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# UNIQUE COMMON FIXED POINTS FOR GENERALIZED MIXED TYPE CONTRACTIVE MAPPINGS ON NON-NORMAL CONE METRIC SPACES

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**Abstract.** Two classes of generalized mixed type contractive conditions are introduced, unique common fixed point theorems for Kannan-mixed type and Chatterjea-mixed type contractive mappings are obtained on non-normal cone metric spaces respectively and the corresponding fixed point theorems and common fixed point theorems are given. The obtained results generalize and improve some common fixed point theorems, especially, Kannan type fixed point theorem and Chatterjea type fixed point theorem.

**Keywords:** cone metric space; mixed type contractive condition; common fixed point; fixed point.

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## 1. Introduction and Preliminaries

In 1968, Kannan[1] obtained the following generalization of Banach contraction principle, that is., Kannam fixed pint theorem:

**Theorem 1.1** Let$X$ be a complete metric space, $f : X \to X$ a map. If there exists $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha [d(x, fx) + d(y, fy)].$$

Then $f$ has a unique fixed point.

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In 2011, Shukla and Tiwari[2] gave thefollowing variant form of Theorem 1.1 as follows:

**Theorem 1.2** Let $X$ be a complete metric space, $f : X \to X$ a map. If there exists $\alpha \in [0, \frac{1}{3})$ such that for each $x, y \in X$,

$$d(fx, fy) \leq \alpha [d(x, fx) + d(y, fy) + d(x, y)].$$

Then $f$ has a unique fixed point.

In 1972, Chatterjea[3] obtain another generalization of Banach contraction principle, that is., Chatterjea fixed point theorem:

**Theorem 1.3** Let $X$ be a complete metric space, $f : X \to X$ a map. If there exists $\alpha \in [0, \frac{1}{2})$ such that $x, y \in X$,

$$d(fx, fy) \leq \alpha [d(x, fy) + d(y, fx)].$$

Then $f$ has a unique fixed point.

In 2010 and 2014, The authors in [4] and [5] used the concept of subsequence convergence[3-4] to obtain the generalizations of Kannan' fixed point theorem as follows respectively:

**Theorem 1.4** Let $X$ be a complete metric space, $T, S : X \to X$ two maps such that $T$ is one to one, continuous and subsequence convergence. If there exists $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$d(TSx, TSy) \leq \alpha [d(Tx, TSx) + d(Ty, TSy)].$$

Then $S$ has a unique fixed point.

**Theorem 1.5** Let $X$ be a complete metric space, $T, f : X \to X$ two maps such that $T$ is one to one, continuous and subsequence convergence. If there exists $\alpha \in [0, \frac{1}{2})$ such that for each $x, y \in X$,

$$F(d(Tfx, Tfy)) \leq \alpha [F(d(Tx, Tfx)) + F(d(Ty, Tfy))],$$

where $F : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function satisfying $F^{-}(0) = \{0\}$. Then $f$ has a unique fixed point.

In 2007, Haung and Zhang[6] introduced the concept of cone metric spaces and obtained several fixed point theorems on normal cone metric spaces. From then, some authors generalized and improved the corresponding results in [6] and discussed the existence problems of fixed points and common fixed points for mappings satisfying contractive or expansive conditions on normal or non-normal cone metric spaces and CMTS[7], see [8-16]. The author in [17] introduced the concept of mixed expansive condition and discussed the existence problems of common fixed point for two mappings satisfying the mixed expansive conditions on cone metric spaces, and further discussed the existence problems of common fixed points for two mappings satisfying generalized II-expansive conditions.

In this paper, by introducing two generalized mixed contractive conditions, we will discuss the existence problems of common fixed point for two mappings and give the existence theorems of common fixed points for mappings with Kannan type and Chatterjea type contractive conditions and also give some new fixed point theorems. The obtained (common) fixed point theorems generalize and improve some well-known (common) fixed point theorems, particularly, Kannan' fixed point theorem(i.e., Theorem 1.1) and Chatterjea' fixed point theorem(i.e., Theorem 1.3).

