY FOR HH-CONVEX FUNCTIONS AND RELATED RESULTS

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Abstract. In this paper we obtain Jensen's inequality for HH-convex functions. Also we get inequalities alike to Hermite-Hadamard inequality for HH-convex functions. Some examples are given.

Keywords: Jensen's inequality; HH-convex; Integral inequality.

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1. INTRODUCTION

Let μ be a positive measure on X such that $\mu(X) = 1$. If f is a real-valued function in $L^1(\mu)$, $a < f(x) < b$ for all $x \in X$ and φ is convex on (a,b), then

$$\varphi\left(\int_X f d\mu\right) \leq \int_X (\varphi.f) d\mu \tag{1}$$

The inequality (1) is known as Jensen's inequality [3],[4].

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Definition 1.1. A function $\varphi : (a,b) \longrightarrow (0,\infty)$, where $0 < a < b \leq \infty$, is called HH-convex (according to the harmonic mean) if the inequality

$$\varphi\left(\frac{1}{\frac{\lambda}{x}+\frac{1-\lambda}{y}}\right) \leq \frac{1}{\frac{\lambda}{\varphi(x)}+\frac{1-\lambda}{\varphi(y)}} \tag{2}$$

or

$$\varphi\left(\frac{xy}{\lambda y+(1-\lambda)x}\right) \leq \frac{\varphi(x)\varphi(y)}{\lambda\varphi(y)+(1-\lambda)\varphi(x)}$$

holds, where $a < x < b$, $a < y < b$, and $0 \leq \lambda \leq 1$.

In this paper we prove Jensen's inequality and alike to Hermite-Hadamard inequality for HH-convex functions. First we need the following theorem.

Theorem 1.2. *A function φ is HH-convex on (a,b) if for $0 < a < s < t < u < b$ the following inequality holds*

$$\frac{\frac{1}{\varphi(s)}-\frac{1}{\varphi(t)}}{\frac{1}{s}-\frac{1}{t}} \leq \frac{\frac{1}{\varphi(t)}-\frac{1}{\varphi(u)}}{\frac{1}{t}-\frac{1}{u}} \tag{3}$$

Proof. Let φ be HH-convex and $\lambda = \dfrac{s(u-t)}{t(u-s)}$, then

$$t = \frac{1}{\frac{\lambda}{s}+\frac{1-\lambda}{u}}.$$

Hence

$$\varphi(t) \leq \frac{1}{\frac{s(u-t)}{t(u-s)}\frac{1}{\varphi(s)}+\frac{u(t-s)}{t(u-s)}\frac{1}{\varphi(u)}}$$

It follows that

$$\frac{1}{\frac{s(u-t)}{t(u-s)}\frac{1}{\varphi(t)}+\frac{u(t-s)}{t(u-s)}\frac{1}{\varphi(t)}} \leq \frac{1}{\frac{s(u-t)}{t(u-s)}\frac{1}{\varphi(s)}+\frac{u(t-s)}{t(u-s)}\frac{1}{\varphi(u)}}$$

$$\Longrightarrow \quad \frac{s(u-t)}{t(u-s)}\frac{1}{\varphi(s)}+\frac{u(t-s)}{t(u-s)}\frac{1}{\varphi(u)} \leq \frac{s(u-t)}{t(u-s)}\frac{1}{\varphi(t)}+\frac{u(t-s)}{t(u-s)}\frac{1}{\varphi(t)}$$

$$\Longrightarrow \quad \frac{s(u-t)}{t(u-s)}\left(\frac{1}{\varphi(s)}-\frac{1}{\varphi(t)}\right) \leq \frac{u(t-s)}{t(u-s)}\left(\frac{1}{\varphi(t)}-\frac{1}{\varphi(u)}\right)$$

since $0 < s < t < u$, we obtain

$$\frac{\frac{1}{\varphi(s)}-\frac{1}{\varphi(t)}}{\frac{1}{s}-\frac{1}{t}} \leq \frac{\frac{1}{\varphi(t)}-\frac{1}{\varphi(u)}}{\frac{1}{t}-\frac{1}{u}}$$

