Available online at http://scik.org

Advances in Inequalities and Applications, 1 (2012), No. 1, 45-52

SOME COMPACT GENERALIZATIONS OF ENESTROM-KAKEYA TYPE THEOREM FOR POLYNOMIALS

B. A. ZARGER*

P.G. Deptt. of Math. University of Kashmir, Srinagar

Abstract. In this paper we present an interesting generalization of Enestrom-Kakeya Theorem which amoung other things yields a number of already known classical results by putting some restrictive conditions on the coefficients of the polynomials.

keywords: coefficients, zeros, parameter

2010 AMS Subject Classification : 20C10,20C15.

1. INTRODUCTION AND STATEMENT OF RESULTS

The following result due to Enestrom-and Kakeya [7] is well-known in the theory of distribution of the zeros of polynomials

THEOREM A. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

is a polynomial of degree n such that

(1) $$a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0,$$

then P(z) does not vanish in $|z| > 1$.

---

*Corresponding author

Received March 26, 2012

Applying this result to P(tz), the following more general result is immediate.

THEOREM B. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

is a polynomial of degree n such that

$$t^n a_n \geq t^{n-1} a_{n-1} \geq ... \geq t a_1 \geq a_0 \geq 0$$

then all the zeros of P(z) lie in $|z| \leq t$.

In the literature [1,2,4,5,8] these exist some extensions and generalizations of Enestrom-Kakeya Theorem, Joyal, Labellel and Rahman[6] extended this theorem to polynomials whose coefficients were monotonic but not necessarily non-negative. Recently Aziz and Zarger[3] relaxed the hypothesis in several ways and among other things proved the following interesting generalization of Theorem A:

THEOREM C. If

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

is a polynomial of degree n such that for some $k \geq 1$.

$$k a_n \geq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0$$

then P(z) has all its zeros in

(2) $$|z + k - 1| \leq k$$

In this paper we start by proving the following result which includes Theorems A,B and C as special cases.

THEOREM 1. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n, if for some real number $k \geq 1$,

(3) $$Max_{|z|=1} \left| (k a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + ... + (a_1 - a_0) z^{n-1} + a_0 z^n \right| \leq M$$

then all the zeros of P(z) lie in the disk

(4) $$|z + k - 1| \leq \frac{M}{|a_n|}$$

REMARK 1. Suppose P(z) satisfies the conditions of Theorem C, then clearly for $|z| = 1$,

$$\left| (k a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) z + ... + a_0 z^n \right|$$

$$\leq |(ka_n - a_{n-1})| + |a_{n-1} - a_{n-2}| + ... + |a_1 - a_0| + |a_0|$$

$$= ka_n - a_{n-1} + a_{n-1} - a_{n-2} + ... + a_1 - a_0 + a_0$$

$$= ka_n$$

We take $M = ka_n$ in Theorem 1, it follows that all the zeors of P(z) satisfying the conditions of Theorem C lie in the circle

$$|z + k - 1| \leq k\,,$$

which is precisely the conclusion of Theorem C.

The following corollary follows by taking $k = \dfrac{a_{n-1}}{a_n}$ in Theorem 1.

COROLLARY 1. If $P(z) = \sum_{j=0}^{n} a_j z^j$, is a polynomial of degree n, such that

$$a_{n-1} \geq a_n \quad and \quad Max_{|z|=1} |(a_{n-1} - a_{n-2})z + ... + a_0 z^n| \leq M_1$$

then all the zeros of P(z) lie in the disk.

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{M_1}{|a_n|}$$

REMARK 2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n satisfying the condition

(05) $$0 < a_n \leq a_{n-1} \geq ... \geq a_1 \geq a_0 \geq 0$$

then $M_1 = Max_{|z|=1} |(a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3})z + ... + a_0 z^n|$

$$\leq (a_{n-1} - a_{n-2}) + (a_{n-2} - a_{n-3}) + ... + a_0$$

$$= a_{n-1}\,.$$

Hence from Corollary 1 it follows that all the zeros of P(z) satisfying (5) lie in the circle.

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{a_{n-1}}{a_n}$$

This result was earlier proved by the authors in [3] Cor. 2.

