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ON NEW INEQUALITIES OF SIMPSON'S TYPE FOR GENERALIZED QUASI CONVEX FUNCTIONS

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Abstract. In this paper, firstly we give a new identity via local fractional integrals. Then we obtained some new Simpson's type integral inequalities for the generalized quasi-convex functions.

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1. Introduction

We will start with the following inequality that is well-known in the literature as Simpson's inequality and has several utilization in different fields of mathematics: Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable mapping on $[a,b]$ and $\|f^{(4)}\|_\infty=\sup\limits_{x\in[a,b]}|f^{(4)}(x)|<\infty$ then the following inequality holds:

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{1}{2880}\|f^{(4)}\|_\infty(b-a)^4.$$

Several researchers make effort to obtain new inequalities related to Simpson inequality. To consult some of them, one can take glance to the papers ([5]-[15])

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Theory of convex sets and convex functions play an important role in different fields of pure and applied sciences. In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative techniques, see [1, 2, 3, 4, 6, 11, 26]. One of these concepts is concept of a quasi-convexity which is well known in the literature.

Many new results were obtained with the help of quasi-convex functions in recent years.(see [7, 14])

In recent years, researchers introduced the generalized convex function on fractal sets as the following: (see [9, 10])

Definition 1.1. [9] Let $f : I \subseteq R \to R^{\alpha}$. For any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, if the following inequality

$$f(\lambda x_1 + \lambda x_2) \leq (\geq) \lambda^{\alpha} f(x_1) + (1-\lambda)^{\alpha} f(x_2)$$

holds, then f is called a generalized convex (or concave) function on I.

We can give two basic examples of generalized convex functions as follows:

1. $f(x) = x^{\alpha p}, x \geq 0, p > 1$;
2. $f(x) = E_{\alpha}(x^{\alpha})$, $x \in R$, where $E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Leffer function.

For some recent results about generalized convex functions, see[9, 10, 16, 19, 25] Now, let us give the notion of generalized quasi-convex function generalizes the notion of generalized convex function.

Definition 1.2. [19] Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ for any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, if the following inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \sup\{f(x), f(y)\}$$

holds , then f is called a generalized quasi convex function on I.

Clearly all generalized convex function is a generalized quasi-convex function but the reverse are not true. Because there exist generalized quasi-convex function which are not generalized convex. In that context, let's give the following example. The function $g : [-3,3] \to \mathbb{R}^{\alpha}$

$$g(t) = \begin{cases} t^{\alpha} - 1, & t \in [-3,-1], \\ t^{2\alpha}, & t \in (-1,3] \end{cases}$$

is not generalized convex function on [-3,3] but if is a generalized quasi-convex function on [-3,3].

2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [21, 22] and so on.

Recently, the theory of Yang's fractional sets [21] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

$\mathbb{Z}^{\alpha}$:The α- type set of integer is defined as the set $\left\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...\pm n^{\alpha},...\right\}$.

$\mathbb{Q}^{\alpha}$:The α-type set of the rational numbers is defined as the set $\left\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}: \ p,q\in\mathbb{Z}, \ q\neq 0\right\}$.

$\mathbb{J}^{\alpha}$:The α- type set of the irrational numbers is defined as the set $\left\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}: \ p,q\in\mathbb{Z}, \ q\neq 0\right\}$

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

1. $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};
2. $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;
3. $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;
4. $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;
5. $a^{\alpha}(b^{\alpha}c^{\alpha})=(a^{\alpha}b^{\alpha})c^{\alpha}$;
6. $a^{\alpha}(b^{\alpha}+c^{\alpha})=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;
7. $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

In [21], local fractional continuity, differentiation and integration and some properties are given as follows.

Definition 2.1. A non-differentiable function $f:R\to R^{\alpha}$, $x\to f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_\alpha(a,b)$.

Definition 2.2. The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by

$$f^{(\alpha)}(x_0)=\left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha},$$

where $\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(1+\alpha)(f(x)-f(x_0))$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^\alpha\ldots D_x^\alpha}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,\ldots$

Definition 2.3. Let $f(x)\in C_\alpha[a,b]$. Then the local fractional integral is defined by,

$${}_aI_b^\alpha f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\to 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^\alpha,$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_{N-1}\}$, where $[t_j,t_{j+1}]$, $j=0,\ldots,N-1$ and $a=t_0<t_1<\cdots<t_{N-1}<t_N=b$ is partition of interval $[a,b]$. Here, it follows that ${}_aI_b^\alpha f(x)=0$ if $a=b$ and ${}_aI_b^\alpha f(x)=-{}_bI_a^\alpha f(x)$ if $a<b$. If for any $x\in[a,b]$, there exists ${}_aI_x^\alpha f(x)$, then we denoted by

$f(x)\in I_x^\alpha[a,b]$.