Conversely let the inequality (3) holds, and $\lambda \in [0,1]$, $0 < a < x < y < b$, then $x \le \frac{1}{\frac{\lambda}{x}+\frac{1-\lambda}{y}} \le y$. By inequality (3) we have

$$\frac{\frac{1}{\varphi(x)} - \frac{1}{\varphi(\frac{xy}{\lambda y+(1-\lambda)x})}}{\frac{1}{x} - \frac{\lambda y+(1-\lambda)x}{xy}} \le \frac{\frac{1}{\varphi(\frac{xy}{\lambda y+(1-\lambda)x})} - \frac{1}{\varphi(y)}}{\frac{\lambda y+(1-\lambda)x}{xy} - \frac{1}{y}}$$

$$\Longrightarrow \quad \frac{\frac{1}{\varphi(x)} - \frac{1}{\varphi(\frac{xy}{\lambda y+(1-\lambda)x})}}{\frac{(1-\lambda)(y-x)}{xy}} \le \frac{\frac{1}{\varphi(\frac{xy}{\lambda y+(1-\lambda)x})} - \frac{1}{\varphi(y)}}{\frac{\lambda(y-x)}{xy}}$$

$$\Longrightarrow \quad \frac{\lambda}{\varphi(x)} - \frac{\lambda}{\varphi\left(\frac{xy}{\lambda y+(1-\lambda)x}\right)} \le \frac{1-\lambda}{\varphi\left(\frac{xy}{\lambda y+(1-\lambda)x}\right)} - \frac{1-\lambda}{\varphi(y)}$$

$$\Longrightarrow \quad \frac{1}{\varphi(\frac{xy}{\lambda y+(1-\lambda)x})} \ge \frac{1-\lambda}{\varphi(y)} + \frac{\lambda}{\varphi(x)}$$

$$\Longrightarrow \quad \varphi(\frac{1}{\frac{\lambda}{x}+\frac{1-\lambda}{y}}) \le \frac{1}{\frac{\lambda}{\varphi(x)}+\frac{1-\lambda}{\varphi(y)}}$$

Thus φ is HH-convex. □

By similar way to the convex functions we can prove if φ is HH-convex on (a,b), then φ is continuous on (a,b).

2. Main Results

Theorem 2.1. *Let μ be a positive measure on a σ-algebra $\mathfrak{m}$ in a set X, so that $\mu(X) = 1$. If f is a real function in $L^1(\mu)$, $0 < a < f(x) < b$ for all $x \in X$, and if φ is HH-convex on (a,b), then*

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) \le \frac{1}{\int_X \frac{d\mu}{\varphi\circ f}} \tag{4}$$

Proof. Put $t = \frac{1}{\int_X \frac{d\mu}{f}}$. Then $a < t < b$. Let

$$M = \sup_{a<s<t} \frac{\frac{1}{\varphi(s)} - \frac{1}{\varphi(t)}}{\frac{1}{s} - \frac{1}{t}}$$

Then M is no larger than any of the quotients on the right side of (3), for any $u \in (t,b)$. It follows that

$$\frac{\frac{1}{\varphi(s)}-\frac{1}{\varphi(t)}}{\frac{1}{s}-\frac{1}{t}} \le M \quad \text{or} \quad \frac{1}{\varphi(s)}-\frac{1}{\varphi(t)} \le M(\frac{1}{s}-\frac{1}{t})$$

Hence, for any $x \in X$, we have

$$\frac{1}{\varphi(f(x))}-\frac{1}{\varphi(t)} \le M(\frac{1}{f(x)}-\frac{1}{t})$$

since φ is continuous, $\varphi \circ f$ is measureable, and since $f \in L^1(\mu)$, $f(x) > a > 0$, so $\frac{1}{f} \in L^1(\mu)$. By integrating both sides with respect to measure μ, we obtain

$$\int_X \frac{d\mu}{\varphi\circ f}-\frac{1}{\varphi(t)} \le M\left(\int_X \frac{d\mu}{f}-\frac{1}{t}\right) \quad (\mu(X)=1)$$