Theorem 1 follows by taking $t = 1$ in the following general result:

THEOREM 2. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n. If for some positive real numbers t and $k \geq 1$

(6) $$Max_{|z|=\frac{1}{t}}|H(z)| \leq M \text{ , where}$$

$$H(z) = \{(tka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})z + ... + (ta_1 - a_0)z^{n-1} + ta_0 z^n\}$$

then all the zeros of P(z) lie in

(7) $$|z + t(k-1)| \leq \frac{M}{|a_n|}$$

Since

$$M \geq Max_{|z|=\frac{1}{t}}|H(z)| \geq \left|H\left(\frac{1}{t}\right)\right|$$

$$= \left|(tka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})\frac{1}{t} + (ta_{n-2} - a_{n-3})\frac{1}{t^2} + ... + (ta_1 - a_0)\frac{1}{t^{n-1}} + ta_0 . \frac{1}{t^n}\right|$$

$$\geq kta_n$$

Theorem 2 follows by taking $R = \frac{1}{t}$ in the following more general result which yields a number of other interesting results for various choices of parameters R and t:

THEOREM 3. Let

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$$

be a polynomial of degree n. If for some positive real numbers t and $k \geq 1$

(8) $$Max_{|z|=R}|H(z)| \leq M \text{ , where}$$

$$H(z) = (kta_n - a_{n-1}) + (ta_{n-1} - a_{n-2})z + ... + ta_0 z^n .$$

then all the zeros of P(z) lie in the circle

(9) $$|z + t(k-1)| \leq \frac{M}{|a_n|} \qquad if \qquad M \geq \left[kt + \left(\frac{1}{R} - t\right)\right]|a_n|$$

and in

(10) $$|z| \leq t(2k-1) + \left(\frac{1}{R} - t\right), \qquad if \qquad M < \left[kt + \left(\frac{1}{R} - t\right)\right]|a_n|.$$

PROOF OF THEOREM 3. Consider the polynomial

$$F(z) = (t-z)P(z)$$

$$= (t-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0)$$

$$= -a_n z^{n+1} + (ta_n - a_{n-1})z^n + (ta_{n-1} - a_{n-2})z^{n-1} + ... + (ta_1 - a_0)z + ta_0$$

then we have

$$G(z) = z^{n+1} F\left(\frac{1}{z}\right)$$

$$= -a_n + (ta_n - a_{n-1})z + (ta_{n-1} - a_{n-2})z^2 + ... + (ta_1 - a_0)z^n + ta_0 z^{n+1}$$

$$= -a_n + ta_n z - kta_n z + (tka_n - a_{n-1})z + (ta_{n-1} - a_{n-2})z^2 + ... + ta_0 z^{n+1}$$

$$= -a_n + ta_n z - kta_n z + zH(z)$$

where

$$H(z) = \{(tka_n - a_{n-1}) + (ta_{n-1} - a_{n-2})z + ... + ta_0 z^n\}$$

Now for $|z| \le R$,

(11) $$|G(z)| \ge |a_n||t(k-1)z + 1| - |z||H(z)|$$

Since

$|H(z)| \le M$ *for* $|z| = R$, *and* $H(z)$ is analytical for $|z| \le R$, therefore by Maximum Modulus Theorem $|H(z)| \le M$ *for* $|z| \le R.$

Using this fact in (11), we get

$$|G(z)| \ge |a_n||t(k-1)z + 1| - |z|M.$$

> 0, if

$$\frac{M}{a_n}|z| < |t(k-1)z + 1| \quad for \quad |z| \le R.$$

Thus in $|z| \le R$,

$$|G(z)| > 0 \quad for \quad z \in E,$$

where

$$E = \left\{ z; \frac{M}{|a_n|}|z| < |t(k-1)z + 1| \right\}$$

we show if $w \in E$, then $|w| \le R$ *if*

$$M \ge \left(kt + \left(\frac{1}{R} - t\right)\right)|a_n|$$

Let $W \in E$, *then*

$$\frac{M}{|a_n|}|w| < |t(k-1)w+1|$$

$$< t(k-1)|w|+1$$

This implies,

$$\left\{\frac{M}{|a_n|} - t(k-1)\right\}|W| < 1$$

or

$$|W| < \frac{|a_n|}{M - t(k-1)|a_n|} \leq R$$

if

$$|a_n| \leq MR - t(k-1)|a_n|R$$

or if,

$$M \geq \frac{|a_n|}{R}\{Rt(k-1)+1\} = |a_n|\left\{kt + \frac{1}{R} - t\right\}$$

