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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR n-TIME DIFFERENTIABLE FUNCTIONS WHICH ARE GENERALIZED (s,m)-PREINVEX FUNCTIONS

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Abstract. Some inequalities of Hermite-Hadamard type for n-time differentiable functions which are generalized (s,m)-preinvex functions are obtained. These results not only extend the results appeared in the literature (see [1]), but also provide new estimates on these types.

Keywords: Hermite-Hadamard type inequality; Hölder's inequality; convex functions; s-convex function in the second sense; m-invex.

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1. Introduction and Preliminaries

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$.

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The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a,b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

In (see [2]) Hermite-Hadamard inequality (1.1) was refined as follows.

Theorem 1.2. *Let $f(x)$ is differentiable on $[a,b]$ such that $|f'(x)|^q$ for $q \geq 1$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q} \tag{1.2}$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}. \tag{1.3}$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [11]) and the references cited therein, also (see [10]) and the references cited therein. For more information on refinements, extensions, generalizations, and other things about Hermite-Hadamard inequality (1.1), please (see [1]) and related references therein.

Now, let us recall some definitions of various convex functions.

Definition 1.3. (see [3]) A function $f : \mathbb{R}_\circ \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x+(1-\lambda)y) \leq \lambda^s f(x)+(1-\lambda)^s f(y) \tag{1.4}$$

for all $x,y \in \mathbb{R}_\circ$, $\lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in (see [3]).

Definition 1.4. (see [7]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details please (see [7], [8]) and the references therein.

Definition 1.5. (see [9]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y). \tag{1.5}$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.6. (see [6]) Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$. For $f : K \longrightarrow \mathbb{R}$ $x, y \in K$ and some fixed $s, m \in (0,1]$, if

$$f(mx + \lambda\eta(y,x,m)) \leq m(1-\lambda)^s f(x) + \lambda^s f(y) \tag{1.6}$$

is valid for all $x, y \in K, \lambda \in [0,1]$, then we say that $f(x)$ is a generalized (s,m)-preinvex function with respect to η.

In (see [4]), the inequality (1.2) was generalized to the case for s-convex functions in the second sense, which can be restated as follows.

Theorem 1.7. *If $f(x)$ is a differentiable function on $[a,b] \subseteq \mathbb{R}_\circ$ such that $f'(x) \in L_1[a,b]$ and $|f'(x)|^q$ for $q \geq 1$ is s-convex function in the second sense on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{(q-1)/q}\left[\frac{s+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q + |f'(b)|^q\right]^{1/q}. \tag{1.7}$$

Motivated by these results, in the present paper, we establish some Hermite-Hadamard type inequalities for n-time differentiable functions which are generalized (s,m)-preinvex functions with respect to η. So, new estimates on these types of Hermite-Hadamard inequalities via classical integrals are provided and the results of (see [1]) are generalized. At the end of the paper, some conclusions are given.

2. Main results

In this section, in order to prove our main results regarding some generalizations of Hermite-Hadamard type inequalities for generalized (s,m)-preinvex functions via classical integrals, we need the following two lemmas:

Lemma 2.8. *Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, $r \in [0,1]$ and let $a,b \in K$, $a < b$ with $ma < ma + \eta(b,a,m)$. Assume that $f : K \longrightarrow \mathbb{R}$ is a mapping such that $f^{(n-1)}(x)$ is absolutely continuous on K° and $f^{(n)}(x)$ for $n \in \mathbb{N}$ exists and is integrable on $[ma, ma + \eta(b,a,m)]$. Then, for each $x \in [ma, ma + \eta(b,a,m)]$, we have that*

$$\frac{f(mb) + rf(mb + \eta(a,b,m))}{r+1} - \frac{1}{\eta(a,b,m)} \int_{mb}^{mb+\eta(a,b,m)} f(x)dx$$
$$-\sum_{k=1}^{n-1} \frac{(-1)^k (k-r)\eta(a,b,m)^k}{(r+1)(k+1)!} f^{(k)}(mb + \eta(a,b,m))$$
$$= \frac{(-1)^n \eta(a,b,m)^n}{(r+1)n!} \int_0^1 t^{n-1}(n - (r+1)t) f^{(n)}(mb + t\eta(a,b,m))dt, \tag{2.8}$$

where in this paper, an empty sum is understood to be nil.