Now set $t = \frac{1}{\int_X \frac{d\mu}{f}}$. It follows that

$$\int_X \frac{d\mu}{\varphi\circ f}-\frac{1}{\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right)} \le 0$$

or

$$\varphi\left(\frac{1}{\int_X \frac{d\mu}{f}}\right) \le \frac{1}{\int_X \frac{d\mu}{\varphi\circ f}}$$

□

Corollary 2.2. *Let $f:[a,b]\longrightarrow(0,\infty)$ $(b>a>0)$ be a continuous function and $\varphi: J\longrightarrow(0,\infty)$ be a HH-convex function on an interval J which includes the image of f. Then*

$$\varphi\left(\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}}\right) \le \frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2(\varphi\circ f)(x)}} \tag{5}$$

Proof. In theorem 2.1, put $X=[a,b]$ and $d\mu = \dfrac{dx}{x^2}$. □

In the following theorem we prove a version for the inverse of Corollay 2.2

Theorem 2.3. *Let $\varphi:(0,\infty)\longrightarrow(0,\infty)$ be a function such that the inequality (5) holds, for every positive real bounded measureable function f. Then φ is HH-convex.*

Proof. Let $\lambda \in [0,1]$, $c,d \in (0,\infty)$. Define

$$f(x)=\begin{cases} c & a\le x<\frac{ab}{\lambda a+(1-\lambda)b} \\ d & \frac{ab}{\lambda a+(1-\lambda)b}\le x\le b \end{cases}$$

we have

$$\begin{aligned}
\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)} &= \frac{ab}{b-a}\left[\int_a^{\frac{ab}{\lambda a+(1-\lambda)b}} \frac{dx}{cx^2}+\int_{\frac{ab}{\lambda a+(1-\lambda)b}}^b \frac{dx}{dx^2}\right] \\
&= \frac{ab}{b-a}\left[\frac{1}{c}\left(-\frac{\lambda a+(1-\lambda)b}{ab}+\frac{1}{a}\right)+\frac{1}{d}\left(-\frac{1}{b}+\frac{\lambda a+(1-\lambda)b}{ab}\right)\right] \\
&= \frac{\lambda}{c}+\frac{1-\lambda}{d}
\end{aligned}$$

Hence

$$\varphi\left(\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}}\right)=\varphi\left(\frac{1}{\frac{\lambda}{c}+\frac{1-\lambda}{d}}\right)=\varphi\left(\frac{cd}{\lambda d+(1-\lambda)c}\right) \tag{*}$$

On the other hand we have

$$\frac{ab}{b-a}\int_a^b \frac{dx}{x^2\varphi(f(x))}=\frac{ab}{b-a}\left[\int_a^{\frac{ab}{\lambda a+(1-\lambda)b}} \frac{dx}{x^2\varphi(c)}+\int_{\frac{ab}{\lambda a+(1-\lambda)b}}^b \frac{dx}{x^2\varphi(d)}\right]=\frac{\lambda}{\varphi(c)}+\frac{1-\lambda}{\varphi(d)}$$

so

$$\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2\varphi(f(x))}}=\frac{1}{\frac{\lambda}{\varphi(c)}+\frac{1-\lambda}{\varphi(d)}}=\frac{\varphi(c)\varphi(d)}{\lambda\varphi(d)+(1-\lambda)\varphi(c)} \tag{**}$$

Now the (*), (**) and (5) show that φ is HH-convex. □

Example 2.4. Let $X=\{x_1,x_2,\dots,x_n\}$, $\mu(\{x_i\})=\dfrac{1}{n}$ and $f(x_i)=a_i>0$. Then (4) becomes

$$\varphi\left(\frac{1}{\frac{1}{n}(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n})}\right)\le\frac{1}{\frac{1}{n}(\frac{1}{\varphi(a_1)}+\frac{1}{\varphi(a_2)}+\cdots+\frac{1}{\varphi(a_n)})}$$

or

$$\varphi\left(\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\right)\le\frac{n}{\frac{1}{\varphi(a_1)}+\frac{1}{\varphi(a_2)}+\cdots+\frac{1}{\varphi(a_n)}} \tag{6}$$