Suppose that $E$ is a real Banach space. The subset $P$ of $E$ is called a cone if and only if $P$ satisfies the following conditions:

(i) $P$ is closed, nonempty and $P \neq \{0\}$;

(ii) for any $a, b \in \mathbb{R}^+ = [0, \infty)$ and $x, y \in P$, $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on $E$ with respective to $P$ by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$ (interior of $P$). In this paper, we assume that $\text{int}P \neq \emptyset$.

The cone $P$ is called normal if there is a number $L > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L\|y\|.$$

The least positive number satisfying the above is then called the normal constant of $P$.

**Definition 1.1** Let $X$ be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, z, y \in X$.

Then $d$ is called a cone metric on $X$, and $(X,d)$ is called a cone metric space. Obviously, A cone metric space is bigger than a metric space.

Let $(X,d)$ be a cone metric space. We say that $\{x_n\}$ is

(e) Cauchy sequence if for every $c \in E$ with $0 \ll c$, there exists an $N$ such that for all $n,m > N$, $d(x_m,x_n) \ll c$;

(f) convergent sequence if for every $c \in E$ with $0 \ll c$, there exists an $N$ such that for all $n > N$ such that $d(x_n,x) \ll c$ for some $x \in E$.

(g) A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$ .

Let $E$ and $F$ are both real Banach spaces, $P$ and $Q$ are cones in $E$ and $F$ respectively, $(X,d)$ and $(Y,\rho)$ are two cone metric spaces with cones $E$ and $F$ respectively, where $d : X \times X \to E$ and $\rho : Y \times Y \to F$ satisfy Definition 1.1. We say that a mapping $f : X \to Y$ is continuous at $x^* \in X$ if for any $c \in F$ with $0 \ll c$, there exists $b \in E$ with $0 \ll b$ such that $d(x,x^*) \ll b$ implies $\rho(fx, fx^*) \ll c$. If $f$ is continuous at any $x \in X$, then we call $f$ is continuous on $X$.

**Lemma 1.1**[14] If a sequence $\{x_n\}$ in a cone metric space $(X,d)$ is convergent, then the limits of $\{x_n\}$ is unique.

**Lemma 1.2**[14] Let $\{x_n\}$ be a sequence in a cone metric space $(X,d)$. If there exists real number sequence $\{a_n\}$ with $a_n \geq 0$ satisfying $\sum a_n < \infty$ and $d(x_n,x_{n+1}) \leq a_n M, \forall n \in \mathbb{N}$, where $M \in P$, then $\{x_n\}$ is a Cauchy sequence.

**Lemma 1.3**[7] Let $(X,d)$ be a cone metric space. Suppose that $a,b,c \in E$, then the following properties hold:

($p_1$) If $a \leq b$ and $b \ll c$, then $a \ll c$,

($p_2$) If $0 \leq a \ll c$ for any $c \in \mathrm{int}P$, then $a = 0$,

($p_3$) If $a \leq ka$, where $a \in P$ and $0 \leq k < 1$, then $a = 0$.

**Lemma 1.4**[7] Let $(X,d)$ and $(Y,\rho)$ be two cone metric spaces and $f : X \to Y$ a map. Then $f$ is continuous at $x^* \in X \Longleftrightarrow$ if any sequence $\{x_n\}$ in $X$ converges to $x^*$, then $fx_n$ converges to $fx^*$.

## 2. Common fixed points

We first give a unique common fixed point theorem for two mappings satisfying a generalized mixed contractive condition.

**Theorem 2.1** Let $(X,d)$ be a complete cone metric space, $f,g : X \to X$ two mappings. Suppose that there exist $\alpha,\beta,\gamma \geq 0$ with $\alpha+\beta+\gamma < 1$ and for any $x,y \in X$ with $x \neq y$,

$$d(fgx,gfy) \leq \alpha\, d(fy,gfy) + \beta\, d(gx,fgx) + \gamma d(fy,gx). \tag{2.1}$$

If $f$ or $g$ is surjective, then $f$ and $g$ have a unique common fixed point.