Proof. We only prove the cases when $n = 1$ and $n = 2$. The proof for $n \geq 3$ is by mathematical induction.

The case $n = 1$. Denote

$$I_n = \int_0^1 t^{n-1}(n - (r+1)t) f^{(n)}(mb + t\eta(a,b,m))dt.$$

By integration by parts, we get

$$I_1 = \int_0^1 (1 - (r+1)t) f'(mb + t\eta(a,b,m))dt$$

$$= (1 - (r+1)t) \frac{f(mb + t\eta(a,b,m))}{\eta(a,b,m)} \Big|_0^1 + \frac{r+1}{\eta(a,b,m)} \int_0^1 f(mb + t\eta(a,b,m))dt$$

$$= -\left[\frac{f(mb)+rf(mb+\eta(a,b,m))}{\eta(a,b,m)}\right] + \frac{r+1}{\eta(a,b,m)^2}\int_{mb}^{mb+\eta(a,b,m)} f(x)dx.$$

So, we have

$$\frac{f(mb)+rf(mb+\eta(a,b,m))}{r+1} - \frac{1}{\eta(a,b,m)}\int_{mb}^{mb+\eta(a,b,m)} f(x)dx = \frac{(-1)\eta(a,b,m)}{r+1}I_1.$$

The case $n=2$. By integration by parts, we get

$$I_2 = \int_0^1 t(2-(r+1)t)f''(mb+t\eta(a,b,m))dt$$

$$= t(2-(r+1)t)\frac{f'(mb+t\eta(a,b,m))}{\eta(a,b,m)}\Big|_0^1 - \frac{2}{\eta(a,b,m)}\int_0^1 (1-(r+1)t)f'(mb+t\eta(a,b,m))dt.$$

So, we have the following recurrent relation

$$I_2 = \frac{(1-r)f'(mb+\eta(a,b,m))}{\eta(a,b,m)} - \frac{2}{\eta(a,b,m)}I_1. \tag{2.9}$$

Using I_1 in equation (2.9), we obtain the following equality

$$\frac{f(mb)+rf(mb+\eta(a,b,m))}{r+1} - \frac{1}{\eta(a,b,m)}\int_{mb}^{mb+\eta(a,b,m)} f(x)dx$$

$$+\frac{1-r}{2(r+1)}\eta(a,b,m)f'(mb+\eta(a,b,m)) = \frac{\eta(a,b,m)^2}{2(r+1)}I_2.$$

□

Remark 2.9. If we take $m=r=1$ and $\eta(a,b,m)=a-mb$ in Lemma 2.8, then equality (2.8) becomes equality as obtained in (see [1], Lemma 2.1).

Lemma 2.10. *Let $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$, and let $a,b\in K$, $a<b$ with $ma<ma+\eta(b,a,m)$. Assume that $f: K\longrightarrow\mathbb{R}$ is n-time differentiable function such that $f^{(n-1)}(x)$ for $n\in\mathbb{N}$ is absolutely continuous on $[ma, ma+\eta(b,a,m)]$. Then the identity*

$$\int_{ma}^{ma+\eta(b,a,m)} f(t)dt = \sum_{k=0}^{n-1}\left[\frac{(ma+\eta(b,a,m)-x)^{k+1}+(-1)^k(x-ma)^{k+1}}{(k+1)!}\right]f^{(k)}(x)$$

$$+(-1)^n\int_{ma}^{ma+\eta(b,a,m)} K_n(x,t,m)f^{(n)}(t)dt, \tag{2.10}$$

holds for all $x\in[ma,ma+\eta(b,a,m)]$*, where the kernel* $K_n:[ma,ma+\eta(b,a,m)]^2\times(0,1]\longrightarrow\mathbb{R}$ *is defined by*

$$K_n(x,t,m):=\begin{cases}\dfrac{(t-ma)^n}{n!}, & t\in[ma,x];\\ \dfrac{(t-ma-\eta(b,a,m))^n}{n!}, & t\in(x,ma+\eta(b,a,m)],\end{cases}\tag{2.11}$$

and n is a natural number, $n\geq 1$.

