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ON SOME INTEGRAL INEQUALITIES FOR GENERALIZED FRACTIONAL INTEGRAL

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Abstract. In this article, we obtain integral inequalities for generalized Riemann-Liouville fractional integrals and Chebyshev functional by using synchronous functions.

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1. Introduction

Let us consider the functional in [1]

$$(1) \qquad T(f,g) := \frac{1}{b-a}\int_b^a f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

where f and g are two synchronous and integrable functions on $[a,b]$.

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Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, ([9]-[12], [15]) and the references therein.

In this paper, we obtain some integral inequalities for (1) type functional via generalized fractional integrals.

2. Preliminaries

Definition 2.1. Let $a,b \in \mathbb{R}$, $a<b$, and $\alpha > 0$. For $f \in L_1(a,b)$

$$\left(J_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt,\ \alpha>0,\ x>a \tag{2}$$

and

$$\left(J_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt,\ b>0,\ b>x. \tag{3}$$

These integrals are called right-sided Riemann-Liouville fractional integral and left-sided Riemann-Liouville fractional integral respectively [2]-[7].

Definition 2.2. Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\alpha>0$. Also let $h(x)$ be an increasing and a positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by [13]

$$\left(J_{a^+,h}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x [h(x)-h(t)]^{\alpha-1} h'(t) f(t)\, dt,\ x \geq a, \tag{4}$$

and

$$\left(J_{b^-,h}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b [h(t)-h(x)]^{\alpha-1} h'(t) f(t)\, dt,\ x \leq b. \tag{5}$$

For (4) and (5)

$$\left(J_{a^+,h}^{\alpha} f\right)(a) = \left(J_{b^-,h}^{\alpha} f\right)(b) = 0.$$

If we take $h(x)=x$ in (4) and (5), we will obtain

$$J_{a^+,h}^{\alpha} = J_{a^+}^{\alpha} \text{ and } J_{b^-,h}^{\alpha} = J_{b^-}^{\alpha}.$$

Also if we choose $h(x)=\dfrac{x^{k+1}}{k+1}$ for $k\geq 0$, then the equalities (4) and (5) will be

$$(J^{\alpha}_{a^{+},k}f)(x)=\tfrac{(k+1)^{1-\alpha}}{\Gamma(\alpha)}\int\limits_{a}^{x}(x^{k+1}-t^{k+1})^{\alpha-1}t^{k}f(t)dt,\ x>a \tag{6}$$

and

$$(J^{\alpha}_{b^{-},k}f)(x)=\tfrac{(k+1)^{1-\alpha}}{\Gamma(\alpha)}\int\limits_{x}^{b}(t^{k+1}-x^{k+1})^{\alpha-1}t^{k}f(t)dt,\ x<b \tag{7}$$

respectively. This kind of generalized fractional integrals are studied in [5], [7], [14] and [16].

For $a=0$ in (4), we can write

$$\left(J^{\alpha}_{0^{+},h}f\right)(x)=\tfrac{1}{\Gamma(\alpha)}\int\limits_{0}^{x}(h(x)-h(t))^{\alpha-1}h^{'}(t)f(t)dt,\ x>0 \tag{8}$$

and

$$\left(J^{0}_{0^{+},h}f\right)(x)=f(x).$$

For the convenience of establishing the results, we give the semigroup property:

$$J^{\alpha}_{a^{+},h}J^{\beta}_{a^{+},h}f(x)=J^{\alpha+\beta}_{a^{+},h}\ f(x),\ \alpha\geq 0,\ \beta\geq 0,$$

which implies the commutative property:

$$J^{\alpha}_{a^{+},h}J^{\beta}_{a^{+},h}f(x)=J^{\beta}_{a^{+},h}J^{\alpha}_{a^{+},h}f(x).$$

From (8), when $f(x)=h(x)$, we get:

$$\begin{aligned}\left(J^{\alpha}_{0^{+},h}h\right)(x)&=\frac{1}{\Gamma(\alpha)}\int\limits_{0}^{x}(h(x)-h(t))^{\alpha-1}h(t)h^{'}(t)\,dt\\&=\frac{(h(x)-h(0))^{\alpha}}{\Gamma(\alpha+2)}\left[h(x)+\alpha h(0)\right].\end{aligned} \tag{9}$$

