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OPTIMAL QUADRATIC BOUNDS FOR THE LARGEST EIGENVALUE OF CORRELATION MATRICES UNDER RESTRICTED INFORMATION

WERMER HÜRLIMANN

Swiss Mathematical Society, University of Fribourg, CH-1700 Fribourg, Switzerland

Abstract: A previous method used for bounding the largest eigenvalue of a 3x3 correlation matrix is extended to higher dimensions. Optimal quadratic bounds by given determinant and traces of the correlation matrix powers are derived for a class of correlation matrices under specific restricted information. Conditions under which these bounds are more stringent than the bounds by Wolkowicz and Styan are determined.

Keywords: correlation matrix; positive semi-definite; characteristic polynomial; largest eigenvalue; quadratic polynomial; inequalities.

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1. Introduction

The topic of bounds on eigenvalues of symmetric matrices has a long history and does not seem to provide ultimate and definitive answers. In some situations optimal bounds have been found. For the set of complex matrices $A=(a_{ij}), 1\le i, j\le n$, with real eigenvalues, Wolkowicz and Styan [14] obtained optimal bounds by given $Tr(A)$ and $Tr(A^2)$. For the same set of

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matrices with positive eigenvalues, Merikoski and Virtanen [9], [10], have studied optimal bounds by given $Tr(A)$ and $\det(A)$. Zhan [15] obtains the optimal bounds for the smallest and largest eigenvalues of real symmetric matrices whose entries belong to a fixed finite interval. However, when restricted to the set of real nxn correlation matrices, these bounds collapse to useless or trivial bounds, as shown in Section 3. Moreover, for correlation matrices $R=(r_{ij}), 1\le i,j\le n$, with unit diagonal elements, one has always $Tr(R)=n$, and the separate knowledge of $Tr(R^2)$ and $\det(R)$ does not provide full information. It is therefore justified to search for further possibly optimal bounds on eigenvalues of correlation matrices.
The present study offers an extension of the method in [5] for bounding the largest eigenvalue of 3x3 correlation matrices. In Theorem 2.1 we derive some new optimal bounds for a class of nxn correlation matrices with restricted information. In Section 3 they are compared to the optimal bounds by Wolkowicz and Styan [14] and found to be more stringent in some specific cases. Section 4 illustrates with numerical comparisons.

2. Quadratic eigenvalue bounds under certain restricted information

Starting point is a real nxn matrix $A=(a_{ij}), 1\le i,j\le n, n\ge 4$, with characteristic polynomial

$$q_n(\lambda)=\sum_{k=0}^{n}(-1)^k e_k \lambda^{n-k}, \qquad (2.1)$$

where $e_n=d=\det(A)$ is the determinant, and the e_k's satisfy Newton's identities

$$ke_k=\sum_{j=1}^{k}(-1)^{j-1}s_j e_{k-j}, \quad k=1,\dots,n-1, \quad e_0=1, \qquad (2.2)$$

where $s_j=Tr(A^j),\ j=1,2,\dots,n-1,$ are the traces of the matrix powers. Each zero λ of this polynomial is called an eigenvalue (abbreviated EV). Restricting the attention to correlation matrices $R=(r_{ij}), 1\le i,j\le n$, with unit diagonal elements, one has $e_1=s_1=Tr(R)=n$. Expressed in terms of the variable $z=\lambda-1$ the polynomial simplifies to the "depressed characteristic polynomial"

$$\begin{aligned} &p_n(z)=q_n(z+1)=z^n-Pz^{n-2}-2Qz^{n-2}+\sum_{k=0}^{n-4}C_k z^{n-4-k}, \\ &P=\tfrac{1}{2}(s_2-n), \quad Q=\tfrac{1}{6}(2n-3s_2+s_3). \end{aligned} \tag{2.3}$$

The coefficients P,Q, and C_k's are uniquely determined by the recursive equations

$$\begin{aligned} &e_2=\tfrac{1}{2}n(n-1)-P, \quad e_3=\tfrac{1}{6}n(n-1)(n-2)-(n-2)P+2Q, \\ &e_{4+s}=\binom{n}{4+s}-\binom{n-2}{2+s}P+\binom{n-3}{1+s}2Q+\sum_{k=0}^{s}(-1)^k\binom{n-4-k}{s-k}C_k, \\ &s=0,\dots,n-k. \end{aligned} \tag{2.4}$$