Proof. We only prove the cases when $n=1$ and $n=2$. The proof for $n\geq 3$ is by mathematical induction.

For $n=1$ we have to prove the equality

$$\int_{ma}^{ma+\eta(b,a,m)}f(t)dt=\eta(b,a,m)f(x)-\int_{ma}^{ma+\eta(b,a,m)}K_1(x,t,m)f'(t)dt,\tag{2.12}$$

where

$$K_1(x,t,m):=\begin{cases}t-ma, & t\in[ma,x];\\ t-ma-\eta(b,a,m), & t\in(x,ma+\eta(b,a,m)].\end{cases}\tag{2.13}$$

Integrating by parts, we have

$$\int_{ma}^{ma+\eta(b,a,m)}K_1(x,t,m)f'(t)dt=\int_{ma}^{x}(t-ma)f'(t)dt$$

$$+\int_{x}^{ma+\eta(b,a,m)}(t-ma-\eta(b,a,m))f'(t)dt=(x-ma)f(x)-\int_{ma}^{x}f(t)dt$$

$$-(x-ma-\eta(b,a,m))f(x)-\int_{x}^{ma+\eta(b,a,m)}f(t)dt.$$

Then equality (2.12) holds.

For $n=2$ we have to prove the equality

$$\int_{ma}^{ma+\eta(b,a,m)}f(t)dt=\eta(b,a,m)f(x)+\left[\frac{(ma+\eta(b,a,m)-x)^2-(x-ma)^2}{2!}\right]f'(x)$$

$$+\int_{ma}^{ma+\eta(b,a,m)}K_2(x,t,m)f''(t)dt,\tag{2.14}$$

where

$$K_2(x,t,m):=\begin{cases}\dfrac{(t-ma)^2}{2!}, & t\in[ma,x];\\ \dfrac{(t-ma-\eta(b,a,m))^2}{2!}, & t\in(x,ma+\eta(b,a,m)].\end{cases}\tag{2.15}$$

Integrating by parts, we have

$$\int_{ma}^{ma+\eta(b,a,m)} K_2(x,t,m) f''(t)dt$$
$$= \int_{ma}^{x} \frac{(t-ma)^2}{2!} f''(t)dt + \int_{x}^{ma+\eta(b,a,m)} \frac{(t-ma-\eta(b,a,m))^2}{2!} f''(t)dt$$
$$= \int_{ma}^{x} f(t)dt + \frac{(x-ma)^2}{2} f'(x) - (x-ma)f(x) + \int_{x}^{ma+\eta(b,a,m)} f(t)dt$$
$$- \frac{(x-ma-\eta(b,a,m))^2}{2} f'(x) + (x-ma-\eta(b,a,m))f(x).$$

Then equality (2.14) follows. □

Remark 2.11. If we take $m=1$ and $\eta(b,a,m) = b - ma$ in Lemma 2.10, then equality (2.10) becomes equality as obtained in (see [5], Lemma 2.1).

Now we are in a position to prove our two theorems.

Theorem 2.12. *Let $f : K = [ma, ma+\eta(b,a,m)] \longrightarrow \mathbb{R}$ be n-time differentiable function on $K \subseteq \mathbb{R}_\circ$ and let $a < b$ with $ma < ma+\eta(b,a,m)$. If $|f^{(n)}(x)|^p$ is a generalized (s,m)-preinvex function on K for $n \geq 2$ and $p \geq 1$, then for some fixed $r \in [0,1]$ and $s,m \in (0,1]$, we have*

$$\left| \frac{f(mb) + rf(mb+\eta(a,b,m))}{r+1} - \frac{1}{\eta(a,b,m)} \int_{mb}^{mb+\eta(a,b,m)} f(x)dx \right.$$
$$\left. - \sum_{k=1}^{n-1} \frac{(-1)^k (k-r)\eta(a,b,m)^k}{(r+1)(k+1)!} f^{(k)}(mb+\eta(a,b,m)) \right|$$