Lemma 2.1.

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x)=g^{(\alpha)}(x)\in C_\alpha[a,b]$, then we have

$${}_aI_b^\alpha f(x)=g(b)-g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x),g(x)\in D_\alpha[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x)\in C_\alpha[a,b]$, then we have

$${}_aI_b^\alpha f(x)g^{(\alpha)}(x)=f(x)g(x)|_a^b-{}_aI_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2.2.

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha}=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)}x^{(k-1)\alpha};$$

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b x^{k\alpha}dx^\alpha=\frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)}\left(b^{(k+1)\alpha}-a^{(k+1)\alpha}\right).$$

Theorem 2.1.[Generalized Hölder inequality][21]

Let $f,g\in C_\alpha[a,b]$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)g(x)|\,(dx)^\alpha \\ &\leq \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p\,(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q\,(dx)^\alpha\right)^{\frac{1}{q}}. \end{aligned} \tag{2.1}$$

It can be found in [21, 22, 23, 24, 25] more detailed information on local fractional calculus. Also, we recommend readers to some new work about local fractional integral inequalities (see [16, 17, 18]).

3. Main results

In order to prove our main theorems, we need the following Lemma.

Lemma 3.1. Let $I \subseteq R$ be an interval , $f : I \subseteq R \to R^\alpha$ (I° is the interior of I) such that $f \in D_\alpha(I^\circ)$ and $f^{(\alpha)} \in C_\alpha[a,b]$ for $a,b \in I^\circ$ with $a<b$. Then the following equality holds:

$$\begin{aligned} &\frac{1}{6^\alpha}\left[f(a)+4^\alpha f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(x) \\ &=\frac{1}{\Gamma(1+\alpha)}\left(\frac{b-a}{2}\right)^\alpha\int_0^1\left[\left(\frac{t}{2}-\frac{1}{3}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right. \\ &\left.+\left(\frac{1}{3}-\frac{t}{2}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right](dt)^\alpha. \end{aligned} \tag{3.1}$$

Proof. Using the local fractional integration by parts, we have

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{t}{2}-\frac{1}{3}\right)^\alpha f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)(dt)^\alpha \\ &= \left(\frac{t}{2}-\frac{1}{3}\right)^\alpha\left(\frac{2}{b-a}\right)^\alpha f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\Big|_0^1 \\ &-\left(\frac{2}{b-a}\right)^\alpha\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{1}{2}\right)^\alpha\Gamma(1+\alpha)f\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)(dt)^\alpha. \end{aligned}$$

Setting $x=\frac{1+t}{2}b+\frac{1-t}{2}a$ and $(dx)^\alpha=\left(\frac{b-a}{2}\right)^\alpha(dt)^\alpha$, which gives

$$\begin{aligned} I_1 &= \left(\frac{2}{b-a}\right)^\alpha\left[\left(\frac{1}{6}\right)^\alpha f(b)+\left(\frac{2}{6}\right)^\alpha f\left(\frac{a+b}{2}\right)\right] \\ &-\left(\frac{2}{b-a}\right)^{2\alpha}\frac{1}{2^\alpha}\Gamma(1+\alpha)\int_{\frac{a+b}{2}}^b f(x)(dx)^\alpha. \end{aligned} \tag{3.2}$$

Similarly, we can show that

$$\begin{aligned}
I_2 &= \frac{1}{\Gamma(1+\alpha)}\int_0^1 \left(\frac{1}{3}-\frac{t}{2}\right)^{\alpha} f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)(dt)^{\alpha} \\
&= \left(\frac{1}{3}-\frac{t}{2}\right)^{\alpha}\left(\frac{2}{a-b}\right)^{\alpha} f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Big|_0^1 \\
&- \left(\frac{2}{a-b}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \left(-\frac{1}{2}\right)^{\alpha}\Gamma(1+\alpha) f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)(dt)^{\alpha}.
\end{aligned}$$