Let $\alpha=0$ in (9), then

$$\left(J^{0}_{0^{+},h}h\right)(x)=h(x).$$

From (8), when $f(x)=x^{\mu}$ and $h(x)=x^{k+1}$ we get:

$$\begin{aligned}&J^{\alpha}_{0^{+},h}(x^{\mu})\\&=\frac{(k+1)^{-\alpha}\Gamma(\frac{k+\mu+1}{k+1})}{\Gamma(\alpha+\frac{k+\mu+1}{k+1})}t^{\alpha(k+1)+\mu},\ \alpha>0;\ k\geq 0,\ \mu>-1,\ t>0.\end{aligned} \tag{10}$$

From (8), when $f(x)=1$ and $h(x)=x^{k+1}$ we get:

$$J^{\alpha}_{0^+,h}(1) \;=\frac{(k+1)^{-\alpha}}{\Gamma(\alpha+1)}t^{\alpha(k+1)},\ \alpha>0;\ k\geq 0,\ \mu>-1,\ t>0. \tag{11}$$

3. Main results

Theorem 3.1. Let f and g be two synchronous functions on $[0,\infty)$. Also let $h(x)$ be an increasing and a positive monotone function on $(a,b]$, having a continuous derivative $h^{'}(x)$ on (a,b). Then for $t>a$, $\alpha>0$;

$$J^{\alpha}_{a^+,h}(fg)(t)\geq\frac{1}{J^{\alpha}_{a^+,h}(1)}J^{\alpha}_{a^+,h}f(t)J^{\alpha}_{a^+,h}g(t). \tag{12}$$

Proof. For f and g synchronous functions, we have

$$(f(\tau)-f(\rho))(g(\tau)-g(\rho))\geq 0. \tag{13}$$

From (13) it can be written as following

$$f(\tau)g(\tau)+f(\rho)g(\rho)\geq f(\tau)g(\rho)+f(\rho)g(\tau). \tag{14}$$

If we multiply two sides of the (14) with $\frac{(h(t)-h(\tau))^{\alpha-1}}{\Gamma(\alpha)}h^{'}(\tau)$, $\tau\in(a,t)$, we get

$$\begin{aligned}&\frac{(h(t)-h(\tau))^{\alpha-1}}{\Gamma(\alpha)}h^{'}(\tau)f(\tau)g(\tau)+\frac{(h(t)-h(\tau))^{\alpha-1}}{\Gamma(\alpha)}h^{'}(\tau)f(\rho)g(\rho)\\ &\geq\frac{(h(t)-h(\tau))^{\alpha-1}}{\Gamma(\alpha)}h^{'}(\tau)f(\tau)g(\rho)+\frac{(h(t)-h(\tau))^{\alpha-1}}{\Gamma(\alpha)}h^{'}(\tau)f(\rho)g(\tau).\end{aligned} \tag{15}$$

Then integrating (15) inequality over (a,t), we obtain:

$$\begin{aligned}&\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h^{'}(\tau)f(\tau)g(\tau)d\tau\\ &+\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h^{'}(\tau)f(\rho)g(\rho)d\tau\\ &\geq\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h^{'}(\tau)f(\tau)g(\rho)d\tau\\ &+\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h^{'}(\tau)f(\rho)g(\tau)d\tau.\end{aligned} \tag{16}$$

Consequently,

$$\begin{aligned}
&J^{\alpha}_{a^+,h}(fg)(t)+f(\rho)g(\rho)\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h'(\tau)\,d\tau\\
&\geq g(\rho)\tfrac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h'(\tau)\,f(\tau)d\tau\\
&+f(\rho)\frac{1}{\Gamma(\alpha)}\int_a^t(h(t)-h(\tau))^{\alpha-1}h'(\tau)\,g(\tau)d\tau.
\end{aligned}\tag{17}$$