$$\leq \frac{|\eta(a,b,m)|^n}{(r+1)n!} \left(\frac{n-r}{n+1} \right)^{1-\frac{1}{p}} \left[P|f^{(n)}(a)|^p + Q|f^{(n)}(b)|^p \right]^{\frac{1}{p}}, \tag{2.16}$$

where

$$P = \frac{n(n+s-r) - s(r+1)}{(n+s)(n+s+1)}, \quad Q = m(n\beta(n,s+1) - (r+1)\beta(n+1,s+1)), \tag{2.17}$$

and

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt \tag{2.18}$$

for $x,y > 0$ is Euler Beta function.

Proof. It follows from Lemma 2.8 that

$$\left|\frac{f(mb)+rf(mb+\eta(a,b,m))}{r+1}-\frac{1}{\eta(a,b,m)}\int_{mb}^{mb+\eta(a,b,m)}f(x)dx\right.$$

$$\left.-\sum_{k=1}^{n-1}\frac{(-1)^k(k-r)\eta(a,b,m)^k}{(r+1)(k+1)!}f^{(k)}(mb+\eta(a,b,m))\right|$$

$$\leq\frac{|\eta(a,b,m)|^n}{(r+1)n!}\int_0^1 t^{n-1}(n-(r+1)t)|f^{(n)}(mb+t\eta(a,b,m))|dt. \tag{2.19}$$

When $p=1$, since $|f^{(n)}(x)|$ is a generalized (s,m)-preinvex function on K, we have

$$|f^{(n)}(mb+t\eta(a,b,m))|\leq t^s|f^{(n)}(a)|+m(1-t)^s|f^{(n)}(b)|.$$

Hence

$$\frac{|\eta(a,b,m)|^n}{(r+1)n!}\int_0^1 t^{n-1}(n-(r+1)t)|f^{(n)}(mb+t\eta(a,b,m))|dt$$

$$\leq\frac{|\eta(a,b,m)|^n}{(r+1)n!}\int_0^1 t^{n-1}(n-(r+1)t)\left[t^s|f^{(n)}(a)|+m(1-t)^s|f^{(n)}(b)|\right]dt$$

$$=\frac{|\eta(a,b,m)|^n}{(r+1)n!}\left[P|f^{(n)}(a)|+Q|f^{(n)}(b)|\right],$$

where P and Q are defined by (2.17). The proof for the case $p=1$ is complete.

When $p>1$, by the well-known Hölder's inequality, we obtain

$$\int_0^1 t^{n-1}(n-(r+1)t)|f^{(n)}(mb+t\eta(a,b,m))|dt\leq\left[\int_0^1 t^{n-1}(n-(r+1)t)dt\right]^{1-\frac{1}{p}}$$

$$\times\left[\int_0^1 t^{n-1}(n-(r+1)t)|f^{(n)}(mb+t\eta(a,b,m))|^p dt\right]^{\frac{1}{p}} \tag{2.20}$$

Since $|f^{(n)}(x)|^p$ is a generalized (s,m)-preinvex function on K, we have

$$|f^{(n)}(mb+t\eta(a,b,m))|^p\leq t^s|f^{(n)}(a)|^p+m(1-t)^s|f^{(n)}(b)|^p.$$

Therefore

$$\int_0^1 t^{n-1}(n-(r+1)t)|f^{(n)}(mb+t\eta(a,b,m))|^p dt$$

$$\leq\int_0^1 t^{n-1}(n-(r+1)t)\left[t^s|f^{(n)}(a)|^p+m(1-t)^s|f^{(n)}(b)|^p\right]dt$$

$$=P|f^{(n)}(a)|^p+Q|f^{(n)}(b)|^p. \tag{2.21}$$

From (2.19), (2.20) and (2.21), it follows that

$$\Bigg|\frac{f(mb)+rf(mb+\eta(a,b,m))}{r+1}-\frac{1}{\eta(a,b,m)}\int_{mb}^{mb+\eta(a,b,m)}f(x)dx$$
$$-\sum_{k=1}^{n-1}\frac{(-1)^k(k-r)\eta(a,b,m)^k}{(r+1)(k+1)!}f^{(k)}(mb+\eta(a,b,m))\Bigg|$$
$$\leq\frac{|\eta(a,b,m)|^n}{(r+1)n!}\left(\frac{n-r}{n+1}\right)^{1-\frac{1}{p}}\left[P|f^{(n)}(a)|^p+Q|f^{(n)}(b)|^p\right]^{\frac{1}{p}},$$