**Proof.** Take any element $x_0 \in X$ and construct a sequence $\{x_n\}_0^\infty$ satisfying

$$x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, n = 0,1,\cdots.$$

If there exists $n$ such that $x_{2n} = x_{2n+1}$, then $d(x_{2n},x_{2n+1}) = 0$, hence by (2.1),

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2}) = d(fgx_{2n-1},gfx_{2n})\\
\leq&\alpha\, d(fx_{2n},gfx_{2n}) + \beta\, d(gx_{2n-1},fgx_{2n-1}) + \gamma(fx_{2n},gx_{2n-1})\\
=&\alpha\, d(x_{2n+1},x_{2n+2}) + \beta\, d(x_{2n},x_{2n+1}) + \gamma(x_{2n+1},x_{2n})\\
=&\alpha\, d(x_{2n+1},x_{2n+2}),
\end{aligned}$$

so $d(x_{2n+1},x_{2n+2}) = 0$ since $\alpha < 1$. Therefore $x_{2n} = x_{2n+1} = x_{2n+2}$. Obviously, $x_{2n}$ is a common fixed point of $f$ and $g$. Similarly, if there exists $n$ such that $x_{2n+1} = x_{2n+2}$, then $x_{2n+1} = x_{2n+2} = x_{2n+3}$, hence $x_{2n+1}$ is a also common fixed point of $f$ and $g$. Hence we may assume that $x_n \neq x_{n+1}$ for all $n = 0,1,2\cdots$.

For any fixed $n = 0,1,2,\cdots$,

$$\begin{aligned}
&d(x_{2n+2},x_{2n+3}) = d(gfx_{2n},fgx_{2n+1})\\
\leq&\alpha\, d(fx_{2n},gfx_{2n}) + \beta\, d(gx_{2n+1},fgx_{2n+1}) + \gamma d(fx_{2n},gx_{2n+1})\\
=&(\alpha+\gamma)\, d(x_{2n+1},x_{2n+2}) + \beta\, d(x_{2n+2},x_{2n+3}),\\
&d(x_{2n+1},x_{2n+2}) = d(gfx_{2n},fgx_{2n-1})\\
\leq&\alpha\, d(fx_{2n},gfx_{2n}) + \beta\, d(gx_{2n-1},fgx_{2n-1}) + \gamma d(fx_{2n},gx_{2n-1})\\
=&\alpha\, d(x_{2n+1},x_{2n+2}) + (\beta+\gamma) d(x_{2n},x_{2n+1}).
\end{aligned}$$

Combining the above two results, we have

$$d(x_{n+1},x_{n+2}) \le h\,d(x_n,x_{n+1}), \forall n=0,1,2,\cdots. \tag{2.2}$$

where $h=\max\{\frac{\alpha+\gamma}{1-\beta},\frac{\beta+\gamma}{1-\alpha}\}\in[0,1)$. Hence by the mathematical induction, we obtain

$$d(x_n,x_{n+1}) \le h^n\,d(x_0,x_1), \forall n=0,1,2,\cdots. \tag{2.3}$$

So there is $u\in X$ such that $x_n\to u$ as $n\to\infty$ by (2.3), Lemma 1.2 and the completeness of $X$.

If $f$ is surjective, then there exists $v\in X$ such that $u=fv$. By (2.1),

$$\begin{aligned}
&d(u,gu)=d(u,gfv)\\
\le& d(u,x_{2n+3})+d(x_{2n+3},gfv)=d(u,x_{2n+3})+d(fgx_{2n+1},gfv)\\
\le& d(u,x_{2n+3})+\alpha\,d(fv,gfv)+\beta\,d(gx_{2n+1},fgx_{2n+1})+\gamma d(fv,gx_{2n+1})\\
=& d(u,x_{2n+3})+\alpha\,d(u,gu)+\beta\,d(x_{2n+2},x_{2n+3})+\gamma d(u,x_{2n+2})\\
\le& d(u,x_{2n+3})+\alpha\,d(u,gu)+\beta\,[d(x_{2n+2},u)+d(u,x_{2n+3})]+\gamma d(u,x_{2n+2}),
\end{aligned}$$

hence

$$d(u,gu)\le \frac{1+\beta}{1-\alpha}d(u,x_{2n+3})+\frac{\beta+\gamma}{1-\alpha}d(u,x_{2n+2}).$$

