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# A COMMON FIXED POINT THEOREM IN INTUITIONISTIC FUZZY METRIC SPACE USING IMPLICIT RELATION

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**Abstract**. In this paper, we prove a common fixed point theorem for subsequential continuous and compatible mappings in Intuitionistic fuzzy metric space using Implicit relation.

**Keywords**: common fixed point; intuitionistic fuzzy metric space; compatible mapping; reciprocal continuity; subcompatible mapping.

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## 1. Introduction

Fuzzy set was defined by Zadeh [1]. Kramosil and Michlek [2] introduced the concept of fuzzy metric space . Later in 1994, A.George and P.Veeramani [3] modified the notion of fuzzy metric space with the help of t-norm.

Subramanyam [4] , Vasuki[5] , Pant and Jha [6] obtained some analogous results proved by Balasubraman et.al.[7].

Jungck [8] introduced the notion of compatible maps for a pair of self maps. Popa ([9],[10]) introduced the idea of implicit function to prove a common fixed point theorem in metric space, Singh and Jain [11] extened the result of Popa ([9],[10]) in fuzzy metric space.Recently in 2009, using the concept of subcompatible maps ,H.Bouhadjera et.al.[12] proved common fixed point theorems.

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Atanassov [13] introduced and studies the concept of intuitionistic fuzzy sets. in 2004, Park [14] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norm and continuous t- conorm.In 2009, Bouhadjera and Godet-Thobie [15] introduced new notions of subcompatibility and subsequential continuity which are respectively weaker than owc and reciprocally continuity. In 2010 and 2011, B.Singh et.al. ( [16],[17],[18]) proved fixed point theorem in fuzzy metric space and Menger space using the concept of semi-compatibility, weak compatibility and compatibility of type ($\beta$ ) respectively.

## 2. Preliminaries

**Definition 2.1** [19] A binary operation $*:[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t –norm if it satisfies the following conditions

( i ) *is associative and commutative,

( ii ) *is continuous,

( iii ) $a*1 = a$, for all $a \in [0,1]$

( iv ) $a*b \leq c*d$, whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

**Definition 2.2** [19] A binary operation $\lozenge:[0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t –conorm if it satisfies the following conditions :

( i ) ◊is associative and commutative,

( ii ) ◊is continuous,

( iii ) $a\lozenge 1 = a$, for all $a \in [0,1]$

( iv ) $a\lozenge b \leq c\lozenge d$, whenever $a \leq c$ and $b \leq d$, for all $a,b,c,d \in [0,1]$.

**Definition 2.3** [20] A 5-tuple (X, M, N, *, ◊) is said to be an Intuitionistic fuzzy mertic space if X is an arbitrary set, * is a continuous t -norm, ◊ is a continuous t –conorm and M,N are fuzzy sets on $X^2 \times [0,\infty)$ satisfying the following conditions.

( i ) $M(x,y,t) + N(x,y,t) \leq 1$, for all $x,y \in X$ and $t > 0$,

( ii ) $M(x,y,0) = 0$, for all $x,y \in X$,

( iii ) $M(x,y,t) = 1$, for all $x,y \in X$ and $t > 0$, iff $x = y$,

( iv ) $M(x,y,t) = M(y,x,t)$, for all $x,y \in X$ and $t > 0$,

( v ) $M(x,y,t)*M(y,z,s) \leq M(x,z,t+s)$, for all $x,y \in X$ and $t ,s > 0$,

( vi ) for all $x,y \in x$, $M(x,y,.) : [0,\infty) \rightarrow [0,1]$ is left continuous,

( vii ) $\lim_{t\rightarrow\infty} M(x,y,t) = 1$, for all $x,y \in X$ and $t > 0$,

( viii ) $N(x,y,0) = 1$, for all $x,y \in X$ ,

( ix ) $N(x,y,t) = 0$, for all $x,y \in X$ and $t > 0$, iff $x = y$,

( x ) $N(x,y,t) = N(y,x,t)$, for all $x,y \in X$ and $t > 0$,

( xi ) $N(x,y,t)*N(y,z,s) \leq N(x,z,t+s)$, for all $x,y \in X$ and $t ,s > 0$,

( xii ) for all $x,y \in x$, $N(x,y,.) : [0,\infty) \rightarrow [0,1]$ is right continuous,

( xiii ) $\lim_{t\rightarrow\infty} N(x,y,t) = 0$, for all $x,y \in X$.

