Available online at http://scik.org
Adv. Inequal. Appl. 2016, 2016:18
ISSN: 2050-7461

POSITIVE SOLUTIONS FOR NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS WITH SINGULARITIES

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Abstract. We study the existence positive periodic solutions to second order differential equations with singular nonlinear perturbations. It is proved that such a problem has at least one positive solutions under reasonable conditions. The proof of the main result relies on a nonlinear alternative principle of Leray-Schauder, together with a truncation technique.

Keywords: positive solution; singular equation; Leray-Schauder alternative principle.

2010 AMS Subject Classification: 34B15, 34B16, 34D20.

1. Introduction

The main aim of this paper is to present some recent existence results for the positive $T-$periodic solutions of second order differential equation

$$-[p(x)y']' + q(x)y = f(x,y) + e(x), \quad 0 \le x \le T, \tag{1}$$

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Received May 18, 2016

and boundary conditions

$$y(0) = y(T), \quad y^{[1]}(0) = y^{[1]}(T), \tag{2}$$

where $y = y(x)$ is a desired solution, and

$$y^{[1]}(x) = p(x)y'(x)$$

denote the quasi-derivative of $y(x)$, we call the periodic boundary conditions which are important representatives of nonseparated boundary conditions. The nonlinearity $f \in C((\mathbb{R}/T\mathbb{Z}) \times (0,\infty),\mathbb{R})$. We are mainly interested in the case that $f(x,y)$ has a repulsive singularity at $y = 0$, which means that

$$\lim_{y\to 0^+} f(x,y) = +\infty, \text{ uniformly in } x.$$

And we will assume that the coefficients $p(x)$ and $q(x)$ of Eq. (1) are real-valued measurable functions defined on $[0,T]$ and satisfy the following condition (H):

(H) $p(x) > 0, q(x) > 0, \int_0^T \frac{1}{p(x)}dx < \infty, \int_0^T q(x)dx < \infty.$

It is well know that second order singular differential equations describe many problems in the applied sciences, such as the Brillouin focusing system and nonlinear elasticity. Therefore, during the last few decades, the study of the existence of periodic solutions for singular differential equations have deserved the attention of many researchers [2, 6, 7, 8, 12, 14, 18, 20]. Recently, it has been found that a particular case of (1), the singular Ermakov-Pinney equation plays an important role in studying the Lyapunov stability of periodic solutions of Lagrangian equations [19, 20].

In the literature, two differential approaches have been used to establish the existence results for singular equations. The first is the variational approach, and the second one is topological methods, including the degree theory [18, 19, 20], the method of upper and lower solutions [1, 12], Schauder's fixed point theorem [7, 13, 5], some fixed point theorems in cones for completely continuous operators [4, 15, 16] and a nonlinear Leray-Schauder alternative principle [9, 10, 17].

The Green function $G(x,s)$ associated with (1) and (2) is positive, it has been proved in [3] with $e(x) \equiv 0$ has at least one positive periodic solution when $f(x,y)$ has a repulsive singularity near $y = 0$ and $f(x,y)$ is superlinear near $y = +\infty$. The proof given in [3] is based on Krasnoselskii fixed point theorem on compression and expansion of cones.

In this paper, we establish the existence of positive periodic solutions to Eq.(1) through a basic application of nonlinear alternative principle of Leray-Schauder, generalizing in several aspects some results in [9, 10]. Our main motivation is to obtain new existence results for positive periodic solutions of the equation

$$-[p(x)y']' + q(x)y = \frac{b(x)}{y^{\alpha}} + \mu c(x)y^{\beta} + e(x), \tag{3}$$

with $b, c, e \in C[0,T]$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ a given parameter.

The rest of this paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, we will state and prove the main results.