Setting $x=\frac{1+t}{2}a+\frac{1-t}{2}b$ and $(dx)^{\alpha}=\left(\frac{a-b}{2}\right)^{\alpha}(dt)^{\alpha}$, which gives

$$\begin{aligned}
I_2 &= \left(\frac{2}{a-b}\right)^{\alpha}\left[\left(-\frac{1}{6}\right)^{\alpha} f(a)-\left(\frac{2}{6}\right)^{\alpha} f\left(\frac{a+b}{2}\right)\right] \\
&+ \left(\frac{2}{a-b}\right)^{2\alpha}\frac{1}{2^{\alpha}}\Gamma(1+\alpha)\int_{\frac{a+b}{2}}^{a} f(x)(dx)^{\alpha}.
\end{aligned} \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\begin{aligned}
I_1+I_2 = & \left(\frac{2}{b-a}\right)^{\alpha}\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha} f\left(\frac{a+b}{2}\right)+f(b)\right] \\
& -\left(\frac{2}{b-a}\right)^{2\alpha}\frac{1}{2^{\alpha}}\Gamma(1+\alpha)\int_a^b f(x)(dx)^{\alpha}
\end{aligned}$$

if we multiple the resulting equality with $\left(\frac{b-a}{2}\right)^{\alpha}$, then we complete the proof.

The next theorems gives a new results of the Simpson inequality for generalized quasi-convex functions:

Theorem 3.1. Let $I\subseteq R$ be an interval , $f: I\subseteq R\to\ R^{\alpha}$ (I° is the interior of I) such that $f\in D_{\alpha}(I^{\circ})$ and $f^{(\alpha)}\in C_{\alpha}[a,b]$ for $a,b\in I^{\circ}$ with $a<b$. If $|f^{(\alpha)}|$ is generalized quasi convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha} f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(x)\right| \\
&\le (b-a)^{\alpha}\left(\frac{5}{18}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sup\left\{|f^{(\alpha)}(a)|,|f^{(\alpha)}(b)|\right\}.
\end{aligned} \tag{3.4}$$

Proof. From Lemma 3.1 and taking modulus, it follows that

$$\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
&\leq \frac{1}{\Gamma(1+\alpha)}\left(\frac{b-a}{2}\right)^{\alpha}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\left.+\left[\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right]\right](dt)^{\alpha}.
\end{aligned} \tag{3.5}$$

$|f^{(\alpha)}|$ is generalized quasi convex, we have

$$\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
&\leq \frac{1}{\Gamma(1+\alpha)}\left(\frac{b-a}{2}\right)^{\alpha}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha} \\
&\times\left[\sup\left\{|f^{(\alpha)}(b)|,|f^{(\alpha)}(a)|\right\}+\sup\left\{|f^{(\alpha)}(a)|,|f^{(\alpha)}(b)|\right\}\right].
\end{aligned} \tag{3.6}$$

Using Lemma 2.2, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}(dt)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}(dt)^{\alpha}=\left(\frac{5}{18}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}. \tag{3.7}$$

So, (3.6) and (3.7), we get

$$\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
\leq\quad &(b-a)^{\alpha}\left(\frac{5}{18}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sup\left\{|f^{(\alpha)}(a)|,|f^{(\alpha)}(b)|\right\}
\end{aligned}$$

which completes of proof.

Theorem 3.2. Let $I\subseteq R$ be an interval , $f:I\subseteq R\to\ R^{\alpha}$ (I° is the interior of I) such that $f\in D_{\alpha}(I^{\circ})$ and $f^{(\alpha)}\in C_{\alpha}[a,b]$ for $a,b\in I^{\circ}$ with $a<b$. If $|f^{(\alpha)}|^{q}$ is generalized quasi convex on $[a,b]$, $\ q>1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
&\leq(b-a)^{\alpha}\left(2^{\alpha}\left[\left(\frac{1}{3}\right)^{(p+1)\alpha}+\left(\frac{1}{6}\right)^{(p+1)\alpha}\right]\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1))\alpha}\right)^{\frac{1}{p}} \\
&\times\left(\sup\left\{|f^{(\alpha)}(a)|^{q},|f^{(\alpha)}(b)|^{q}\right\}\right)^{\frac{1}{q}}
\end{aligned} \tag{3.8}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 3.1 and using the generalized Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
&\leq \frac{1}{\Gamma(1+\alpha)}\left(\frac{b-a}{2}\right)^{\alpha}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right](dt)^{\alpha} \\
&\leq \left(\frac{b-a}{2}\right)^{\alpha}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{p\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}}\right. \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{p\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}} \\
&\left.\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Since $|f^{(\alpha)}|^q$ is generalized quasi-convex on $[a,b]$, we have