So we have

$$J^{\alpha}_{a^+,h}(fg)(t)+f(\rho)g(\rho)J^{\alpha}_{a^+,h}(1)\geq g(\rho)J^{\alpha}_{a^+,h}f(t)+f(\rho)J^{\alpha}_{a^+,h}g(t).\tag{18}$$

Now multiplying two sides of (18) by $\frac{(h(t)-h(\rho))^{\alpha-1}}{\Gamma(\alpha)}h'(\rho)$, $\rho\in(a,t)$, we obtain:

$$\begin{aligned}
&\frac{(h(t)-h(\rho))^{\alpha-1}}{\Gamma(\alpha)}h'(\rho)\,J^{\alpha}_{a^+,h}(fg)(t)\\
&+\frac{(h(t)-h(\rho))^{\alpha-1}}{\Gamma(\alpha)}h'(\rho)\,f(\rho)g(\rho)J^{\alpha}_{a^+,h}(1)\\
&\geq\frac{(h(t)-h(\rho))^{\alpha-1}}{\Gamma(\alpha)}h'(\rho)\,g(\rho)J^{\alpha}_{a^+,h}f(t)\\
&+\frac{(h(t)-h(\rho))^{\alpha-1}}{\Gamma(\alpha)}h'(\rho)\,f(\rho)J^{\alpha}_{a^+,h}g(t).
\end{aligned}\tag{19}$$

By integrating to (19) over (a,t), we get:

$$\begin{aligned}
&J^{\alpha}_{a^+,h}(fg)(t)\int_a^t\frac{(h(t)-h(\rho))^{\alpha-1}}{\Gamma(\alpha)}h'(\rho)\,d\rho\\
&+\frac{J^{\alpha}_{a^+,h}(1)}{\Gamma(\alpha)}\int_a^t f(\rho)g(\rho)(h(t)-h(\rho))^{\alpha-1}h'(\rho)\,d\rho\\
&\geq\frac{J^{\alpha}_{a^+,h}f(t)}{\Gamma(\alpha)}\int_a^t(h(t)-h(\rho))^{\alpha-1}h'(\rho)\,g(\rho)d\rho\\
&+\frac{J^{\alpha}_{a^+,h}g(t)}{\Gamma(\alpha)}\int_a^t(h(t)-h(\rho))^{\alpha-1}h'(\rho)\,f(\rho)d\rho.
\end{aligned}\tag{20}$$

This inequality is can be written as the following at the same time,

$$J^{\alpha}_{a^+,h}(fg)(t) \geq \frac{1}{J^{\alpha}_{a^+,h}(1)} J^{\alpha}_{a^+,h} f(t) J^{\alpha}_{a^+,h} g(t). \tag{21}$$

So the proof is completed.

Theorem 3.2. Let f and g be two synchronous functions on $[a,b]$. Then for $t > a$, $\alpha > 0$, and $\beta > 0$,

$$J^{\beta}_{a^+,h}(1) J^{\alpha}_{a^+,h}(fg)(t) + \frac{(h(t)-h(a))^{\alpha}}{\Gamma(\alpha+1)} J^{\beta}_{a^+,h}(fg)(t)$$
$$\geq J^{\alpha}_{a^+,h} f(t) J^{\beta}_{a^+,h} g(t) + J^{\alpha}_{a^+,h} g(t) J^{\beta}_{a^+,h} f(t).$$

Proof. If we multiply two sides of (18) by $\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho)$, we obtain:

$$\begin{aligned}
&\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) J^{\alpha}_{a^+,h}(fg)(t) \\
&+\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) g(\rho) J^{\alpha}_{a^+,h}(1) \\
&\geq \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) g(\rho) J^{\alpha}_{a^+,h} f(t) \\
&+\frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) J^{\alpha}_{a^+,h} g(t).
\end{aligned} \tag{22}$$