2. Preliminaries

Let us denote $u(t)$ and $v(t)$ by the solutions of the following homogeneous equations

$$-[p(x)y']' + q(x)y = 0, \quad 0 \le x \le T,$$

satisfying the initial conditions

$$u(0) = 1, u^{[1]}(0) = 0, v(0) = 0, v^{[1]}(0) = 1,$$

and set

$$D = u(T) + v^{[1]}(T) - 2. \tag{4}$$

Lemma 2.1[3] For the solution $y(x)$ of the boundary value problem

$$\begin{cases} -[p(x)y']' + q(x)y = h(x), \quad 0 \le x \le T, \\ y(0) = y(T), \quad y^{[1]}(0) = y^{[1]}(T), \end{cases} \tag{5}$$

the formula

$$y(x) = \int_0^T G(x,s)h(s)ds, \quad x \in [0,T],$$

hold, where

$$G(x,s) = \frac{v(T)}{D}u(x)u(s) - \frac{u^{[1]}(T)}{D}v(x)v(s) + \begin{cases} \frac{v^{[1]}(T)-1}{D}u(x)v(s) - \frac{u(T)-1}{D}u(s)v(x), & 0 \le s \le x \le T, \\ \frac{v^{[1]}(T)-1}{D}u(s)v(x) - \frac{u(T)-1}{D}u(x)v(s), & 0 \le x \le s \le T, \end{cases}$$

is the Green function, the number D is defined by (4).

Lemma 2.2 [3] Under condition (H), the Green's function $G(x,s)$ of the boundary value problem (5) is positive, i.e., $G(x,s) > 0$ for $x,s \in [0,T]$.

In other words, the anti-maximum principle holds. Under this assumption, let us defined the function

$$\gamma(x) = \int_0^T G(x,s)e(s)ds,$$

which is the unique $T-$periodic solution of the linear equation

$$-[p(x)y']' + q(x)y = e(x).$$

We denote

$$A = \min_{0 \le s,x \le T} G(x,s), \quad B = \max_{0 \le s,x \le T} G(x,s), \quad \sigma = A/B. \tag{6}$$

Thus $B > A > 0$ and $0 < \sigma < 1$.

Remark 2.3 The existence results are based on the positivity of $G(x,s)$, which plays a very important role in employing nonlinear alternative principle of Leray-Schauder. If $p(x) = 1, q(x) = m^2 > 0$, then the Green's function $G(x,s)$ of the boundary value problem (5) has the form

$$G(x,s) = \begin{cases} \frac{e^{m(x-s)} + e^{m(T-x+s)}}{2m(e^{mT}-1)}, & 0 \le s \le x \le T, \\ \frac{e^{m(s-x)} + e^{m(T+x-s)}}{2m(e^{mT}-1)}, & 0 \le x \le s \le T. \end{cases}$$

It is obvious that $G(x,s) > 0$ for $0 \le s,x \le T$, and a direct calculation shows that

$$A = \frac{e^{mT/2}}{m(e^{mT}-1)}, B = \frac{1+e^{mT}}{2m(e^{mT}-1)}, \sigma = \frac{2e^{mT/2}}{1+e^{mT}} < 1.$$

Let $X = C[0,T]$, we suppose that $f : [0,T] \times \mathbb{R} \to [0,\infty)$ is a continuous function. Define an operator:

$$(Ty)(x) = \int_0^T G(x,s)f(s,y(s))ds$$

for $y \in X$ and $x \in [0,T]$.

Lemma 2.4 T is well defined.

Proof. We only need to show that $T(X) \subset X$. Let $y \in X$, then we have

$$\begin{aligned}\min_{0\le x\le T}(Ty)(x) &= \min_{0\le x\le T}\int_0^T G(x,s)f(s,y(s))ds\\ &\ge A\int_0^T f(s,y(s))ds\\ &\ge \sigma \max_{0\le x\le T}\int_0^T G(x,s)f(s,y(s))ds\\ &= \sigma\|Ty\|.\end{aligned}$$

This implies that $T(X) \subset X$ and the proof is completed.