Now we inestigate this inequality for $\varphi(x)=x^{\gamma}$ and $\varphi(x)=e^{\frac{1}{x}}$

(i) $\varphi(x)=x^{\gamma}$ is HH-convex on $(0,\infty)$ for $0\leq\gamma\leq 1$, because $\frac{x^2\varphi'(x)}{\varphi^2(x)}=\gamma x^{1-\gamma}$ is increasing (see [1]). The inequality (6) implies that

$$\left(\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\right)^{\gamma}\leq\frac{n}{\frac{1}{a_1^{\gamma}}+\frac{1}{a_2^{\gamma}}+\cdots+\frac{1}{a_n^{\gamma}}}$$

put $\frac{1}{a_i}=p_i$ $(i=1,2,\ldots,n)$. It follows that

$$\left(\frac{n}{p_1+p_2+\cdots+p_n}\right)^{\gamma}\leq\frac{n}{p_1^{\gamma}+p_2^{\gamma}+\cdots+p_n^{\gamma}}$$

or

$$\frac{p_1+p_2+\cdots+p_n}{n}\geq\left(\frac{p_1^{\gamma}+p_2^{\gamma}+\cdots+p_n^{\gamma}}{n}\right)^{\frac{1}{\gamma}} \tag{7}$$

The number $C_{\gamma}=\left(\frac{p_1^{\gamma}+p_2^{\gamma}+\cdots+p_n^{\gamma}}{n}\right)^{\frac{1}{\gamma}}$ is termed the mean power of numbers $p_1,p_2,\ldots,p_n$ of order γ. Inequality (7) shows that for $0\leq\gamma\leq 1$, $C_{\gamma}\leq C_1$.

Now let $0\leq\gamma=\frac{\alpha}{\beta}\leq 1$, then (7) becomes,

$$\frac{p_1+p_2+\cdots+p_n}{n}\geq\left(\frac{p_1^{\frac{\alpha}{\beta}}+p_2^{\frac{\alpha}{\beta}}+\cdots+p_n^{\frac{\alpha}{\beta}}}{n}\right)^{\frac{\beta}{\alpha}}$$

Put $p_i^{\frac{1}{\beta}}=q_i$ $(i=1,2,\ldots,n)$. It follows that

$$\left(\frac{q_1^{\beta}+q_2^{\beta}+\cdots+q_n^{\beta}}{n}\right)^{\frac{1}{\beta}}\geq\left(\frac{q_1^{\alpha}+q_2^{\alpha}+\cdots+q_n^{\alpha}}{n}\right)^{\frac{1}{\alpha}}$$

So if $0\leq\alpha\leq\beta$, then $C_{\alpha}\leq C_{\beta}$. By HH-concavity of $\varphi(x)=x^{\gamma}$ on $(0,\infty)$ for $\gamma<0$, and $\gamma>1$ and similar way we can prove for $\alpha<0<\beta$ and $\alpha<\beta<0$ we have

$$C_{\alpha}<C_{\beta}$$

Thus the mean power of order γ monotonically increasing together with γ.

(ii) $\varphi(x)=e^{\frac{1}{x}}$ is HH-concave on $(0,\infty)$, because $\frac{x^2\varphi'(x)}{\varphi(x)}=-e^{-\frac{1}{x}}$ is decreasing (see [1]). The inequality (6) follows that

$$e^{\frac{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}{n}}\geq\frac{n}{\frac{1}{e^{\frac{1}{a_1}}}+\frac{1}{e^{\frac{1}{a_2}}}+\cdots+\frac{1}{e^{\frac{1}{a_n}}}}$$

put $\frac{1}{e^{\frac{1}{x_i}}} = p_i$, $i = 1, 2, \ldots, n$. Hence

$$\sqrt[n]{\frac{1}{p_1 p_2 \ldots p_n}} \geq \frac{n}{p_1 + p_2 + \cdots + p_n}$$

so

$$\sqrt[n]{p_1 p_2 \ldots p_n} \leq \frac{p_1 + p_2 + \cdots + p_n}{n}$$

That is, the geametric mean of positive numbers is not greater than the arithmetic mean of the same numbers.