Integrating to (22) over (a,t), we get:

$$\begin{aligned}
&\int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) J^{\alpha}_{a^+,h}(fg)(t) dt \\
&+\int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) g(\rho) J^{\alpha}_{a^+,h}(1) dt \\
&\geq \int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) g(\rho) J^{\alpha}_{a^+,h} f(t) dt \\
&+\int_a^t \frac{(h(t)-h(\rho))^{\beta-1}}{\Gamma(\beta)} h'(\rho) f(\rho) J^{\alpha}_{a^+,h} g(t) dt.
\end{aligned} \tag{23}$$

Consequently,

$$J^{\beta}_{a^+,h}(1)J^{\alpha}_{a^+,h}(fg)(t)+J^{\alpha}_{a^+,h}(1)J^{\beta}_{a^+,h}(fg)(t) \tag{24}$$
$$\geq J^{\alpha}_{a^+,h}f(t)J^{\beta}_{a^+,h}g(t)+J^{\alpha}_{a^+,h}g(t)J^{\beta}_{a^+,h}f(t).$$

This is the proof of the theorem.

Remark 3.3. *Applying Theorem* 3.2 *for* $\alpha=\beta$, *we obtain Theorem* 3.1.

Theorem 3.4. Let $(f_i)_{i=1,...n}$ be n positive increasing functions on $[0,\infty)$. Then for all $t>a,\ \alpha>0$,

$$J^{\alpha}_{a^+,h}(\prod_{i=1}^{n}f_i)(t)\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{1-n}\left(\prod_{i=1}^{n}J^{\alpha}_{a^+,h}f_i\right)(t) \tag{25}$$

Proof. We will prove this theorem by induction.It is clear that for $n=1$ and all $t>0,\ \alpha>0$, we have $J^{\alpha}_{a^+,h}(f_1)(t)\geq J^{\alpha}_{a^+,h}f_1(t)$. And for $n=2$, we obtain (12),

$$J^{\alpha}_{a^+,h}(f_1f_2)(t)\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}\left(J^{\alpha}_{a^+,h}f_1\right)(t)\left(J^{\alpha}_{a^+,h}f_2\right)(t) \tag{26}$$

Now assume that(induction hypothesis)

$$J^{\alpha}_{a^+,h}(\prod_{i=1}^{n-1}f_i)(t)\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{2-n}\left(\prod_{i=1}^{n-1}J^{\alpha}_{a^+,h}f_i\right)(t) \tag{27}$$

If $(f_i)_{i=1,...n}$ are positive increasing functions, then $\left(\prod_{i=1}^{n-1}f_i\right)(t)$ is an increasing function. So we can use Theorem 3.1 for functions $\prod_{i=1}^{n-1}f_i=g$, and $f_n=f$, therefore we obtain

$$J^{\alpha}_{a^+,h}(\prod_{i=1}^{n}f_i)(t)=J^{\alpha}_{a^+,h}(fg)(t)\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}\left(\prod_{i=1}^{n-1}J^{\alpha}_{a^+,h}f_i\right)(t)\left(J^{\alpha}_{a^+,h}f_n\right)(t). \tag{28}$$

By (27)

$$J^{\alpha}_{a^+,h}(\prod_{i=1}^{n}f_i)(t)\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}\left(J^{\alpha}_{a^+,h}(1)\right)^{2-n}\left(\prod_{i=1}^{n-1}J^{\alpha}_{a^+,h}f_i\right)(t)\left(J^{\alpha}_{a^+,h}f_n\right)(t). \tag{29}$$

This completes the proof.

Theorem 3.5. Let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h^{'}(x)$ on (a,b). If f is an increasing and g is a differentiable functions and there exist a real number $mh^{'}(t) := \inf_{t\geq 0} g'(t)$ on $[0,+\infty)$. Then for all $t\in[a,b]$ and $\alpha>0$,

$$J^{\alpha}_{a^+,h}(fg)(t) \tag{30}$$
$$\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h}f(t) J^{\alpha}_{a^+,h}g(t) - \frac{mh(t)}{\alpha+1} J^{\alpha}_{a^+,h}f(t) + mJ^{\alpha}_{a^+,h}(hf)(t).$$