3. Main results

In this section, we state and prove the new existence results for (1). In order to prove our main results, the following nonlinear alternative of Leray-Schauder is need, which can be found in [11]. Let us define the function $\omega(x) = \int_0^T G(x,s)ds$ and use $\|\cdot\|_1$ denote the usual L^1- norm over $(0,T)$, by $\|\cdot\|$ the supremum norm of $C[0,T]$. For a given function $a \in L^1[0,T]$ essentially bounded, we denote the essential supremum and infimum of a by a^* and a_*, respectively.

Lemma 3.1 Assume Ω is a relatively compact subset of a convex set E in a normed space X. Let $T : \overline{\Omega} \to E$ be a compact map with $0 \in \Omega$. Then one of the following two conclusions holds:

(i) T has at least one fixed point in $\overline{\Omega}$.

(ii) There exist $u \in \partial\Omega$ and $0 < \lambda < 1$ such that $u = \lambda Tu$.

Now we present our main existence result of positive solution to problem (1).

Theorem 3.2 Suppose that (1) satisfies (H) and $f(x,y)$ satisfies the following.

(H_1) There exists constants $\sigma > 0$ and $\nu \ge 1$ such that

$$f(x,y) \ge \sigma x^{-\nu} \quad \text{for all } x \in [0,T].$$

(H_2) There exist continuous, non-negative functions $g(y)$, $h(y)$ such that

$$f(x,y) \le g(y) + h(y), \quad \text{for all } (x,y) \in [0,T] \times (0,\infty],$$

where $g(y) > 0$ is non-increasing, $h(y)/g(y)$ is non-decreasing in $y \in (0,\infty)$.

(H$_3$) There exists a positive number r such that $\sigma r + \gamma_* > 0$, and

$$\frac{r}{g(\sigma r+\gamma_*)\left\{1+\frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\}} > \|\omega\|.$$

Then for each $e \in C(\mathbb{R}/T\mathbb{Z},\mathbb{R})$, (1) has at least one positive T-periodic solution y with $y(x) > \gamma(x)$ for all x and $0 < \|y-\gamma\| \le r$.

Proof. Since (H$_3$) holds, let $N_0 = \{n_0, n_0+1, \cdots\}$, we can choose $n_0 \in \{1,2,\cdots\}$ such that $\frac{1}{n_0} < \sigma r + \gamma_*$ and

$$\|\omega\| g(\sigma r+\gamma_*)\left\{1+\frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\} + \frac{1}{n_0} < r.$$

To show (1) has a positive solution, we should only show that

$$-[p(x)y']' + q(x)y = f(x, y(x)+\gamma(x)), \tag{7}$$

has a positive solution y satisfying (2). If it is right, then $k(x) = y(x)+\gamma(x)$ is a solution of (1) since

$$\begin{aligned} -[p(x)k']' + q(x)k &= -[p(x)(y(x)+\gamma(x))']' + q(x)(y(x)+\gamma(x)) \\ &= f(x,k(x)) + e(x). \end{aligned}$$

Consider the family of equations

$$-[p(x)y']' + q(x)y = \lambda f_n(x, y(x)+\gamma(x)) + \frac{q(x)}{n}, \tag{8}$$

where $\lambda \in [0,1]$, $n \in N_0$ and

$$f_n(x,y) = \begin{cases} f(x,y) & \text{if } y \ge 1/n, \\ f(x,1/n) & \text{if } y \le 1/n. \end{cases}$$

Problem (8)-(2) is equivalent to the following fixed point of the operator equation

$$y = T_n y \tag{9}$$

where T_n is a continuous and completely continuous operator defined by

$$T_n y(x) = \lambda \int_0^T G(x,s) f_n(s, y(s)+\gamma(x)) ds + \frac{1}{n},$$

and we used the fact

$$\int_0^T G(x,s)q(s)ds \equiv 1. \quad \text{(see Lemma 2.1 with } h=q\text{)}$$