**Remark 2.1[20]**

In intuitionistic metric fuzzy space (X ,M ,*) is an intuitionistic fuzzy space of the form (X ,M ,1- M ,*, ◊) , such that t-norm * and t-conorm ◊ are associated as $x◊y = 1-((1-x) * (1-y))$ for all $x,y \in X$ .

**Remark 2.2[20]**

In intuitionistic fuzzy metric space (X ,M ,N ,*,◊) , M(x,y,*) is non-decreasing and N(x,y,◊) is Non-increasing for all $x,y \in X$ .

**Example 2.1 –** Let (X ,d) be a metric space . Define $a*b = ab$ and $a◊ b = \min \{1,a + b\}$ for all a,b $\in [0,1]$ and let $M_d$ and $N_d$ be a fuzzy sets on $X^2 \times (0,\infty)$ defined as

$$M_d (x,y,t) = t / t + d(x,y) , N_d(x,y,t) =d(x,y) / t + d(x,y)$$

Then (X , $M_d$ ,$N_d$ ,*,◊) is an intuitioistic fuzzy metric space.

**Definition 2.4[20]** Let (X , M , N ,*,◊) be an intuitioistic fuzzy metric space . Then

( 1 ) A sequence $\{x_n\}$ in X is set to be convergent to a point x in X iff $\text{Lim}_{n\rightarrow\infty} M(x_n , x,t) = 1$ and

$$\lim_{n\rightarrow\infty} N(x_n , x,t) = 0, \text{ for all } t > 0.$$

**Definition 2.5[20]** A pair of self mappings ( P ,Q ) of a intuitionistic fuzzy metric space ( X , M , N ,*,◊) is said to be compatible if $\lim_{n\rightarrow\infty} M(PQx_n , QPx_n , t) = 1$ and $\lim_{n\rightarrow\infty} N(PQx_n , QPx_n , t) = 0$ , for all t > o, Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\rightarrow\infty} Px_n = \text{Lim}_{n\rightarrow\infty} Qx_n = z, \text{ for some } z \in X.$$

**Definition 2.6[20]**A pair of self mappings ( P ,Q ) of a intuitionistic fuzzy metric space ( X , M , N ,*,◊) is said to be semi compatible if $\lim_{n\rightarrow\infty} PQx_n = Qx$ , When ever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\rightarrow\infty} Px_n = \text{Lim}_{n\rightarrow\infty} Qx_n = x, \text{ for some } x \in X.$$

**Definition 2.7[21]** A pair (P,Q) of self mappings defined on an Intuitionistic fuzzy metric space (X,M,N,*,◊) is said to be Subcompatible if and only if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some $z \in X$ , and

$$\lim_{n\to\infty} M(PQx_n, QPx_n,t) = 1 \text{ , for all } t > 0.$$

**Definition 2.8 [4]** A pair (P,Q) of self mappings defined on an Intuitionistic fuzzy metric space (X,M,N,*,◊) is said to be reciprocally continuous if for a sequence $\{x_n\}$ in X , $\lim_{n\to\infty} PQx_n = Pz$ and $\lim_{n\to\infty} QPx_n = Qz$. Whenever

$$\lim_{n\to\infty} Px_n = z = \lim_{n\to\infty} Qx_n \text{ for some } z \in X .$$

**Definition 2.9 [21]** A pair (P,Q) of self mappings defined on an Intuitionistic fuzzy metric space (X,M,N,*,◊) is said to be Subsequentially continuous if and only if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n = z$, for some

$$z \in X \text{ , and } \lim_{n\to\infty} PQx_n = Pz \text{ and } \lim_{n\to\infty} QPx_n = Qz.$$

**Implicit Relation 2.10 -** Let $\Phi$ be the set of all real continuous function $\emptyset : [0,1]^4 \to R$ , non – decreasing in first argument ,and satisfying the following conditions :

(i). For $u,v \geq 0$ , $\emptyset(u,v,u,v) \geq 0$ *or* $\emptyset(u,v,v,u) \geq 0$ implies that $u \geq v$,

(ii). $\emptyset(u,u,1,1) \geq 0$ implies that $u \geq 1$.