Since $x_n$ converges to $u$, for any $c\gg 0$ there exists natural number $N$ such that for any $n>N$,

$$\frac{1+\beta}{1-\alpha}d(u,x_{2n+3})\ll\frac{c}{2},\ \frac{\beta+\gamma}{1-\alpha}d(u,x_{2n+2})\ll\frac{c}{2},$$

hence for $n>N$,

$$d(u,gu)\ll c.$$

so $u=gu$ by Lemma 1.3. Since $\beta<1$ and

$$\begin{aligned}
&d(u,fu)=d(gu,fu)=d(gfv,fgu)=d(fgu,gfv)\\
\le&\alpha d(fv,gfv)+\beta d(gu,fgu)+\gamma d(fv,gu)\\
=&\alpha d(u,gu)+\beta d(u,fu)+\gamma d(u,gu)=\beta d(u,fu),
\end{aligned}$$

we have $u=fu$. Hence $u$ is a common fixed point of $f$ and $g$.

Suppose that $u^*$ is another common fixed point of $f$ and $g$, then $d(u,u^*)>0$ and by (2.1),

$$d(u,u^*)=d(fgu,gfu^*)\le\alpha\,d(fu^*,gfu^*)+\beta\,d(gu,fgu)+\gamma d(fu^*,gu)=\gamma d(u,u^*).$$

Hence $d(u,u^*) = 0$ since $\gamma < 1$, this is a contradiction. So $u$ is the unique common fixed point of $f$ and $g$. Similarly, we can prove the same conclusion for the case of $g$ being surjective.

**Remark 2.1** If $\alpha = \beta = 0$, then Theorem 2.1 is the contractive version of Theorem 2.1 in [17]. But in [17], ont only $f$ and $g$ are both surjective, but also $f$ or $g$ is continuous.

Letting $\alpha = \beta$ and $\gamma = 0$ or $\alpha = \beta = \gamma$ in Theorem 2.1, we obtain the following common fixed point theorems for two mappings satisfying Kannan-mixed contractive condition and variant Kannan-mixed contractive condition respectively.

**Theorem 2.2** Let $(X,d)$ be a complete cone metric space, $f,g : X \to X$ two mappings. Suppose that there exists $\alpha \in [0,\frac{1}{2})$ such that

$$d(fgx,gfy) \le \alpha\,[d(fy,gfy)+d(gx,fgx)],\ \forall x,y \in X, x \neq y.$$

If $f$ or $g$ is surjective, then $f$ and $g$ have a unique common fixed point.

**Theorem 2.3** Let $(X,d)$ be a complete cone metric space, $f,g : X \to X$ two mappings. Suppose that there exists $\gamma \in [0,\frac{1}{3})$ such that

$$d(fgx,gfy) \le \gamma[d(fy,gfy)+d(gx,fgx)+d(fy,gx)],\ \forall x,y \in X, x \neq y.$$

If $f$ or $g$ is surjective, then $f$ and $g$ have a unique common fixed point.

**Remark 2.2** Using Theorem 2.2 and Theorem 2.3, we can obtain many fixed point theorems. For example, letting $f = 1_X$ or $g = 1_X$ in Theorem 2.2 and Theorem 2.3, we obtain that $g$ or $f$ has a unique fixed point. In this case, $g$ and $f$ need not be surjective since $f = 1_X$ or $g = 1_X$ is surjective.

The following result is a new version of Theorem 2.1 under the continuity of mappings .

**Theorem 2.4** Let $(X,d)$ be a complete cone metric space, $f,g : X \to X$ two continuous mappings. If there exist $\alpha,\beta,\gamma \ge 0$ such that $\alpha+\beta+\gamma < 1$ and (2.1) also holds for any $x,y \in X$ with $x \neq y$, then $f$ and $g$ have a unique common fixed point.

**Proof.** As the proof of Theorem 2.1, we can construct a sequence $\{x_n\}_0^\infty$ such that $x_n$ converges to $u$. Since $x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}$, using the continuity of $f$ and $g$ and Lemma 1.4, we obtain $u = fu = gu$. The uniqueness follows from the proof of Theorem 2.1.

**Remark 2.3** Theorem 2.4 needs the continuity of $f$ and $g$, but does not need the surjective condition of $f$ and $g$. Hence Theorem 2.1 and Theorem 2.4 are two different conclusions, they can not be comparable.

Letting $f=g$ and $\alpha=\beta=0$ or $\alpha=\beta,\gamma=0$ in Theorem 2.4, we obtain two fixed point theorems.

