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A NOTE ON $|A|_k$ SUMMABILITY FACTORS OF INFINITE SERIES

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Abstract. In this note we give an improvement toa recent result obtained by Savas and Rhoade cocerning $|A|_k$ summability of infinite series .

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1. Introduction

Let T be a lower triangular matrix, (s_n) a sequence, and

$$T_n := \sum_{v=0}^{n} t_{nv} s_v . \tag{1.1}$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} n^{k-1} \left| \Delta T_{n-1} \right|^k < \infty . \tag{1.2}$$

Given any lower triangular matrix T one can associate the matrices $\overline{T}$ and $\hat{T}$, with entries defined by

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$$\bar{t}_{nv} = \sum_{i=v}^{n} t_{ni}, \quad n, i = 0,1,2..., \qquad \hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}$$

respectively. With $s_n = \sum_{i=0}^{n} a_i \lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i . \tag{1.3}$$

$$Y_n := t_n - t_{n-1} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i = \sum_{i=0}^{n} \hat{t}_{ni} a_i \lambda_i, \quad as \ \bar{t}_{n-1,n} = 0. \tag{1.4}$$

$$X_n := u_n - u_{n-1} = \sum_{i=0}^{n} \hat{u}_{ni} a_i \mu_i . \tag{1.5}$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n.

We assume that (p_n) is a sequence of positive real numbers such that

$$P_n = p_0 + p_1 + ... + p_n \to \infty \ as \ n \to \infty.$$

In the special case when $t_{nv} = p_v / P_n$, summability $|T|_k$ reduces to $|\bar{N}, p_n|_k$ summability.

Generalizing the result of [2], Rhoads and Savas [3] proved the following result

Theorem 1.1. *Let A be a triangle with nonnegative entries satisfying*

(i) $\bar{a}_{n0} = 1, \ n = 0,1,...,$

(ii) $a_{n-1,v} \geq a_{nv} \quad for \ n \geq v+1,$

(iii) $na_{nn} = O(1),\ 1 = O(na_{nn})$,

(iv) $\Delta(1/a_{nn}) = O(1)$,

(v) $\sum_{v=0}^{n} a_{vv}|a_{n,v+1}| = O(a_{nn})$.

If (X_n) *is a positive nondecreasing sequence and the sequences* (λ_n) *and* (β_n) *satisfy*

(vi) $|\Delta\lambda_n| \le \beta_n$,

(vii) $\lim \beta_n = 0$,

(viii) $|\lambda_n| X_n = O(1)$,

(ix) $\sum_{n=1}^{\infty} n|\Delta\beta_n| X_n < \infty$,

(x) $T_n := \sum_{v=1}^{n}\left(|s_v|^k / v\right) = O(X_n)$,

then the series $\sum (a_n\lambda_n)/na_{nn}$ *is summable* $|A|_k,\ k \ge 1$.

The object of this paper is to give two improvements to theorem 1.1 as follows

1. Replacing the four conditions (vi)-(ix) by two conditions ,
2. By weakening the condition (x),

and adding a simple condition. In fact we prove the theorem without any loss of powers through estimation. In [3], through the proof, there is a loss of some powers through estimation. For example $|\lambda_n|^k$ is replaced by the factor $|\lambda_n|$ as $|\lambda_n| = O(1)$, and in such case we are losing the power $|\lambda_n|^{k-1}$ without any advantage.

In what follows we prove the following

Theorem 1.2. *Let A be a triangle with nonnegative entries satisfying*

(i) $\bar{a}_{n0} = 1,\ n = 0, 1, \ldots,$

(ii) $a_{n-1,v} \geq a_{nv}$ *for* $n \geq v+1,$

(iii) $na_{nn} = O(1),\ 1 = O(na_{nn}),$

(iv) $\Delta(1/a_{nn}) = O(1),$

(v) $\sum_{v=1}^{n} a_{vv} |\hat{a}_{n,v+1}| = O(a_{nn}).$

(vi) $\sum_{n=v+1}^{\infty} n^{k-1} |\hat{a}_{n,v+1}|^k = O(1).$

If (X_n) is a positive nondecreasing sequence and the sequence (λ_n) satisfy

(vii) $\lambda_n \to 0,$ *as* $n \to \infty$

(viii) $\sum_{n=1}^{\infty} n\left|\Delta^2\lambda_n\right| X_n < \infty,$

and

(ix) $T_n := \sum_{v=1}^{n}\left(\left|s_v\right|^k / vX_v^{k-1}\right) = O\left(X_n\right),$

then the series $\sum\left(a_n\lambda_n\right)/ na_{nn}$ *is summable* $\left|A\right|_k,\ k \ge 1.$