Now we show $\|y\| \neq r$ for any fixed point y of (9). If not, assume that y is a fixed point of (9) such that $\|y\| = r$. Note that

$$\begin{aligned}
y(x) - \frac{1}{n} &= \lambda \int_0^T G(x,s) f_n(s, y(s)+\gamma(s))ds \\
&\geq \lambda A \int_0^T f_n(s, y(s)+\gamma(s))ds \\
&= \sigma B \lambda \int_0^T f_n(s, y(s)+\gamma(s))ds \\
&\geq \sigma \max_{x\in[0,T]} \left\{ \lambda \int_0^T G(x,s) f_n(s, y(s)+\gamma(s))ds \right\} \\
&= \sigma \|y - \frac{1}{n}\|.
\end{aligned}$$

By the choice of n_0, $\frac{1}{n} \leq \frac{1}{n_0} < \sigma r + \gamma_*$. Hence, we have

$$y(x) \geq \sigma \|y - \frac{1}{n}\| + \frac{1}{n} \geq \sigma(\|y\| - \frac{1}{n}) + \frac{1}{n} \geq \sigma r, \quad \text{for} \quad 0 \leq x \leq T. \tag{10}$$

Therefore,

$$y(x) + \gamma(x) \geq \sigma r + \gamma_* \geq \frac{1}{n}.$$

Thus from condition (H_2) , for all $x \in [0,T]$,

$$\begin{aligned}
y(x) &= \lambda \int_0^T G(x,s) f_n(s, y(s)+\gamma(s))ds + \frac{1}{n} \\
&= \lambda \int_0^T G(x,s) f(s, y(s)+\gamma(s))ds + \frac{1}{n} \\
&\leq \int_0^T G(x,s) f(s, y(s)+\gamma(s))ds + \frac{1}{n} \\
&\leq \int_0^T G(x,s) g(y(s)+\gamma(s)) \left\{ 1 + \frac{h(y(s)+\gamma(s))}{g(y(s)+\gamma(s))} \right\} + \frac{1}{n} \\
&\leq g(\sigma r + \gamma_*)) \left\{ 1 + \frac{h(r+\gamma^*)}{g(r+\gamma^*)} \right\} \int_0^T G(x,s)ds + \frac{1}{n} \\
&\leq g(\sigma r + \gamma_*)) \left\{ 1 + \frac{h(r+\gamma^*)}{g(r+\gamma^*)} \right\} \|\omega\| + \frac{1}{n_0}.
\end{aligned}$$

Therefore,

$$r = \|y\| \le g(\sigma r + \gamma_*)\left\{1 + \frac{h(r+\gamma^*)}{g(r+\gamma^*)}\right\}\|\omega\| + \frac{1}{n_0}.$$

This is a contradiction, so $\|y\| \neq r$.

Using Lemma 3.1, the Leray-Schauder alternative principle guarantees that

$$y = T_n y$$

has a fixed point, denoted by y_n, i.e., equation

$$-[p(x)y']' + q(x)y = f_n(x, y(x) + \gamma(x)) + \frac{q(x)}{n}, \tag{11}$$

has a periodic solution y_n with $\|y_n\| < r$.

In order to pass the solutions of the truncation equation (11) to that of the original equation (7), we need the fact $\|y_n'\|$ is bounded. Now we show that

$$\|y_n'\| \le L_1 r$$

for a solution $y(x)$ of equation (11).

Integrating (11) from 0 to T , we obtain

$$\int_0^T q(x)y_n(x)dx = \int_0^T \left[f_n(x, y_n(x) + \gamma(x)) + \frac{q(x)}{n}\right] dx.$$

Since $y_n(0) = y_n(T)$, there exists $x_0 \in [0, T]$ such that $y_n'(x_0) = 0$, therefore

$$\begin{aligned}
\left|p(x)y_n'(x)\right| &= \left|\int_{x_0}^x (p(s)y_n'(s))' ds\right| \\
&= \left|\int_{x_0}^x \left[q(s)y(s) - f_n(s, y_n(s) + \gamma(s)) - \frac{q(s)}{n}\right] ds\right| \\
&\le \int_0^T \left[q(s)y(s) + f_n(s, y_n(s) + \gamma(s)) + \frac{q(s)}{n}\right] ds \\
&= 2\int_0^T q(s)y_n(s)ds \\
&\le 2r\int_0^T q(s)ds.
\end{aligned}$$