The system (2.4) is obtained through binomial expansion of the polynomial $p_n(\lambda-1)$ and comparison with (2.1). In particular, solving (2.4) for P,Q using (2.2) yields the values in (2.3). The set of correlation matrices is uniquely determined by the set of $\frac{1}{2}n(n-1)$ upper diagonal elements $r=(r_{ij})\in[-1,1]^{\frac{1}{2}n(n-1)}$, $1\le i<j\le n$, the n-dimensional elliptope, denoted by E_n. It is known that, up to permutations, one has $r\in E_n$ if, and only if, the following representation holds (see [4], Theorem 3.1):

$$\begin{aligned} &r_{in}=x_{in}\in[-1,1], \quad i=1,\dots,n-1, \quad n\ge 2, \\ &r_{in-1}=x_{in}x_{n-1n}+x_{in-1}y_{in-1,n}, \quad x_{in-1}\in[-1,1], \quad i=1,\dots,n-2, \quad n\ge 3 \\ &r_{in-k}=x_{in}x_{n-kn}+\sum_{j=2}^{k}x_{in-j+1}x_{n-kn-j+1}\prod_{\ell=n-j+2}^{n}y_{in-k,\ell}+x_{in-k}\prod_{\ell=n-k+1}^{n}y_{in-k,\ell}, \\ &x_{in-k}\in[-1,1], \quad i=1,\dots,n-k-1, \quad k=2,\dots,n-2, \quad n\ge 4, \end{aligned} \tag{2.5}$$

with the abbreviation $y_{ij,\ell}=y_{ij,\ell}(x_{i\ell},x_{j\ell})=\sqrt{(1-x_{i\ell}^2)(1-x_{j\ell}^2)}$. The formulas (2.5) are useful for algorithmic generation of arbitrary correlation matrices, and some applications of them have been discussed in [6], [7]. On the other hand, through straightforward calculation one obtains

$$s_2 = Tr(R^2) = n + 2 \cdot \sum_{i<j} r_{ij}^2, \quad s_3 = Tr(R^3) = n + 6 \cdot \sum_{i<j} r_{ij}^2 + 6 \cdot \sum_{i<j<k} r_{ij} r_{ik} r_{jk}, \tag{2.6}$$

Comparing (2.3) with (2.6) one sees that the first two coefficients of the depressed characteristic polynomial are given by

$$P = \sum_{i<j} r_{ij}^2, \quad Q = \sum_{i<j<k} r_{ij} r_{ik} r_{jk}. \tag{2.7}$$

We ask for possibly optimal bounds for the largest EV (abbreviated LEV) of a nxn correlation matrix by given $(d, s_1, s_2, ..., s_{n-1})$, or equivalently by given $(P, Q, C_0, ..., C_{n-4})$, which is the maximum available information. The following sharp inequality, which characterizes the semi-definite property of a correlation matrix, plays a crucial role in the analysis. For all $r \in E_n$ the determinant of the correlation matrix is non-negative, i.e., one has

$$d = \det(R) = (-1)^n p_n(-1) = 1 - P + 2Q + \sum_{k=0}^{n-4} (-1)^k C_k \geq 0. \tag{2.8}$$

In the following, we assume that $r = (r_{ij}) \neq (0)$ does not generate the identity correlation matrix, which implies that the LEV, denoted by λ_1, satisfies the condition $\lambda_1 > 1$. Therefore, we search for a positive zero $z = \lambda - 1 > 0$ of the depressed characteristic polynomial (2.3). Generalizing the method in [5] to arbitrary dimensions $n \geq 4$ (assumed throughout), we make use of the determinant identity (2.8) in two different ways. First, insert the relationship $P = 1 + 2Q + \sum_{k=0}^{n-4} (-1)^k C_k - d$ into (2.3) to see that a positive zero of $p_n(z) = 0$ must satisfy the identity

$$(z^2 - z - 2Q)z^{n-3}(z+1) = -(d - \sum_{k=0}^{n-4}(-1)^k C_k)z^{n-2} - \sum_{k=0}^{n-4} C_k z^{n-4-k}. \qquad (2.9)$$

Similarly, insert the relationship $2Q = d - (1-P) - \sum_{k=0}^{n-4}(-1)^k C_k$ into (2.3) to see that such a zero must also satisfy the identity

$$(z^2 - z - (P-1))z^{n-3}(z+1) = (d - \sum_{k=0}^{n-4}(-1)^k C_k)z^{n-3} - \sum_{k=0}^{n-4} C_k z^{n-4-k}. \qquad (2.10)$$