where P and Q are defined by (2.17). This completes the proof of Theorem 2.12. □

Remark 2.13. If we take $m=r=1$ and $\eta(a,b,m)=a-mb$ in Theorem 2.12, then inequality (2.16) becomes inequality as obtained in (see [1], Theorem 1.1).

Theorem 2.14. *Let $f:K=[ma,ma+\eta(b,a,m)]\longrightarrow\mathbb{R}$ be n-time differentiable function on $K\subseteq\mathbb{R}_\circ$ and let $a<b$ with $ma<ma+\eta(b,a,m)$. If $|f^{(n)}(x)|^p$ is a generalized (s,m)-preinvex function on K for $n\geq 1$ and $p\geq 1$, then for some fixed $s,m\in(0,1]$, we have*

$$\Bigg|\frac{1}{\eta(b,a,m)}\sum_{k=0}^{n-1}\left[\frac{\left[1+(-1)^k\right]\eta(b,a,m)^{k+1}}{2^{k+1}(k+1)!}\right]f^{(k)}\left(ma+\frac{\eta(b,a,m)}{2}\right)$$
$$-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(t)dt\Bigg|$$

(2.22)
$$\leq M\left[N|f^{(n)}(ma)|^p+\frac{m+1}{n+s+1}\left|f^{(n)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|^p+mN|f^{(n)}(ma+\eta(b,a,m))|^p\right]^{\frac{1}{p}},$$

where

$$M=\frac{1}{2n!}\left(\frac{2}{n+1}\right)^{1-\frac{1}{p}}\left(\frac{\eta(b,a,m)}{2}\right)^n \text{ and } N=\beta(n+1,s+1). \tag{2.23}$$

Proof. Choosing $x=ma+\frac{\eta(b,a,m)}{2}$ in Lemma 2.10 yields

$$\frac{1}{\eta(b,a,m)}\sum_{k=0}^{n-1}\left[\frac{\left[1+(-1)^k\right]\eta(b,a,m)^{k+1}}{2^{k+1}(k+1)!}\right]f^{(k)}\left(ma+\frac{\eta(b,a,m)}{2}\right)$$
$$-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(t)dt=\frac{(-1)^{n+1}}{n!\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}S(t,m)f^{(n)}(t)dt,$$

where

$$S(t,m) := \begin{cases} (t-ma)^n, & t \in \left[ma, ma + \frac{\eta(b,a,m)}{2}\right]; \\ (ma+\eta(b,a,m)-t)^n, & t \in \left(ma + \frac{\eta(b,a,m)}{2}, ma+\eta(b,a,m)\right]. \end{cases}$$

From this, we have

$$\left| \frac{1}{\eta(b,a,m)} \sum_{k=0}^{n-1} \left[\frac{\left[1+(-1)^k\right]\eta(b,a,m)^{k+1}}{2^{k+1}(k+1)!} \right] f^{(k)}\left(ma + \frac{\eta(b,a,m)}{2}\right) \right.$$

$$\left. -\frac{1}{\eta(b,a,m)} \int_{ma}^{ma+\eta(b,a,m)} f(t)dt \right| \leq \frac{1}{n!\eta(b,a,m)} \int_{ma}^{ma+\eta(b,a,m)} |S(t,m)||f^{(n)}(t)|dt. \tag{2.24}$$

When $p = 1$, by (2.24), we have

$$\int_{ma}^{ma+\eta(b,a,m)} |S(t,m)||f^{(n)}(t)|dt = \int_{ma}^{ma+\frac{\eta(b,a,m)}{2}} (t-ma)^n |f^{(n)}(t)|dt$$

$$+ \int_{ma+\frac{\eta(b,a,m)}{2}}^{ma+\eta(b,a,m)} (ma+\eta(b,a,m)-t)^n |f^{(n)}(t)|dt.$$