So,

$$|y_n'(x)| \le \frac{2r\int_0^T q(s)ds}{p(x)} \le \frac{2r\|q\|_1}{\min\limits_{0\le x\le T} p(x)} := L_1 r.$$

In the next lemma, we will show that $y_n(x)+\gamma(x)$ have a uniform positive lower bound, i.e., there exists a constant $\delta > 0$, independent of $n \in N_0$, such that

$$y_n(x)+\gamma(x) \geq \delta \tag{12}$$

for all $n \in N_0$.

The fact $\|y_n\| < r$ and $\|y_n'\| \leq L_1 r$ show that $\{y_n\}_{n\in N_0}$ is a bounded and equi-continuous family on $[0,T]$. Thus the Arzela–Ascoli Theorem guarantees that $\{y_n\}_{n\in N_0}$ has a subsequence, $\{y_{n_i}\}_{i\in\mathbb{N}}$ converging uniformly on $[0,T]$ to a function $y \in X$. Moreover, y_{n_i} satisfies the integral equation

$$y_{n_i}(x) = \int_0^T G(x,s) f(s, y_{n_i}(s)+\gamma(s))ds + \frac{1}{n_i}.$$

Letting $i \to \infty$, we arrive at

$$y(x) = \int_0^T G(x,s) f(s, y(s)+\gamma(s))ds,$$

where the uniform continuity of $f(x,y)$ on $[0,T]\times[\delta, r+\gamma^*]$. Therefore, y is a positive periodic solution of (1).

Lemma 3.3 There exist a constant $\delta > 0$ such that any solution y_n of (11) satisfies (12) for all n large enough.

Proof. By condition (H_3), there exists $R_1 \in (0, R_0)$ and a continuous function $\widetilde{g}_0$ such that

$$f(x,y) - q(x)y \geq \sigma R_1^{-\nu} \geq \max\{q^*(r+\gamma^*), r\|q\|\} \tag{13}$$

for all $(x,y) \in [0,T]\times(0,R_1]$.

Choose $n_1 \in N_0$ such that $1/n_1 \leq R_1$ and let $N_1 = \{n_1, n_1+1, \ldots\}$. For $n \in N_1$, let

$$\alpha_n = \min_{0\leq x\leq T}[y_n(x)+\gamma(x)] \quad \text{and} \quad \beta_n = \max_{0\leq x\leq T}[y_n(x)+\gamma(x)].$$

We first show that $\beta_n > R_1$ for all $n \in N_1$. If not, suppose that $\beta_n \leq R_1$ for some $n \in N_1$. Then from (13), it is easy to verify

$$f_n(x, y_n(x)+\gamma(x)) > r\|q\|,$$

Integrating (11) from 0 to T, we deduce that

$$\begin{aligned} 0 &= \int_0^T \left(-[p(x)y_n'(x)]' + q(x)y_n(x) - f_n(x, y_n(x)+\gamma(x)) - \frac{q(x)}{n}\right) dx \\ &= \int_0^T q(x)y_n(x)dx - 1/n \int_0^T q(x)dx - \int_0^T f_n(x, y_n(x)+\gamma(x))dx \\ &< \int_0^T q(x)y_n(x)dx - r\|q\|T \le 0. \end{aligned}$$

This is a contradiction. Thus $\beta_n > R_1$.