**Theorem 2.5** Let $(X,d)$ be a complete cone metric space, $f:X\to X$ continuous. If there exists $\alpha\in[0,1)$ such that

$$d(f^2x,f^2y)\leq\alpha\, d(fx,fy),\forall x,y\in X,\, x\neq y.$$

Then $f$ has a unique fixed point.

**Theorem 2.6** $(X,d)$ be a complete cone metric space, $f:X\to X$ continuous. If there exists $\alpha\in[0,\frac{1}{2})$ such that

$$d(f^2x,f^2y)\leq\alpha\,[d(fx,f^2x)+d(fy,f^2y)],\forall x,y\in X,\, x\neq y.$$

Then $f$ has a unique fixed point.

**Example 2.1** Let $E=C^1_{\mathbb{R}}[0,1]$, Define a norm on $E$ by $\parallel x\parallel=\parallel x\parallel_\infty+\parallel x'\parallel_\infty$ for each $x\in E$. Then $E$ is a Banach space. Let $P=\{x\in E: x\geq 0\}$, then $P$ is a non-normal cone subset in $E$[18].

Let $X=\{a,b,c\}$ and define $d: X\times X\to E$ as follows: for any $t\in[0,1]$,

$$d(a,a)(t)=d(b,b)(t)=d(c,c)(t)=0,\ d(a,b)(t)=d(b,a)(t)=2e^t,$$

$$d(a,c)(t)=d(c,a)(t)=3e^t,\ d(b,c)(t)=d(c,b(t))=3.5e^t.$$

Then obviously $(X,d)$ is a complete non-normal cone metric space. Define a mapping $f:X\to X$ as follows: $fa=a, fb=c, fc=a$, then $f$ is not surjective continuous mapping. Sine $f^2x=a$ for all $x\in X$, it is easy to know that $f$ satisfies the contractive condition in Theorem 2.6, hence $f$ has a unique fixed point $a$. On the other hand, for $x=a$ and $y=b$,

$$d(fa,fb)(t)=d(a,c)(t)=3e^t,\ [d(a,fa)+d(b,fb)](t)=d(b,c)(t)=3.5e^t,$$

hence $d(fa,fb)\leq\alpha\,[d(a,fa)+d(b,fb)]$ does not hold for any $\alpha\in[0,\frac{1}{2})$, that is., $f$ does not satisfy Kannan' contractive condition(i.e., the contractive condition in Theorem 1.1). Hence we can't use Kannan' fixed point theorem to determine the existence of fixed point of $f$. So

Theorem 2.6 generalize and improve Kannan' fixed point theorem. Similarly, Theorem 2.5 also generalize and improve Banach contraction principle.

Next, we will give another unique fixed point theorem for two mappings satisfying another generalized mixed contractive condition.

**Theorem 2.7** Let $(X,d)$ be a complete cone metric space, $f,g: X \to X$ two mappings. Suppose that there exist $\alpha,\beta,\gamma \geq 0$ such that $2\max\{\alpha,\beta\}+\gamma<1$ and for any $x,y \in X$ with $x \neq y$,

$$d(fgx,gfy) \leq \alpha\, d(fy,fgx)+\beta\, d(gx,gfy)+\gamma d(fy,gx). \tag{2.4}$$

If $f$ or $g$ is surjective, then $f$ and $g$ have a unique common fixed point

**Proof.** Take any $x_0 \in X$ and construct a sequence $\{x_n\}$ as follows:

$$x_{2n+1}=fx_{2n},\, x_{2n+2}=gx_{2n+1},\, n=0,1,\cdots.$$

If there exists $n$ such that $x_{2n}=x_{2n+1}$, then $d(x_{2n},x_{2n+1})=0$, hence by (2.4),

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2})=d(fgx_{2n-1},gfx_{2n})\\
\leq&\alpha\, d(fx_{2n},fgx_{2n-1})+\beta\, d(gx_{2n-1},gfx_{2n})+\gamma d(fx_{2n},gx_{2n-1})\\
=&\beta\,[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})]+\gamma d(x_{2n+1},x_{2n})\\
=&\beta\, d(x_{2n+1},x_{2n+2}),
\end{aligned}$$

which implies that $x_{2n+2}=x_{2n+1}=x_{2n}$ since $\beta<1$. It is easy to know that $x_{2n}$ is a common fixed point of $f$ and $g$. Similarly, if there exists $n$ such that $x_{2n+1}=x_{2n+2}$, then $x_{2n+1}$ is the common fixed point of $f$ and $g$. Hence we can assume that $x_n \neq x_{n+1}$ for all $n=0,1,2\cdots$.