Since $|f^{(n)}(t)|$ is a generalized (s,m)-preinvex function on K, we have

$$\int_{ma}^{ma+\eta(b,a,m)} |S(t,m)||f^{(n)}(t)|dt \leq \int_{ma}^{ma+\frac{\eta(b,a,m)}{2}} (t-ma)^n$$

$$\times \left\{ \left[\frac{ma + \frac{\eta(b,a,m)}{2} - t}{\frac{\eta(b,a,m)}{2}} \right]^s |f^{(n)}(ma)| + m \left[\frac{t-ma}{\frac{\eta(b,a,m)}{2}} \right]^s \left| f^{(n)}\left(ma + \frac{\eta(b,a,m)}{2}\right) \right| \right\} dt$$

$$+ \int_{ma+\frac{\eta(b,a,m)}{2}}^{ma+\eta(b,a,m)} (ma+\eta(b,a,m)-t)^n$$

$$\times \left\{ \left[\frac{ma+\eta(b,a,m)-t}{\frac{\eta(b,a,m)}{2}} \right]^s \left| f^{(n)}\left(ma + \frac{\eta(b,a,m)}{2}\right) \right| \right.$$

$$\left. + m \left[\frac{t-ma-\frac{\eta(b,a,m)}{2}}{\frac{\eta(b,a,m)}{2}} \right]^s \left| f^{(n)}(ma+\eta(b,a,m)) \right| \right\} dt$$

It is not difficult to calculate the above integrals. Then using inequality (2.24), we get

$$\left| \frac{1}{\eta(b,a,m)} \sum_{k=0}^{n-1} \left[\frac{\left[1+(-1)^k\right]\eta(b,a,m)^{k+1}}{2^{k+1}(k+1)!} \right] f^{(k)}\left(ma + \frac{\eta(b,a,m)}{2}\right) \right.$$

$$\left. -\frac{1}{\eta(b,a,m)} \int_{ma}^{ma+\eta(b,a,m)} f(t)dt \right|$$

$$\leq M\left[N|f^{(n)}(ma)|+\frac{m+1}{n+s+1}\left|f^{(n)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|+mN\left|f^{(n)}(ma+\eta(b,a,m))\right|\right]^{\frac{1}{p}},$$

where

$$M=\frac{1}{2n!}\left(\frac{\eta(b,a,m)}{2}\right)^{n} \text{ and } N=\beta(n+1,s+1).$$

When $p>1$, by the well-known Hölder's inequality, we obtain

$$\int_{ma}^{ma+\eta(b,a,m)}|S(t,m)||f^{(n)}(t)|dt$$

$$\leq\left[\int_{ma}^{ma+\eta(b,a,m)}|S(t,m)|dt\leq\right]^{1-\frac{1}{p}}\times\left[\int_{ma}^{ma+\eta(b,a,m)}|S(t,m)||f^{(n)}(t)|^{p}dt\right]^{\frac{1}{p}},$$

where

$$\int_{ma}^{ma+\eta(b,a,m)}|S(t,m)||f^{(n)}(t)|^{p}dt=\int_{ma}^{ma+\frac{\eta(b,a,m)}{2}}(t-ma)^{n}|f^{(n)}(t)|^{p}dt$$
$$+\int_{ma+\frac{\eta(b,a,m)}{2}}^{ma+\eta(b,a,m)}(ma+\eta(b,a,m)-t)^{n}|f^{(n)}(t)|^{p}dt.$$