**Example** Define $\emptyset(a,b,c,d) = 15a-13b+5c-7d$. Then $\emptyset \in \Phi$.

## 3. Main Result

**Theorem 3.1-** Let P, Q, S and T be four self maps of an Intuitionistic Fuzzy metric space (X,M,N,*,◊ ) with continuous t-norm * and continuous t- conorm ◊ defined by $t*t \geq t$ , and $(1-t) \lozenge (1-t) \leq (1-t)$ respectively for all $t \in [0,1]$ . If the pairs (P,S) and (Q,T) are subsequential continuous and compatible mappings ,then

(a). For any $x,y \in X$, $\emptyset, \Psi\psi \in \Phi$ , and for all $t > 0$ ,

$$\emptyset[ M(Px,Qy,t), M(Px,Ty,t),M(Sx,Px,t),M(Ty,Qy,t) ] \geq 0 \text{ . } and$$

$$\psi[N(Px,Qy,t), N(Px,Ty,t),N(Sx,Px,t),N(Ty,Qy,t)] \leq 0.$$

then the pairs (P,S) and (Q,T) have a point of coincidence. Then P,Q,S and T have a unique common fixed point.

**Proof-** Since the pairs (P,S) and (Q,T) are subsequential continuous and compatible mappings , then there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Sx_n = z$ , for some $z \in X$ , and $\lim_{n\to\infty} M(PSx_n, SPx_n ,t) =1$ , for all $t > 0$.

$\lim_{n\to\infty} Qy_n = \lim_{n\to\infty} Ty_n = w$ , for some $w \in X$ , and $\lim_{n\to\infty} M(QTy_n , TQy_n,t) =1$ , for all $t > 0$ .

Therefore $Pz = Sz$ and $Qw = Tw$ , …(1)

that is z is the coincidence point of the pair (P,S) and w is the coincidence point of the pair (Q,T).

Now , we will prove that $z = w$ .

Put $x =x_n$ and $y =y_n$ in inequality 3.1 (a) , we get

$\emptyset [ M( Px_n, Qy_n,t) , M(Px_n, Ty_n,t), M(Sx_n ,Px_n ,t) , M(Ty_n,Qy_n,t)] \geq 0$ , and

$\psi\Psi [ N( Px_n, Qy_n,t) , N(Px_n, Ty_n,t), N(Sx_n ,Px_n ,t) ,N(Ty_n,Qy_n,t)] \leq 0$

Taking lim $n\to\infty$ , we get

$\emptyset[ M(z,w,t),M(z,w,t), M(z,z,t),M(w,w,t)]\geq 0,$ and

$\psi[N(z,w,t), N(z,w,t),N(z,z,t), N(w,w,t)] \leq 0.$

$\emptyset[ M(z,w,t), M(z,w,t),1,1] \geq 0,$ and $\psi [N(z,w,t),N(z,w,t),1,1] \leq 0$

In view of 2.10 (ii) , $z = w$ .

Again , we claim that $Pz = z$ .

By putting $x =z$ and $y = y_n$ , in inequality 3.1 (a), we get

$\emptyset[ M(Pz,Qy_n,t), M(Pz,Ty_n,t), M(Sz,Pz,t),M(Ty_n,Qy_n,t)] \geq 0$ , and

$\psi[ N(Pz,Qy_n,t), N(Pz,Ty_n,t) ,N(Sz,Pz,t), N(Ty_n,Qy_n,t)] \leq 0.$

Taking limit $n\to\infty$ , we get

$\emptyset[ M(Pz,w,t),M(Pz,w,t),M(Sz,Pz,t), M(w,w,t)]\geq 0$ , and

$\psi[ N(Pz,w,t), N(Pz,w,t),N( Sz,Pz,,t),N(w,w,t)] \leq 0$ .

$\emptyset[M(Pz,w,t), M(Pz,w,t),1,1] \geq 0$ , and $\psi[ N(Pz,w,t), N(Pz,w,t),1,1] \leq 0.$

Since $w = z$ , then we get

$\emptyset[M(Pz,z,t), M(Pz,z,t),1,1]\geq 0$ , and $\Psi[N(Pz,z,t),N(Pz,z,t),1,1] \leq 0.$

From 2.12(ii) , we get $Pz = z$ . Since $Pz = Sz$ , combining both results , we get

$Pz = z = Sz$ …(2)

Now , again we claim that $Qw = z$ .

substitute $x =z$ and $y = w$ in inequality 3.1(a), we get

$\emptyset[M(Pz,Qw,t), M(Pz,Tw,t),M(Sz,Pz,t),M(Tw,Qw,t)] \geq 0$ , and

$\psi[N(Pz,Qw,t),N( Pz,Tw,t),N(Sz,Pz,t),N(Tw,Qw,t)] \leq 0$ .