In the following theorem we obtain inequalities alike to Hermite-Hadamard inequality for HH-convex functions.

Theorem 2.5. *Let $f : [a,b] \longrightarrow (0,\infty)$ $(b > a > 0)$ be a HH-convex function. Then the following inequalities hold:*

(i) $f(\frac{2ab}{a+b}) \leq \frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$

(ii) $f(\frac{2ab}{a+b}) \leq \frac{ab}{b-a}\int_a^b \frac{2f(x)f(\frac{abx}{x(a+b)-ab})}{f(x)+f(\frac{abx}{x(a+b)-ab})}\frac{dx}{x^2} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$

Proof. (i) The inequality (5) follows that

$$\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}} \geq f\left(\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^3}}\right) = f\left(\frac{2ab}{a+b}\right)$$

on the other hand by change of variable $x = \frac{ab}{ta+(1-t)b} = \frac{ab}{t(a-b)+b}$, $dx = \frac{ab(b-a)}{(t(a-b)+b)^2}dt$ and HH-convexity of f we get

$$\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)} = \int_0^1 \frac{dt}{f(\frac{ab}{ta+(1-t)b})} = \int_0^1 \frac{dt}{f(\frac{1}{\frac{t}{b}+\frac{1-t}{a}})} \geq \int_0^1 \frac{dt}{\frac{1}{\frac{t}{f(b)}+\frac{1-t}{f(a)}}}$$
$$= \int_0^1 \frac{t(f(a)-f(b))+f(b)}{f(a)f(b)}dt = \frac{f(a)+f(b)}{2f(a)f(b)}$$

so

$$\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$$

(ii) Since f is HH-convex, we have

$$f(\frac{2ab}{a+b}) = f(\frac{2}{\frac{1}{a}+\frac{1}{b}}) = f\left(\frac{2}{(\frac{t}{a}+\frac{1-t}{b})+(\frac{t}{b}+\frac{1-t}{a})}\right)$$
$$\leq \frac{2f(\frac{ab}{tb+(1-t)a})f(\frac{ab}{ta+(1-t)b})}{f(\frac{ab}{tb+(1-t)a})+f(\frac{ab}{ta+(1-t)b})}$$

By integrating both sides and HH-convexity f we obtain

$$f(\frac{2ab}{a+b}) \leq \int_0^1 \frac{2f(\frac{ab}{tb+(1-t)a})f(\frac{ab}{ta+(1-t)b})}{f(\frac{ab}{tb+(1-t)a})+f(\frac{ab}{ta+(1-t)b})}$$
$$= \int_0^1 \frac{2}{\frac{1}{f(\frac{ab}{ta+(1-t)b})}+\frac{1}{f(\frac{ab}{tb+(1-t)a})}}dt$$
$$\leq 2\int_0^1 \frac{dt}{\frac{1}{\frac{f(a)f(b)}{tf(a)+(1-t)f(b)}}+\frac{1}{\frac{f(a)f(b)}{tf(b)+(1-t)f(a)}}}$$
$$= 2\int_0^1 \frac{f(a)f(b)}{f(a)+f(b)}dt = \frac{2f(a)f(b)}{f(a)+f(b)}$$