For any fixed $n$, by (2.4),

$$\begin{aligned}
&d(x_{2n+2},x_{2n+3})=d(gfx_{2n},fgx_{2n+1})\\
\leq&\alpha\, d(fx_{2n},fgx_{2n+1})+\beta\, d(gx_{2n+1},gfx_{2n})+\gamma d(fx_{2n},gx_{2n+1})\\
=&(\alpha+\gamma)\, d(x_{2n+1},x_{2n+2})+\alpha\, d(x_{2n+2},x_{2n+3}),
\end{aligned}$$

hence

$$d(x_{2n+2},x_{2n+3}) \leq \frac{\alpha+\gamma}{1-\alpha} d(x_{2n+1},x_{2n+2}). \tag{2.5}$$

By (2,4) again,

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2}) = d(gfx_{2n},fgx_{2n-1})\\
\leq&\alpha\, d(fx_{2n},fgx_{2n-1}) + \beta\, d(gx_{2n-1},gfx_{2n}) + \gamma d(fx_{2n},gx_{2n-1})\\
=&\beta\, d(x_{2n+1},x_{2n+2}) + (\beta+\gamma)d(x_{2n},x_{2n+1}),
\end{aligned}$$

hence

$$d(x_{2n+1},x_{2n+2}) \leq \frac{\beta+\gamma}{1-\beta} d(x_{2n},x_{2n+1}). \tag{2.6}$$

Let $h = \{\frac{\alpha+\gamma}{1-\alpha}, \frac{\beta+\gamma}{1-\beta}\}$, then $h \in [0,1)$ and combining (2.5) and (2.6), we obtain

$$d(x_{n+1},x_{n+2}) \leq h\, d(x_n,x_{n+1}), \forall n = 0,1,2,\cdots. \tag{2.7}$$

Hence by using the mathematical induction, we have

$$d(x_n,x_{n+1}) \leq h^n\, d(x_0,x_1), \forall n = 0,1,2,\cdots. \tag{2.8}$$

So there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$ by (2.8), Lemma 1.2 and completeness of $X$.

If $f$ is surjective, then there exists $v \in X$ such that $u = fv$. By (2.4),

$$\begin{aligned}
&d(u,gu) = d(u,gfv)\\
\leq&d(u,x_{2n+3}) + d(x_{2n+3},gfv)\\
\leq&d(u,x_{2n+3}) + d(fgx_{2n+1},gfv)\\
\leq&d(u,x_{2n+3}) + \alpha\, d(fv,fgx_{2n+1}) + \beta\, d(gx_{2n+1},gfv) + \gamma d(fv,gx_{2n+1})\\
=&d(u,x_{2n+3}) + \alpha\, d(u,x_{2n+3}) + \beta\, d(x_{2n+2},gu) + \gamma d(u,x_{2n+2})\\
\leq&(1+\alpha)d(u,x_{2n+3}) + \beta[d(u,x_{2n+2}) + d(u,gu)] + \gamma d(u,x_{2n+2}),
\end{aligned}$$

hence

$$d(u,gu) \leq \frac{1+\alpha}{1-\beta} d(u,x_{2n+3}) + \frac{\beta+\gamma}{1-\beta} d(u,x_{2n+2}).$$

Since $x_n$ converges to $u$, for any $c \gg 0$ there exists $N \in \mathbb{N}$ such that for any $n > N$,

$$\frac{1+\alpha}{1-\beta} d(u,x_{2n+3}) \ll \frac{c}{2}, \qquad \frac{\beta+\gamma}{1-\beta} d(u,x_{2n+2}) \ll \frac{c}{2},$$

hence $d(u,gu) \ll c$. Therefore $u = gu$. And by (2.4) again,

$$\begin{aligned} d(u,fu) &= d(fu,gu) = d(fgu,gfv) \\ &\leq \alpha d(fv,fgu) + \beta d(gu,gfv) + \gamma d(fv,gu) \\ &= \alpha d(u,fu) + \beta d(gu,gu) + \gamma d(u,gu) = \alpha d(u,fu), \end{aligned}$$

hence $u = fu$ since $\alpha \in [0,1)$. So $u$ is the common fixed point of $f$ and $g$.