In a first step, we determine conditions under which the left-hand sides of (2.9)-(2.10) satisfy the following two quadratic inequalities (Equations (2.5) in [5])

$$\text{(I)} \quad z^2 - z - 2Q \le 0, \quad \text{(II)} \quad z^2 - z - (P-1) \ge 0. \qquad (2.11)$$

Lemma 2.1. *Assume that* $d - \sum_{k=0}^{n-4}(-1)^k C_k = 1 - P + 2Q \ge 0$. *If* $C_k \ge 0$, $k = 0,...,n-4$, *then the inequality* (I) *is fulfilled. If* $C_k \le 0$, $k = 0,...,n-4$, *then the inequality* (II) *is fulfilled.*

Proof. Under the stated conditions, a positive zero $z > 0$ of $p_n(z) = 0$ necessarily satisfies the inequality (I) of (2.11) in virtue of the identity (2.9). Similarly, the inequality (II) of (2.11) is satisfied under the given conditions in virtue of the identity (2.10). ◊

Clearly, the inequalities (I) and (II) are equivalent with the following quadratic EV inequalities

$$\text{(I)} \quad \lambda^2 - 3\lambda + 2(1-Q) \le 0, \quad \text{(II)} \quad \lambda^2 - 3\lambda - (P-3) \ge 0. \qquad (2.12)$$

From the elementary analytical properties of quadratic polynomials, one knows that (2.12) implies the inequalities

$$\tfrac{1}{2}(3 - \sqrt{1+8Q}) \le \lambda \le \tfrac{1}{2}(3 + \sqrt{1+8Q}) \quad \text{provided} \quad 1 + 8Q \ge 0 \quad \text{for (I)}, \qquad (2.13)$$

and

$$\lambda \leq \tfrac{1}{2}(3-\sqrt{4P-3}) \text{, or } \lambda \geq \tfrac{1}{2}(3+\sqrt{4P-3}) \quad \text{provided} \quad 4P-3 \geq 0 \quad \text{for (II).} \tag{2.14}$$

These inequalities lead to the following optimal bounds under specific restricted information.

Theorem 2.1. (*Optimal quadratic bounds for the LEV of non-trivial correlation matrices*) *Assume that* $d - \sum_{k=0}^{n-4} (-1)^k C_k = 1 - P + 2Q \geq 0$. *Two cases are possible.*

Case 1: *optimal quadratic upper bound*

If $C_k \geq 0, k = 0,...,n-4$, *and* $1+8Q \geq 0$, *then one has the best upper bound*

$$\lambda_1 \leq \tfrac{1}{2}(3+\sqrt{1+8Q}). \tag{2.15}$$

Case 2: *optimal quadratic lower bound*

If $C_k \leq 0, k = 0,...,n-4$, *and* $4P-3 \geq 0$, *then one has the best lower bound*

$$\lambda_1 \geq \tfrac{1}{2}(3+\sqrt{4P-3}). \tag{2.16}$$

Moreover, the upper and lower bound are attained if, and only if, one has $C_k = 0, k = 0,...,n-4, d = 0$ *, and the largest eigenvalue is given by* $\lambda_1 = \tfrac{1}{2}(3+\sqrt{1+8Q}) = \tfrac{1}{2}(3+\sqrt{4P-3})$.

Proof. Since the correlation matrix is not the identity matrix $z = \lambda_1 - 1$ is a positive zero of the depressed characteristic polynomial (2.3). The assumptions of Lemma 2.1 are fulfilled, hence (I), (II) in (2.12) are satisfied for $\lambda = \lambda_1$. The bounds (2.15)-(2.16) follow from the bounds (2.13)-(2.14) under the stated conditions. In both cases, equality is attained if, and only if, one has $C_k = 0, k = 0,...,n-4, d = 0$, or equivalently by (2.8) one has $C_k = 0, k = 0,...,n-4$, and $1+8Q = 4P-3 \geq 0$. In this situation, the upper and the lower bounds coincide with the LEV. ◊

It is important to note that the obtained optimal quadratic bounds are only valid upon restricted information on correlation matrices in terms of the coefficients $(P, Q, C_0, ..., C_{n-4})$ of the depressed characteristic polynomial (2.3). Besides the common inequality $1+8Q \geq 4P-3$ the following information is needed. For the upper bound the inequality $1+8Q \geq 0$ and the non-negativity of the coefficients C_k, $k = 0,...,n-4$, are required, and for the lower bound the inequality $4P-3 \geq 0$ and the non-positivity of C_k, $k = 0,...,n-4$, must hold.