Thus if

$$M \geq \left(kt + \left(\frac{1}{R} - t\right)\right)|a_n|$$

then in $|z| \leq R,$

$$|G(z)| > 0, if$$

$$\frac{M}{a_n}|z| < |t(k-1)z+1|$$

This shows that all the zeros of G(z) lie in the region defined by

$$\frac{M}{|a_n|}|z| \geq |t(k-1)z+1|$$

Replacing z by $\frac{1}{z}$ and noting that $F(z) = z^{n+1}G(z),$ it follows that all the zeros of F(z) lie in

$$|z + t(k-1)| \leq \frac{M}{|a_n|}.$$

Since all the zeros of P(z) are also the zeros of F(z), we conclude that all the zeros of P(z) lie in

(12) $$|z+t(k-1)|\le \frac{M}{a_n} \quad if \quad M\ge \left\{kt+\left(\frac{1}{R}-t\right)\right\}|a_n| .$$

We now assume that

$$M<\left(\frac{Rt(k-1)+1}{R}\right)|a_n|$$

then for $|z|\le R$,

$$|G(z)|=|-a_n+(ta_n-kta_n)z+zH(z)|$$

$$\ge |a_n|\left\{1-|z|\left(t(k-1)+\frac{|H(z)|}{|a_n|}\right)\right\}$$

$$\ge |a_n|-|z|\left\{t(k-1)+\frac{M}{|a_n|}\right\}$$

$$\ge |a_n|\left\{1-|z|\left(t(k-1)+\left(\frac{1}{R}-t\right)+tk\right)\right\}$$

>0, if

$$|z|<\frac{1}{t(2k-1)+\left(\frac{1}{R}-t\right)}(\le R) .$$

Tthis shows that all the zeros of G(z) lie in the region

$$|z|\ge \frac{1}{t(2k-1)+\left(\frac{1}{R}-t\right)}$$

Replacing z by $\frac{1}{z}$ and as before noting that

$$F(z)=z^{n+1}G\left(\frac{1}{z}\right)$$

it follows that all the zeros of F(z) and hence all the zeros of P(z) lie in the circle

(13) $$|z|\le t(2k-1)+\left(\frac{1}{R}-t\right)$$

if

$$M<\left\{\left(\frac{1}{R}-t\right)+kt\right\}|a_n|$$

From (12) and (13), the desired result follows.

REFERENCES

[1] A. Aziz and Q.G Mohammad, On the zeros of certain class of polynomials and related analytic functions, J. Math. Anal.Appl. 75 (1980), 495-502.

[2] A. Aziz and W.M.Shah, On the zeros of polynomials and related analytic functions, Glasnik Matematicki, 33 (1998), 173-189.

[3] A. Aziz and B. A. Zargar , Some Extension of Enestrom - Kakeya Theorem, Glasnik Matematicki, 31 (1996) 239-244

[4] K.K. Dewan and M. Bidkham, on the Enestrom-Kakeya Theorem, J.Math Anal. Appl. 180(1993), 29-36.

[5] N.K. Govil and Q.I Rahman, On the Enestrom-Kakeya Theorem, Tohoku Math.J. 20 (1968), 126 – 136.

[6] A. Joyal.G.Labelle and Q.I. Rahman, On the Location of the zeros of polynomials, Canad, Math. Bull 10 (1967), 53 – 63

[7] M. Marden, Geometry of polynomials, Math surveys No 3. Amer Math. Soc. (R.I. Providence) 1966.

[8] G.V Milovanoic , D.S. Mitrinovic, Th.M Rassias. Topics in polynomials, Extremal problems, Inequalities, zeros (World Scientific, Singapore, (1994);