Proof. Consider the given function $H(t)=g(t)-mh(t)$. It is clear that H is an increasing function and differentiable on $[0,+\infty)$. Then using Theorem 3.1 we obtain

$$J^{\alpha}_{a^+,h}(Hf)(t) = J^{\alpha}_{a^+,h}((g(t)-mh(t))f(t))$$
$$\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h}f(t)\left[J^{\alpha}_{a^+,h}g(t) - mJ^{\alpha}_{a^+,h}h(t)\right]$$
$$\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h}f(t) J^{\alpha}_{a^+,h}g(t) \tag{31}$$
$$-\frac{m\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}(h(t)-h(a))^{\alpha}(h(t)+\alpha h(a))}{\Gamma(\alpha+2)} J^{\alpha}_{a^+,h}f(t)$$
$$\geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h}f(t) J^{\alpha}_{a^+,h}g(t) - \frac{m(h(t)+\alpha h(a))}{\alpha+1} J^{\alpha}_{a^+,h}f(t).$$

Also,

$$J^{\alpha}_{a^+,h}(Hf)(t)$$
$$= J^{\alpha}_{a^+,h}((g(t)-mh(t))f(t)) \tag{32}$$
$$= J^{\alpha}_{a^+,h}(fg)(t) - mJ^{\alpha}_{a^+,h}(hf)(t)$$

From (31) and (32), we get:

$$J^{\alpha}_{a^+,h}(fg)(t) \geq \left(J^{\alpha}_{a^+,h}(1)\right)^{-1} J^{\alpha}_{a^+,h}f(t) J^{\alpha}_{a^+,h}g(t) \tag{33}$$
$$-\frac{m(h(t)+\alpha h(a))}{\alpha+1} J^{\alpha}_{a^+,h}f(t) + mJ^{\alpha}_{a^+,h}(hf)(t).$$

This is the proof of theorem.

Corollary 3.6. Let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h^{'}(x)$ on (a,b). If f is an increasing and g is a differentiable functions on $[0,+\infty)$. Then for all $t\in[a,b]$ and $\alpha>0$,

I. If there exist real numbers $m_1h'(t):=\inf_{t\geq 0} f'(x)$, and $m_2h'(t):=\inf_{t\geq 0} g'(t)$. Then we have:

$$J^{\alpha}_{a^+,h}(fg)(t)-m_1J^{\alpha}_{a^+,h}(hg)(t)-m_2J^{\alpha}_{a^+,h}(hf)(t)+m_1m_2J^{\alpha}_{a^+,h}h(t)^2$$

$$\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}\Big[J^{\alpha}_{a^+,h}f(t)J^{\alpha}_{a^+,h}g(t)-m_1J^{\alpha}_{a^+,h}h(t)J^{\alpha}_{a^+,h}g(t) \tag{34}$$

$$-m_2J^{\alpha}_{a^+,h}h(t)J^{\alpha}_{a^+,h}f(t)+m_1m_2\left(J^{\alpha}_{a^+,h}h(t)\right)^2\Big].$$

II. If there exist real numbers $M_1h'(t):=\sup_{t\geq 0} f'(x)$, and $M_2h'(t):=\sup_{t\geq 0} g'(t)$. Then we have:

$$J^{\alpha}_{a^+,h}(fg)(t)-M_1J^{\alpha}_{a^+,h}(hg)(t)-M_2J^{\alpha}_{a^+,h}(hf)(t)+M_1M_2\left(J^{\alpha}_{a^+,h}h(t)\right)^2$$

$$\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}\Big[J^{\alpha}_{a^+,h}f(t)J^{\alpha}_{a^+,h}g(t)-M_1J^{\alpha}_{a^+,h}h(t)J^{\alpha}_{a^+,h}g(t) \tag{35}$$

$$-M_2J^{\alpha}_{a^+,h}h(t)J^{\alpha}_{a^+,h}f(t)+M_1M_2\left(J^{\alpha}_{a^+,h}h(t)\right)^2\Big].$$