3. Analytical comparison results

It is interesting to compare the new optimal bounds with related results, which deal, however, all with larger sets of matrices. For the set of complex matrices $A = (a_{ij}), 1 \leq i < j \leq n$, of arbitrary dimensions with real eigenvalues, Wolkowicz and Styan [14] obtained optimal bounds by given $Tr(A)$ and $Tr(A^2)$, called hereafter *WS bounds*. Although this is quite restrictive incomplete information for arbitrary nxn correlation matrices, a detailed comparison with the WS bounds is instructive and provided below. In contrast to this, for the same set of matrices with positive eigenvalues, the bounds by Merikoski and Virtanen [9] by given $Tr(A)$ and $\det(A)$, hereafter called *MV bounds*, are not optimal, that is not attained for a specific matrix with the given properties. Even more, the best possible bounds cannot in general be expressed algebraically, as shown in Merikoski and Virtanen [10]. More recently, Zhan [15] obtains the optimal bounds for the smallest and largest eigenvalues of real symmetric matrices whose entries belong to a fixed finite interval. However, when restricted to the set of real nxn correlation matrices, the Zhan bounds collapse to useless or trivial bounds (see Zhan [15], Corollary 2 (ii), p.854, Theorem 5 (ii), pp. 854-855).

Restricting the attention to correlation matrices, the WS bounds depend on the squared Frobenius norm $Tr(R^2) = \|R\|_F^2$ only, or equivalently on P. Since the new bounds of Theorem 2.1

depend on (P,Q) and some additional restrictions, it is interesting to analyze the conditions under which the one bounds are more stringent than the others. For nxn correlation matrices the WS bounds are (see Wolkowicz and Styan [14], equation (2.3)):

$$1+\tfrac{1}{n-1}\cdot\sqrt{2\tfrac{n-1}{n}P}\le\lambda_1\le 1+\sqrt{2\tfrac{n-1}{n}P}\,. \tag{3.1}$$

When refereeing to the bounds in (3.1), the notation $\lambda_1^{m,WS}$ and $\lambda_1^{M,WS}$ is used for the lower respectively upper bound. Similarly, the notations λ_1^m and λ_1^M are used for the lower and upper bounds in (2.15) and (2.16).

Theorem 3.1. *Under the required restricted information, the WS bounds compare with the quadratic bounds of Theorem 2.1 as follows:*

Upper bound:

Case (a): $P<\tfrac{1}{4}(\tfrac{n}{n-1})\cdot(1+4Q) \quad\Rightarrow\quad \lambda_1^{M,WS}<\lambda_1^M$

With $P^+=\tfrac{1}{4}(\tfrac{n}{n-1})\cdot(1+4Q+\sqrt{1+8Q})$, one has

Case (b): $\tfrac{1}{4}(\tfrac{n}{n-1})\cdot(1+4Q)\le P<P^+ \quad\Rightarrow\quad \lambda_1^{M,WS}<\lambda_1^M$

Case (c): $P\ge P^+ \quad\Rightarrow\quad \lambda_1^{M,WS}\ge\lambda_1^M$

Lower bound: $\lambda_1^{m,WS}\le\lambda_1^m$

Proof. A case by case analysis based on Theorem 2.1 and (3.1) is required. For the upper bound one has $\lambda_1^{M,WS}\ge\lambda_1^M$ if, and only if, the inequality $\sqrt{1+8Q}\le 4(\tfrac{n-1}{n})P-\{1+4Q\}$ is fulfilled. In Case (a) this cannot be fulfilled, hence $\lambda_1^{M,WS}<\lambda_1^M$. Otherwise, the preceding inequality holds if, and only if, one has

$$4(\tfrac{n-1}{n})^2 P^2 - 2(\tfrac{n-1}{n})\{1+4Q\}P + 4Q^2 \geq 0 .$$

This quadratic polynomial in P has the positive reduced discriminant $\Delta = (\tfrac{n-1}{n})^2 \cdot (1+8Q)$. Therefore, the inequality holds exactly if $P \geq P^+$, where P^+ is the non-negative zero of the quadratic polynomial, which is Case (c). The remaining situation is Case (b). For the lower bound $\lambda_1^{m,WS} \leq \lambda_1^m$ holds if, and only if $4\{\tfrac{1}{2}n(n-1)-1\}P - n(n-1) + n(n-1)\sqrt{4P-3} \geq 0$. This inequality is always fulfilled because $4P-3 \geq 0$ and $n \geq 4$ by assumption. ◊