To prove (12), we first show

$$y_n(x) + \gamma(x) \ge \frac{1}{n}, \quad 0 \le x \le T \quad \text{for} \quad n \in N_1. \tag{14}$$

Let $N_1 = P \cup Q$; here $\alpha_n \ge R_1$ if $n \in P$, and $\alpha_n < R_1$ if $n \in Q$. If $n \in P$, it is easy to verify (14) is satisfied. We now show (14) holds if $n \in Q$. If not, suppose there exists $n \in Q$ with

$$\alpha_n = \min_{0 \le x \le T}[y_n(x) + \gamma(x)] = y_n(c_n) + \gamma(c_n) < \frac{1}{n}.$$

for some $c_n \in [0, T]$. As $\alpha_n = y_n(c_n) + \gamma(c_n) < R_1$, by $\beta_n > R_1$, there exists $c_n \in [0, T]$ (without loss of generality, we assume $a_n < c_n$) such that $y_n(a_n) + \gamma(a_n) = R_1$ and $y_n(x) + \gamma(x) \le R_1$ for $a_n \le x \le c_n$.

From (13), we easily show that

$$f_n(x, y(x)+\gamma(x)) > q(x)(y_n(x)+\gamma(x)) + e(x) \quad \text{for} \quad x \in [a_n, c_n].$$

Using Eq.(11) for $y_n(x)$, we have, for $x \in [a_n, c_n]$,

$$\begin{aligned} \left[-p(x)(y_n'(x)+\gamma'(x))\right]' &= -\left[p(x)y_n'(x)\right]' - \left[p(x)\gamma'(x)\right]' \\ &= -q(x)y_n(x) + f_n(x, y_n(x)+\gamma(x)) + q(x)/n + e(x) \\ &> q(x)/n \ge 0. \end{aligned}$$

As $y_n'(c_n) + \gamma'(c_n) = 0, p(x) > 0$, so $y_n'(x) + \gamma'(x) < 0$ for all $x \in [a_n, c_n)$ and the function $\nu_n := y_n + \gamma$ is strictly decreasing on $[a_n, c_n]$. We use η_n to denote the inverse function of y_n restricted to $[a_n, c_n]$. Thus there exists $b_n \in (a_n, c_n)$ such that $y_n(b_n) + \gamma(b_n) = \frac{1}{n}$ and

$$y_n(x) + \gamma(x) \le \frac{1}{n} \quad \text{for} \quad c_n \ge x \ge b_n, \quad \frac{1}{n} \le y_n(x) + \gamma(x) \le R_1 \quad \text{for} \quad b_n \ge x \ge a_n.$$

By using the method of substitution we obtain

$$\begin{aligned}
\int_{1/n}^{R_1} f(\eta_n(\nu),\nu)d\nu &= \int_{b_n}^{a_n} f(x,y_n(x)+\gamma(x))(y_n'(x)+\gamma'(x))dx \\
&= \int_{b_n}^{a_n} f_n(x,y_n(x)+\gamma(x))(y_n'(x)+\gamma'(x))dx \\
&= \int_{b_n}^{a_n} (-[p(x)(y_n'(x))]'+q(x)y_n'(x)-q(x)/n)(y_n'(x)+\gamma'(x))dx \\
&= \int_{b_n}^{a_n} (-[p(x)(y_n'(x))]'(x)(y_n'(x)+\gamma'(x))dx+\int_{b_n}^{a_n} (q(x)y_n(x)-q(x)/n)(y_n'(x)+\gamma'(x))dx.
\end{aligned}$$

By the facts $\|y_n\|<r$, $\|y_n'\|$ are bounded, one can easily obtain that the right side of the above equality is bounded. As a consequence, there exists $L>0$ such that

$$\int_{1/n}^{R_1} f(\eta_n(y),y)dy \le L.$$

On the other hand, by (H_1), we can choose $n_2 \in N_1$ large enough such that

$$\int_{1/n}^{R_1} f(\eta_n(y),y)dy \ge \sigma \int_{1/n}^{R_1} y^{-\nu}dy > L$$

for all $n \in N_2 = \{n_2, n_2+1, \ldots\}$. This is a contradiction. So (14) hold.