On the other hand by change of variable

$$\frac{ab}{ta+(1-t)b} = \frac{ab}{t(a-b)+b} = x, \quad \frac{ab(b-a)}{(t(a-b)+b)^2}dt = dx$$

we see that

$$\int_0^1 \frac{2f(\frac{ab}{tb+(1-t)a})f(\frac{ab}{ta+(1-t)b})}{f(\frac{ab}{tb+(1-t)a})+f(\frac{ab}{ta+(1-t)b})}dt = \frac{ab}{b-a}\int_a^b \frac{2f(x)f(\frac{abx}{x(a+b)-ab})}{f(x)+f(\frac{abx}{x(a+b)-ab})}\frac{dx}{x^2}$$

The proof is complete. □

Corollary 2.6. *Let $f:[a,b]\longrightarrow(0,\infty)$ $(b>a>0)$ be a HH-convex function. Then the following inequalities hold:*

$$f(\frac{2ab}{a+b}) \leq \frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}}$$
$$\leq \frac{ab}{b-a}\int_a^b \frac{2f(x)f(\frac{abx}{x(a+b)-ab})}{f(x)+f(\frac{abx}{x(a+b)-ab})}\frac{dx}{x^2} \leq \frac{2f(a)f(b)}{f(a)+f(b)}$$

Proof. By theorem 2.5 it is sufficient that prove the middle part.

By change of variable $x = \frac{abt}{t(a+b)-ab}$, we see that

$$\int_a^b \frac{dx}{x^2 f(x)} = \int_a^b \frac{dx}{x^2 f(\frac{abx}{x(a+b)-ab})}.$$

Hence

$$\begin{aligned}\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)} &= \frac{ab}{2(b-a)}[\int_a^b \frac{dx}{x^2 f(x)} + \int_a^b \frac{dx}{x^2 f(x)}] \\ &= \frac{ab}{2(b-a)}[\int_a^b \frac{dx}{x^2 f(x)} + \int_a^b \frac{dx}{x^2 f(\frac{abx}{x(a+b)-ab})}] \\ &= \frac{ab}{2(b-a)}\int_a^b \left(\frac{1}{f(x)} + \frac{1}{f(\frac{abx}{x(a+b)-ab})}\right)\frac{dx}{x^2}.\end{aligned}$$

Put $h(x) = \frac{1}{f(x)} + \frac{1}{f(\frac{abx}{x(a+b)-ab})}$, $X = [a,b]$ and $d\mu = \frac{dx}{x^2}$. Thus

$$\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)} = \frac{1}{2}\int_X h d\mu.$$

On the other hand by these notations we see that

$$\frac{ab}{b-a}\int_a^b \frac{2f(x)f(\frac{abx}{x(a+b)-ab})}{f(x)+f(\frac{abx}{x(a+b)-ab})}\frac{dx}{x^2} = \frac{2ab}{b-a}\int_a^b \frac{1}{\frac{1}{f(\frac{abx}{x(a+b)-ab})}+\frac{1}{f(x)}}\frac{dx}{x^2} = 2\int_X \frac{d\mu}{h}.$$

By Holder's inequality we have

$$\begin{aligned}1 &= \int_X d\mu = \int_X \sqrt{h}\frac{1}{\sqrt{h}}d\mu \le \left(\int_X (\sqrt{h})^2 d\mu\right)^{\frac{1}{2}}\left(\int_X (\frac{1}{\sqrt{h}})^2 d\mu\right)^{\frac{1}{2}} \\ &= (\int_X h d\mu)^{\frac{1}{2}}\left(\int_X \frac{d\mu}{h}\right)^{\frac{1}{2}},\end{aligned}$$

so

$$1 \le \int_X h d\mu \int_X \frac{d\mu}{h} \quad \text{or} \quad \frac{1}{\frac{1}{2}\int_X h d\mu} \le 2\int_X \frac{d\mu}{h}.$$

Thus

$$\frac{1}{\frac{ab}{b-a}\int_a^b \frac{dx}{x^2 f(x)}} \le \frac{ab}{b-a}\int_a^b \frac{2f(x)f(\frac{abx}{x(a+b)-ab})}{f(x)+f(\frac{abx}{x(a+b)-ab})}\frac{dx}{x^2}.$$

□

Conflict of Interests

The author declares that there is no conflict of interests.

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