According to Theorem 3.1 the new upper bound is more stringent than the WS upper bound in Case (c). The new lower bound is always more stringent than the WS lower bound. Of course these statements hold only under the conditions of Theorem 2.1. In particular, the two new bounds can hold simultaneously only if they are equal, and in this situation they coincide with the LEV. Similar comparison statements can be made for other LEV bounds. For example, one can compare Theorem 2.1 with the MV bounds in Merikoski and Virtanen [9], Theorems 1, 2, 3, or with Theorem 2.1 in Huang and Wang [2]. It might also be useful to compare the new lower bounds with the classical lower bound $\lambda_1 \geq 1 + \tfrac{2}{n}\sum_{i<j} r_{ij}$ and its improvement in Walker and Van Mieghem [13], or with the lower bound by Sharma et al. [11], Theorem 3.1. We note that these few further possibilities do certainly not exhaust the list of possible LEV bounds.

4. Some numerical comparisons

For an easy algorithmic generation and a more precise analysis of the conditions in Theorem 2.1, one might parameterize elements $r \in E_n$ as univariate functions of $x = r_{12}$ and use the representation (2.5) such that

$$\begin{aligned} &r(x) = (x, r_{ij}, 1 \leq i < j \leq n, j > 2), \quad x \in [\gamma^-, \gamma^+], \\ &\gamma^{\pm} = x_{1n}x_{2n} + \sum_{j=2}^{n-2} x_{1n-j+1}x_{2n-j+1} \prod_{\ell=n-j+2}^{n} y_{12,\ell} \pm \prod_{\ell=3+1}^{n} y_{12,\ell} . \end{aligned} \tag{4.1}$$

The main coefficients (P,Q) in (2.7) are parameterized as follows:

$$P(x)=x^2+\sum_{1\le i<j\le n,\, j>2} r_{ij}^2,\quad Q(x)=x\cdot\sum_{k>2} r_{1k}r_{2k}+\sum_{i<j<k,(i,j)\neq(1,2)} r_{ij}r_{ik}r_{jk}. \tag{4.2}$$

The coefficients $C_k,\, k=0,...,n-4$ can also be parameterized as univariate functions $C_k(x)$. In Table 4.1 below the calculated largest eigenvalues are also expressed as function of $x=r_{12}$.

To illustrate numerically, we focus on the special case $n=4$. The coefficients of the depressed quartic in (2.3) are given by

$$\begin{aligned}&P=r_{12}^2+r_{13}^2+r_{14}^2+r_{23}^2+r_{24}^2+r_{34}^2,\quad Q=r_{12}r_{13}r_{23}+r_{12}r_{14}r_{24}+r_{13}r_{14}r_{34}+r_{23}r_{24}r_{34},\\ &C_0=r_{12}^2r_{34}^2+r_{13}^2r_{24}^2+r_{14}^2r_{23}^2-2(r_{12}r_{13}r_{24}r_{34}+r_{12}r_{14}r_{23}r_{34}+r_{13}r_{14}r_{23}r_{24}).\end{aligned} \tag{4.3}$$

For (P,Q) this is (2.7). For the remaining coefficient note that $C_0=d-1+P-2Q$ by (2.8) and that the determinant is given by (use Proposition 2.1 and 2.2 in [3])

$$\begin{aligned}d&=(1-r_{13}^2)(1-r_{23}^2)(1-r_{34}^2)-(1-r_{13}^2)(r_{24}-r_{23}r_{34})^2-(1-r_{23}^2)(r_{14}-r_{13}r_{34})^2\\ &-(1-r_{34}^2)(r_{12}-r_{13}r_{23})^2+2(r_{12}-r_{13}r_{23})(r_{14}-r_{13}r_{34})(r_{24}-r_{23}r_{34}).\end{aligned} \tag{4.4}$$

Since the LEV is the largest root of a quartic polynomial, a lot of formulas exist to calculate it. In particular, it is possible to combine Ferrari's method with a resolvent cubic, whose roots can be expressed exactly using the trigonometric Vieta formula (see [5], Section 4, for the latter). Following Tignol [12], Section 3.2, one gets the roots of the depressed quartic equation (2.3) as follows.