Since $|f^{(n)}(t)|^{p}$ is a generalized (s,m)-preinvex function on K, we have

$$\int_{ma}^{ma+\eta(b,a,m)}|S(t,m)||f^{(n)}(t)|^{p}dt\leq\int_{ma}^{ma+\frac{\eta(b,a,m)}{2}}(t-ma)^{n}$$
$$\times\left\{\left[\frac{ma+\frac{\eta(b,a,m)}{2}-t}{\frac{\eta(b,a,m)}{2}}\right]^{s}|f^{(n)}(ma)|^{p}+m\left[\frac{t-ma}{\frac{\eta(b,a,m)}{2}}\right]^{s}\left|f^{(n)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|^{p}\right\}dt$$
$$+\int_{ma+\frac{\eta(b,a,m)}{2}}^{ma+\eta(b,a,m)}(ma+\eta(b,a,m)-t)^{n}$$
$$\times\left\{\left[\frac{ma+\eta(b,a,m)-t}{\frac{\eta(b,a,m)}{2}}\right]^{s}\left|f^{(n)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|^{p}\right.$$
$$\left.+m\left[\frac{t-ma-\frac{\eta(b,a,m)}{2}}{\frac{\eta(b,a,m)}{2}}\right]^{s}\left|f^{(n)}(ma+\eta(b,a,m))\right|^{p}\right\}dt$$

It is not difficult to calculate the above integrals. Then using inequality (2.24), we get

$$\left|\frac{1}{\eta(b,a,m)}\sum_{k=0}^{n-1}\left[\frac{\left[1+(-1)^{k}\right]\eta(b,a,m)^{k+1}}{2^{k+1}(k+1)!}\right]f^{(k)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right.$$
$$\left.-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(t)dt\right|$$
$$\leq M\left[N|f^{(n)}(ma)|^{p}+\frac{m+1}{n+s+1}\left|f^{(n)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|^{p}+mN\left|f^{(n)}(ma+\eta(b,a,m))\right|^{p}\right]^{\frac{1}{p}},$$

where

$$M = \frac{1}{2n!}\left(\frac{2}{n+1}\right)^{1-\frac{1}{p}}\left(\frac{\eta(b,a,m)}{2}\right)^{n} \text{ and } N = \beta(n+1,s+1).$$

□

Remark 2.15. If we take $m = 1$ and $\eta(b,a,m) = b - ma$ in Theorem 2.14, then inequality (2.22) becomes inequality as obtained in (see [1], Theorem 1.2).

3. Corollaries

In order to show that inequalities (2.16) and (2.22) generalize and refine those known inequalities, some corollaries are deduced from inequalities (2.16) and (2.22) as follows.
Letting $s = 1$ in Theorem 2.12. Then, we have the following Corollary.

Corollary 3.16. *Let $f : K = [ma, ma+\eta(b,a,m)] \longrightarrow \mathbb{R}$ be n-time differentiable function on $K \subseteq \mathbb{R}_\circ$ and let $a < b$ with $ma < ma+\eta(b,a,m)$. If $|f^{(n)}(x)|^p$ is a generalized $(1,m)$-preinvex function on K for $n \geq 2$ and $p \geq 1$, then for some fixed $r \in [0,1]$ and $m \in (0,1]$, we have*

$$\left|\frac{f(mb)+rf(mb+\eta(a,b,m))}{r+1} - \frac{1}{\eta(a,b,m)}\int_{mb}^{mb+\eta(a,b,m)} f(x)dx \right.$$

$$\left. - \sum_{k=1}^{n-1} \frac{(-1)^k(k-r)\eta(a,b,m)^k}{(r+1)(k+1)!} f^{(k)}(mb+\eta(a,b,m)) \right|$$

$$\leq \frac{|\eta(a,b,m)|^n}{(r+1)n!}\left(\frac{n-r}{n+1}\right)^{1-\frac{1}{p}}\left[P|f^{(n)}(a)|^p + Q|f^{(n)}(b)|^p\right]^{\frac{1}{p}}, \tag{3.25}$$

where

$$P = \frac{n(n-r+1)-(r+1)}{(n+1)(n+2)}, \quad Q = m(n\beta(n,2)-(r+1)\beta(n+1,2)), \tag{3.26}$$

Remark 3.17. If we take $m = r = 1$ and $\eta(a,b,m) = a - mb$ in Corollary 3.16, then inequality (3.25) becomes inequality as obtained in (see [1], Corollary 4.1).