Using (1) and (2) , we get

$\emptyset[M(z,Qw,t), M(z,Qw,t), M(Pz,Pz,t), M(Qw,Qw,t)] \geq 0$ , and ψ[N(z,Qw,t), N(z,Qw,t), N(Pz,Pz,t),N(Qw,Qw,t)] $\leq 0$ .

∅[M(z,Qw,t), M(z,Qw,t), 1,1] $\geq 0$ , and ψ[N(z,Qw,t),N(z,Qw,t),1,1] $\leq 0$ .

From 2.10 (ii) , we get Qw = z . Since w = z , then we get Qz = z .

Since Qz = Tz , hence from this we conclude that Qz = z = Tz.

Therefore , combining all the result , we get

z = Pz = Sz = Qz = Tz .

That is z is a common fixed point of P, Q, S and T.

**Uniqueness –** Let u be another common fixed point of P, Q , S and T . Then

Pu = Su = Qu = Tu = u.

Put x = z and y = u in inequality 3.1(a) , we get

∅[M(Pz,Qu,t), M(Pz,Tu,t), M(Sz,Pz,t), M(Tu,Qu,t) ] $\geq 0$ , and

ψ [N(Pz,Qu,t),N(Pz,Tu,t),N(Sz,Pz,t),N(Tu,Qu,t)] $\leq$ 0.

∅[$M$(z,u,t),M(z,u,t), M(z,z,t), M(u,u,t)] $\geq 0$ , and

ψ[N(z,u,t), N(z,u,t), N(z,z,t), N(u,u,t)] $\leq 0$ .

∅[M(z,u,t) ,M(z,u,t), 1,1] $\geq 0$ , and ψ[N(z,u,t),N(z,u,t),1,1] $\leq 0$ .

$From$ 2.10 (ii) , we get z = u .

Therefore , uniqueness follows.

**Corollary 3.2** Let P , Q and S be three self-maps of an Intuitionistic fuzzy metric space (X,M,N,*,◊) with continuous

t-norm * and continuous t- conor m ◊ defined by t *t $\geq t$ and (1-t) ◊(1-t) $\leq$ (1-t) , for all t∈[0,1] . If the pairs (P,S) and

(Q,T) are subsequential continuous and compatible mappings , then for some ∅ , Ψ ∈ $\Phi$ and for all x,y ∈ X , and every t > 0 ,

∅[ $M(Px,Qy,t)$, M(Sx,Px,t),M(Sy,Qy,t),M(Sx,Qy,t),M(Sy,Px,t)] $\geq 0$, and

$\psi\Psi$ [ $N(Px,Qy,t), N(Sx,Px,t), N(Sy,Qy,t), N(Sx,Qy,t), N(Sy,Px,t)] \leq 0$.

then $(P,S)$ and (Q,S) have a coincidence points. Then P, Q and S have a unique common fixed point.

**Corollary 3.3** Let P ,Q ,S and T be four self maps of an Intuitionistic fuzzy metric space (X,M,N,*,◊) with continuous t-norm * and continuous t- conor m ◊ defined by t *t $\geq t$ and (1-t) ◊(1-t) $\leq$ (1-t) , for all t∈[0,1] . If the pairs (P,S) and (Q,T) are subsequential continuous and compatible mappings , then for some ∅ , Ψ ∈ $\Phi$ and for all x,y ∈ X , and every t > 0 ,

$$\emptyset[\, M(Px,Qy,t),\ 1/2\{ M(Sx,Px,t) + M(Ty,Qy,t)\} , M(Sx,Qy,t), M(Ty,Px,t)] \geq 0\ ,\ \ and$$

$$\psi[N(Px,Qy,t),\ 1/2\{N (Sx,Px,t) + N(Ty,Qy,t)\} , N(Sx,Qy,t), N(Ty,Px,t)] \leq 0\ .$$

$Then$ $(P,S)$ and (Q,T) have a coincidence points . Then $\mathrm{P}, \mathrm{Q}, S$ and $\mathrm{T}$ a unique common fixed point.

**Conflicts of Interests**

The author declares that there is no conflict of interests

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