For the non-trivial case $Q\neq 0$ the roots $z_1\ge z_2\ge z_3\ge z_4$ of (2.3) are given by

$$\begin{aligned} z_1 &= \sqrt{\tfrac{1}{2}u} + \sqrt{\tfrac{1}{2}(P-u+\tfrac{2Q}{\sqrt{2u}})}, \quad z_2 = \sqrt{\tfrac{1}{2}u} - \sqrt{\tfrac{1}{2}(P-u+\tfrac{2Q}{\sqrt{2u}})}, \\ z_3 &= -\sqrt{\tfrac{1}{2}u} + \sqrt{\tfrac{1}{2}(P-u-\tfrac{2Q}{\sqrt{2u}})}, \quad z_4 = -\sqrt{\tfrac{1}{2}u} - \sqrt{\tfrac{1}{2}(P-u-\tfrac{2Q}{\sqrt{2u}})}, \end{aligned} \tag{4.5}$$

where $u = u_1$ is the largest non-negative solution of the resolvent cubic

$$u^3 - Pu^2 + \left((\tfrac{P}{2})^2 - C_0\right)u - \tfrac{1}{2}Q^2 = 0\,. \tag{4.6}$$

Setting $u = w + \frac{1}{3}P$ one must solve the depressed cubic

$$w^3 + pw + q = 0, \quad p = -\left(\tfrac{1}{12}P^2 + C_0\right), \quad q = \tfrac{1}{108}P^3 - \tfrac{1}{3}PC_0 - \tfrac{1}{2}Q^2. \tag{4.7}$$

The trigonometric Vieta formulas, which solve (4.7), read:

$$\begin{aligned} &w_1 = 2a\cdot\cos\alpha, \quad w_2 = -a\cdot(\cos\alpha - \sqrt{3}\sin\alpha), \quad w_3 = -a\cdot(\cos\alpha + \sqrt{3}\sin\alpha), \\ &a = \sqrt{\tfrac{1}{3}\left(\tfrac{1}{12}P^2 + C_0\right)}, \quad \alpha = \tfrac{1}{3}\arccos\left(\frac{1}{2}\,\frac{\tfrac{1}{2}Q^2 + \tfrac{1}{3}PC_0 - \tfrac{1}{108}P^3}{\left(\tfrac{1}{3}\left(\tfrac{1}{12}P^2 + C_0\right)\right)^{1.5}}\right) \end{aligned} \tag{4.8}$$

The use of analytical formulas to compute the eigenvalues of a 4x4 matrix is found in several papers (e.g. Ichige et al. [8]).

On the other hand, another quite recent and attractive evaluation of the LEV, which can be applied to correlation matrices of any dimension, is the limiting Bernoulli type ratio approximation formula in Cirnu [1], Theorem 2.1 and Section 3. For an arbitrary correlation matrix $R = (r_{ij}), 1 \le i, j \le n$, one has the limiting formula

$$\lambda_1 = \lim_{k\to\infty} \frac{Tr(R^{k+1})}{Tr(R^k)}\,. \tag{4.9}$$

The Table 4.1 below provides a typical selection of numerical examples for Theorem 3.1. In particular, the new optimal quadratic lower bound offers a substantial improvement over the WS lower bound.

Table 4.1: Numerical comparison of LEV bounds

Upper Bound	$r=(x,r_{13},r_{14},r_{23},r_{24},r_{34})\in E_4$	$\lambda_1(x)$	$\lambda_1^M(x)$	$\lambda_1^{M,WS}(x)$
(b)	$(-1,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},1)$	3.41421	3.56155	**3.44949**
(c)	$(0,\frac{1}{2},\sqrt{\frac{2}{3}},\frac{\sqrt{3}}{3},\frac{1}{2}\sqrt{\frac{2}{3}},\frac{\sqrt{3}}{3},0)$	2	**2**	2.22474
	$(0.51158,0.92173,\frac{1}{8},0.54426,\frac{1}{4},\frac{1}{2})$	2.51286	**2.523**	2.61353
	$(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1)$	3	**3**	3.12132
Lower Bound	$r=(x,r_{13},r_{14},r_{23},r_{24},r_{34})\in E_4$	$\lambda_1(x)$	$\lambda_1^m(x)$	$\lambda_1^{m,WS}(x)$
(b)	$(-1,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},1)$	3.41421	**3.30278**	1.8165
(c)	$(0,\frac{1}{2},\sqrt{\frac{2}{3}},\frac{\sqrt{3}}{3},\frac{1}{2}\sqrt{\frac{2}{3}},\frac{\sqrt{3}}{3},0)$	2	**2**	1.40285
	$(0.51158,0.92173,\frac{1}{8},0.54426,\frac{1}{4},\frac{1}{2})$	2.51286	**2.4928**	1.53784
	$(0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1)$	3	**3**	1.70711

Conflicts of Interests

The author declares that there is no conflict of interests

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