If $u^*$ is another common fixed point of $f$ and $g$, then $u \neq u^*$. By (2.4),

$$d(u,u^*) = d(fgu,gfu^*) \leq \alpha\, d(fu^*,fgu) + \beta\, d(gu,gfu^*) + \gamma d(fu^*,gu) \leq (\alpha+\beta+\gamma)\, d(u.u^*).$$

Hence $d(u,u^*) = 0$ since $\alpha+\beta+\gamma \leq 2\max\{\alpha,\beta\}+\gamma < 1$, which is a contradiction. Therefore $u$ is the unique common fixed point of $f$ and $g$. Similarly, we can obtain the same result for the case of $g$ being surjective.

Using Theorem 2.7, we obtain following common fixed point theorems for two mappings satisfying Chatterjea-mixed contractive condition and variant Chatterjea-mixed contractive condition respectively.

**Theorem 2.8** Let $(X,d)$ be a complete cone metric space, $f,g : X \to X$ two mappings. Suppose that there exists $\alpha \in [0,\frac{1}{2})$ such that

$$d(fgx,gfy) \leq \alpha\,[d(fy,fgx) + d(gx,gfy)], \forall x,y \in X, x \neq y.$$

If $f$ or $g$ is surjective, then $f$ and $g$ have a unique common fixed point.

**Proof** Let $\alpha = \beta, \gamma = 0$ in Theorem 2.7, then the conclusion follows from Theorem 2.7.

**Theorem 2.9** Let $(X,d)$ be a complete cone metric space, $f,g : X \to X$ two mappings. Suppose that there exists $\gamma \in [0,\frac{1}{3})$ such that

$$d(fgx,gfy) \leq \gamma[d(fy,fgx) + d(gx,gfy) + d(fy,gx)], \forall x,y \in X, x \neq y.$$

If $f$ or $g$ is surjective, then $f$ and $g$ have a unique common fixed point.

**Proof** Let $\alpha = \beta = \gamma$ in Theorem 2.7, then the conclusion follows from Theorem 2.7.

Similarly, we give a new version of Theorem 2.7 under the continuous condition of the mappings.

**Theorem, 2.10** Let $(X,d)$ be a complete cone metric space, $f,g:X\to X$ two continuous mappings. If there exists $\alpha,\beta,\gamma\geq 0$ such that $2\max\{\alpha,\beta\}+\gamma<1$ and (2.4) holds for any $x,y\in X$ with $x\neq y$. Then $f$ and $g$ have a unique common fixed point.

Taking $f=g$ and $\alpha=\beta,\gamma=0$ in Theorem 2.10, we obtain the following generalized and improved form of Theorem 1.3.

**Theorem 2.11** Let $(X,d)$ be a complete cone metric space, $f:X\to X$ a continuous mapping. If there exists $\alpha\in[0,\frac{1}{2})$ such that

$$d(f^2x,f^2y)\leq\alpha\,[d(fx,f^2y)+d(fy,f^2x)],\forall\,x,y\in X,\,x\neq y.$$

Then $f$ has a unique fixed point.

**Example 2.2** Consider the cone metric space $X$ and the mapping $f$ in Example 2.1. Obviously, $f$ satisfies all conditions of Theorem 2.11, hence $f$ has a unique fixed point $a$. But for $x=b,y=c$,

$$d(fb,fc)(t)=d(c,a)(t)=3e^t,\ [d(b,fc)+d(c,fb)](t)=d(b,a)(t)+d(c,c)(t)=2e^t,$$

hence $d(fb,fc)\leq\alpha\,[d(b,fc)+d(c,fb)]$ does not hold for any $\alpha\in[0,\frac{1}{2})$, that is., $f$ does not satisfy Chatterjea' contractive condition(i.e., the contractive condition in Theorem 1.3). Hence we can't use Chatterjea' fixed point theorem to determine the existence of fixed point of $f$. So Theorem 2.11 generalize and improve Chatterjea' fixed point theorem.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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