Finally, we will show that (12) is right in $n \in Q$. Notice that estimate (14) and employ the method of substitution, we obtain

$$\begin{aligned}
\int_{\alpha_n}^{R_1} f(\eta_n(y),y)dy &= \int_{c_n}^{a_n} f(x,y_n(x)+\gamma(x))(y_n'(x)+\gamma'(x))dx \\
&= \int_{c_n}^{a_n} f_n(x,y_n(x)+\gamma(x))(y_n'(x)+\gamma'(x))dx \\
&= \int_{c_n}^{a_n} (-[p(x)y_n'(x)]'+q(x)y_n(x)-q(x)/n)(y_n'(x)+\gamma'(x))dx.
\end{aligned}$$

Obviously, the right-hand side of the above equality is bounded. On the other hand, by (H_1),

$$\int_{\alpha_n}^{R_1} f(\eta_n(y),y)dy \ge \sigma \int_{\alpha_n}^{R_1} y^{-\nu}dy \to +\infty$$

if $\alpha_n \to 0^+$. Thus we know that $\alpha_n \ge \delta$ for some constant $\delta > 0$, the proof is completed.

Corollary 3.4 Assume that there exist continuous functions $d, \hat{d}$ and $\lambda > 0$ such that

$$0 \le \frac{\hat{d}(x)}{y^\lambda} \le f(x,y) \le \frac{d(x)}{y^\lambda} \text{for all} y > 0 and x \in [0,T].$$

Then problem (1) has at least one positive solution.

Proof. We will apply Theorem 3.1, (H_1) and (H_2) are satisfied if we take

$$h(y)=0, \quad g(y)=d(x)y^{-\lambda}.$$

The existence condition (H_3) become

$$r(\sigma r+\gamma_*)^{\lambda} > \sup_{0\le x\le T}\int_0^1 G(x,s)d(s)ds \tag{15}$$

for some $r>0$. Since $\lambda>0$ and $y(x)>0$, we can choose $r>0$ large enough such that (15) is satisfied.

Corollary 3.5 Let the nonlinearity in (1) be

$$f(x,y)=b(x)y^{-\alpha}+\mu c(x)y^{\beta}, \quad 0\le x\le T,$$

where $\alpha>0, \beta\ge 0, b(x), c(x)\in C[0,T]$ are non-negative functions and $b(x)>0$ for all x, and μ is a positive parameter. Then

(i) if $\beta<1$, then (3) has at least one positive solution for each $\mu>0$,

(ii) if $\beta\ge 1$, then (3) has at least one positive solution for each $0<\mu<\mu_1$, where μ_1 is some positive constant.

Proof. We will apply Theorem 3.1. If we take

$$g(y)=b_0y^{-\alpha}, \quad h(y)=\mu c_0 y^{\beta},$$

where

$$b_0=\max_{0\le x\le T} b(x)>0, \quad c_0=\max_{0\le x\le T} c(x)>0,$$

then (H_2) is satisfied.

Now the existence condition (H_3) become

$$u<\frac{r(\sigma r+\gamma_*)^{\alpha}-\|w\|b_0}{\|w\|c_0(r+\gamma^*)^{\alpha+\beta}}.$$

for some $r>0$, so (3) has at least one positive periodic solution for

$$0<\mu<\mu_1:=\sup_{r>0}\frac{r(\sigma r+\gamma_*)^{\alpha}-\|w\|b_0}{\|w\|c_0(r+\gamma^*)^{\alpha+\beta}}.$$

Note that $\mu_1=\infty$ if $\beta<1$ and $\mu_1<\infty$ if $\beta\ge 1$. We have the desired results.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (No.11461016), the Scientific Research Foundation of Hainan University (kyqd1544, hdjy1639), Hainan Natural Science Foundation(No.20167246), Scientific Research Development Fund of Wenzhou Medical University(No:QTJ11014) and Zhejiang Provincial Department of Education Research Project(No:Y201328047).

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