$$\begin{aligned}
&\leq \left(\frac{b-a}{2}\right)^{\alpha}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{p\alpha}(dt)^{\alpha}\right)^{\frac{1}{p}} \\
&\times\left[\left(\sup\left\{|f^{(\alpha)}(b)|^q,|f^{(\alpha)}(a)|^q\right\}\right)^{\frac{1}{q}}+\left(\sup\left\{|f^{(\alpha)}(a)|^q,|f^{(\alpha)}(b)|^q\right\}\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.9}$$

Using Lemma 2.2, we have

$$\begin{aligned}
\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{p\alpha}(dt)^{\alpha} &= \int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{p\alpha}(dt)^{\alpha} \\
&= 2^{\alpha}\left[\left(\frac{1}{3}\right)^{(p+1)\alpha}+\left(\frac{1}{6}\right)^{(p+1)\alpha}\right]\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}.
\end{aligned} \tag{3.10}$$

So, (3.9) and (3.10), we get(3.8), which completes of proof.

Theorem 3.3. Let $I\subseteq R$ be an interval , $f:I\subseteq R\to R^{\alpha}$ (I° is the interior of I) such that $f\in D_{\alpha}(I^{\circ})$ and $f^{(\alpha)}\in C_{\alpha}[a,b]$ for $a,b\in I^{\circ}$ with $a<b$. If $|f^{(\alpha)}|^q$ is generalized quasi convex on $[a,b]$, $q\geq 1$, then the following inequality holds:

$$
\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
&\leq (b-a)^{\alpha}\left(\frac{5}{18}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\sup\left\{|f^{(\alpha)}(a)|^{q},|f^{(\alpha)}(b)|^{q}\right\}\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.11}
$$

Proof. From Lemma 3.1 and generalized power-mean inequality, we have

$$
\begin{aligned}
&\left|\frac{1}{6^{\alpha}}\left[f(a)+4^{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}f(x)\right| \\
&\leq \frac{1}{\Gamma(1+\alpha)}\left(\frac{b-a}{2}\right)^{\alpha}\int_0^1\left[\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|\right. \\
&\left.+\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|\right](dt)^{\alpha} \\
&\leq \left(\frac{b-a}{2}\right)^{\alpha}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}} \\
&+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}} \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}\left|f^{(\alpha)}\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.12}
$$

Since $|f^{(\alpha)}|^{q}$ is generalized quasi-convex on $[a,b]$, we have

$$
\begin{aligned}
&\leq \left(\frac{b-a}{2}\right)^{\alpha}\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}}\right. \\
&\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}\sup\left\{|f^{(\alpha)}(b)|^{q},|f^{(\alpha)}(a)|^{q}\right\}\right)^{\frac{1}{q}}(dt)^{\alpha} \\
&+\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}(dt)^{\alpha}\right)^{1-\frac{1}{q}} \\
&\left.\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}\sup\left\{|f^{(\alpha)}(a)|^{q},|f^{(\alpha)}(b)|^{q}\right\}\right)^{\frac{1}{q}}(dt)^{\alpha}\right].
\end{aligned} \tag{3.13}
$$

Using Lemma 2.2, we have

$$
\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{t}{2}-\frac{1}{3}\right|^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|\frac{1}{3}-\frac{t}{2}\right|^{\alpha}=\left(\frac{5}{18}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}. \tag{3.14}
$$

So, (3.13) and (3.14), we get (3.11), which completes of proof.

Remark 3.1. Theorem 3.3 is equal to theorem (3.1) for $q = 1$.

Remark 3.2. Theorem 3.1 is equal to theorem 4 in [20] for $\alpha = 1$.

Remark 3.2. Theorem 3.2 is equal to theorem 5 in [20] for $\alpha = 1$.

Remark 3.3. Theorem 3.3 is equal to theorem 6 in [20] for $\alpha = 1$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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