Corollary 3.18. *Under assumptions of Corollary 3.16 with $n = 2$, we have*

$$\left|\frac{f(mb)+rf(mb+\eta(a,b,m))}{r+1} - \frac{1}{\eta(a,b,m)}\int_{mb}^{mb+\eta(a,b,m)} f(x)dx\right.$$

$$+\frac{1-r}{2(r+1)}\eta(a,b,m)f'(mb+\eta(a,b,m))\Bigg|$$

$$\leq \frac{\eta(a,b,m)^2}{2(r+1)}\left(\frac{2-r}{3}\right)^{1-\frac{1}{p}}\left[\frac{(5-3r)|f''(a)|^p+m(3-r)|f''(b)|^p}{12}\right]^{\frac{1}{p}}, \tag{3.27}$$

Remark 3.19. If we take $m=r=1$ and $\eta(a,b,m)=a-mb$ in Corollary 3.18, then inequality (3.27) becomes inequality as obtained in (see [1], Corollary 4.2).

Letting $s=1$ in Theorem 2.14. Then, we have the following Corollary.

Corollary 3.20. *Let $f:K=[ma,ma+\eta(b,a,m)]\longrightarrow\mathbb{R}$ be n-time differentiable function on $K\subseteq\mathbb{R}_\circ$ and let $a<b$ with $ma<ma+\eta(b,a,m)$. If $|f^{(n)}(x)|^p$ is a generalized $(1,m)$-preinvex function on K for $n\geq 1$ and $p\geq 1$, then for some fixed $m\in(0,1]$, we have*

$$\Bigg|\frac{1}{\eta(b,a,m)}\sum_{k=0}^{n-1}\left[\frac{\left[1+(-1)^k\right]\eta(b,a,m)^{k+1}}{2^{k+1}(k+1)!}\right]f^{(k)}\left(ma+\frac{\eta(b,a,m)}{2}\right)$$

$$-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(t)dt\Bigg|$$

$$\leq M\left[N|f^{(n)}(ma)|^p+\frac{m+1}{n+2}\left|f^{(n)}\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|^p+mN|f^{(n)}(ma+\eta(b,a,m))|^p\right]^{\frac{1}{p}}, \tag{3.28}$$

where

$$M=\frac{1}{2n!}\left(\frac{2}{n+1}\right)^{1-\frac{1}{p}}\left(\frac{\eta(b,a,m)}{2}\right)^n \ \text{and}\ N=\beta(n+1,2). \tag{3.29}$$

Remark 3.21. If we take $m=1$ and $\eta(b,a,m)=b-ma$ in Corollary 3.20, then inequality (3.28) becomes inequality as obtained in (see [1], Corollary 4.3).

Corollary 3.22. *Under assumptions of Corollary 3.20 with $n=1$, we have*

$$\left|f\left(ma+\frac{\eta(b,a,m)}{2}\right)-\frac{1}{\eta(b,a,m)}\int_{ma}^{ma+\eta(b,a,m)}f(t)dt\right|$$

$$\leq \frac{\eta(b,a,m)}{4}\left[\frac{|f'(ma)|^p+2(m+1)\left|f'\left(ma+\frac{\eta(b,a,m)}{2}\right)\right|^p+m|f'(ma+\eta(b,a,m))|^p}{6}\right]^{\frac{1}{p}}. \tag{3.30}$$

Remark 3.23. If we take $m=1$ and $\eta(b,a,m)=b-ma$ in Corollary 3.22, then inequality (3.30) becomes inequality as obtained in (see [1], Corollary 4.4).

Remark 3.24. For $M\in\mathbb{R}$ and $p\geq 1$, if $|f^{(n)}(x)|^p\leq M$, then by our theorems mentioned in this paper we can get some special kinds of Hermite-Hadamard type inequalities.

4. Conclusions

In this paper, we investigated Hermite-Hadamard type inequalities for the functions which their derivatives of order n are generalized (s,m)-preinvex functions via classical integrals. Some known results are improved (see [1]), and we provide new estimates on these Hermite-Hadamard type inequalities. Also, these results can be applied to find new inequalities for special means such as geometric, arithmetic, logarithmic means, etc. This can be done if we substitute $\eta(a,b,m)$ or $\eta(b,a,m)$ with known special means in our theorems mentioned in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests.

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