Proof. Consider the given function $F(t)=f(t)-m_1h(t)$ and $G(t)=g(t)-m_2h(t)$. It is clear that F and G are an increasing function and differentiable on $[0,+\infty)$. Then using Theorem 3.1 we obtain

$$\begin{aligned}
J^{\alpha}_{a^+,h}(FG)(t) &= J^{\alpha}_{a^+,h}\left(f(t)-m_1h(t)\right)\left(g(t)-m_2h(t)\right)\\
&\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}J^{\alpha}_{a^+,h}\left(f(t)-m_1h(t)\right)J^{\alpha}_{a^+,h}\left(g(t)-m_2h(t)\right)\\
&\geq\left(J^{\alpha}_{a^+,h}(1)\right)^{-1}\Big[J^{\alpha}_{a^+,h}f(t)J^{\alpha}_{a^+,h}g(t)-m_1J^{\alpha}_{a^+,h}h(t)J^{\alpha}_{a^+,h}g(t)\\
&\quad -m_2J^{\alpha}_{a^+,h}f(t)J^{\alpha}_{a^+,h}h(t)+m_1m_2\left(J^{\alpha}_{a^+,h}h(t)\right)^2\Big]
\end{aligned}$$

Therefore

$$J_{a^+,h}^{\alpha}(fg)(t) - m_1 J_{a^+,h}^{\alpha}(hg)(t) - m_2 J_{a^+,h}^{\alpha}(hf)(t) + m_1 m_2 J_{a^+,h}^{\alpha}(h(t))^2$$

$$\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1}\Big[J_{a^+,h}^{\alpha}f(t)J_{a^+,h}^{\alpha}g(t) - m_1 J_{a^+,h}^{\alpha}h(t)J_{a^+,h}^{\alpha}g(t)$$

$$-m_2 J_{a^+,h}^{\alpha}f(t)J_{a^+,h}^{\alpha}h(t) + m_1 m_2\left(J_{a^+,h}^{\alpha}h(t)\right)^2\Big].$$

This is the proof of (I).

Consider the given function $F(t) = f(t) - M_1 h(t)$, $G(t) = g(t) - M_2 h(t)$. It is clear that F and G are an increasing function and differentiable on $[0, +\infty)$. Then using Theorem 3.1 we obtain

$$\begin{aligned} J_{a^+,h}^{\alpha}(FG)(t) &= J_{a^+,h}^{\alpha}(f(t) - M_1 h(t))(g(t) - M_2 h(t)) \\ &\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1} J_{a^+,h}^{\alpha}(f(t) - M_1 h(t)) J_{a^+,h}^{\alpha}(g(t) - M_2 h(t)) \\ &\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1}\Big[J_{a^+,h}^{\alpha}f(t)J_{a^+,h}^{\alpha}g(t) - M_1 J_{a^+,h}^{\alpha}h(t)J_{a^+,h}^{\alpha}g(t) \\ &\quad -m_2 J_{a^+,h}^{\alpha}f(t)J_{a^+,h}^{\alpha}h(t) + M_1 M_2\left(J_{a^+,h}^{\alpha}h(t)\right)^2\Big] \end{aligned}$$

Therefore

$$J_{a^+,h}^{\alpha}(fg)(t) - M_1 J_{a^+,h}^{\alpha}(hg)(t) - M_2 J_{a^+,h}^{\alpha}(hf)(t) + M_1 M_2 J_{a^+,h}^{\alpha}(h(t))^2$$

$$\geq \left(J_{a^+,h}^{\alpha}(1)\right)^{-1}\Big[J_{a^+,h}^{\alpha}f(t)J_{a^+,h}^{\alpha}g(t) - M_1 J_{a^+,h}^{\alpha}h(t)J_{a^+,h}^{\alpha}g(t)$$

$$-M_2 J_{a^+,h}^{\alpha}f(t)J_{a^+,h}^{\alpha}h(t) + M_1 M_2\left(J_{a^+,h}^{\alpha}h(t)\right)^2\Big].$$

This is the proof of (II).

Conflict of Interests

The authors declare that there is no conflict of interests.

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