We have to mention that whenever $X_n \to \infty,$ condition (vii) of theorem 1.2 is weaker than condition (viii) of theorem 1.1. For if (viii) is satisfied, then $X_n \to \infty$ implies that $\lambda_n \to 0,$ while if (vii) is satisfied, that is $\lambda_n \to 0,$ then by choosing

$$\lambda_n = n^{-1/2},\ X_n = n^{\in+(1/2)},\ \in > 0,$$

we obtain $\left|\lambda_n\right| X_n = O(n^{\in}) \ne O(1).$

Lemma 1.2. *Condition* (ix) *of theorem 1.2 is weaker than condition* (x) *of theorem 1.1.*

Proof. If (x) holds, then we have

$$\sum_{n=1}^{m}\frac{\left|s_n\right|^k}{nX_n^{k-1}} = O\left(\frac{1}{X_1^{k-1}}\right)\sum_{n=1}^{m}\frac{1}{n}\left|s_n\right|^k = O\left(X_m\right),$$

while if (ix) is satisfied then,

$$\sum_{n=1}^{m}\frac{1}{n}|s_n|^k = \sum_{n=1}^{m}\frac{1}{nX_n^{k-1}}|s_n|^k X_n^{k-1}$$

$$= \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}\frac{|s_v|^k}{vX_v^{k-1}}\right)\Delta X_n^{k-1} + \left(\sum_{n=1}^{m}\frac{|s_n|^k}{nX_n^{k-1}}\right)X_m^{k-1}$$

$$= \mathrm{O}(1)\sum_{n=1}^{m-1}X_n\left|\Delta X_n^{k-1}\right| + \mathrm{O}(X_m)X_m^{k-1}$$

$$= \mathrm{O}(X_{m-1})\sum_{n=1}^{m-1}\left(X_{n+1}^{k-1} - X_n^{k-1}\right) + \mathrm{O}\left(X_m^k\right)$$

$$= \mathrm{O}(X_{m-1})\left(X_m^{k-1} - X_1^{k-1}\right) + \mathrm{O}\left(X_m^k\right)$$

$$= \mathrm{O}\left(X_m^k\right).$$

Lemma 1.3. *Conditions* (vii)-(viii) *of theorem 1.2 imply that*

$$nX_n|\Delta\lambda_n| = \mathrm{O}(1), \tag{1.6}$$

$$\sum_{n=1}^{\infty}X_n|\Delta\lambda_n| < \infty, \tag{1.7}$$

and (1.7) *implies*

$$|\lambda_n|X_n = \mathrm{O}(1). \tag{1.8}$$

Proof. Since $\lambda_n \to 0$, then $\Delta\lambda_n \to 0$, and hence

$$nX_n|\Delta\lambda_n| = nX_n\sum_{v=n}^{\infty}\Delta|\Delta\lambda_v| = \mathrm{O}(1)\,nX_n\sum_{v=n}^{\infty}\left|\Delta^2\lambda_v\right| = \mathrm{O}(1)\sum_{v=n}^{\infty}vX_v\left|\Delta^2\lambda_v\right| = \mathrm{O}(1).$$

$$\sum_{n=1}^{m}X_n|\Delta\lambda_n| = \sum_{n=1}^{m-1}\left(\sum_{v=1}^{n}X_v\right)\Delta|\Delta\lambda_n| + \left(\sum_{n=1}^{m}X_n\right)|\Delta\lambda_m|$$

$$= \mathrm{O}(1)\sum_{n=1}^{m-1}nX_n\left|\Delta^2\lambda_n\right| + \mathrm{O}(1)\,mX_m|\Delta\lambda_m| = \mathrm{O}(1)$$

As $\lambda_n \to 0$,

$$|\lambda_n|\,X_n = X_n\sum_{v=n}^{\infty}\Delta|\lambda_n| = \mathrm{O}(1)\sum_{v=n}^{\infty}X_v|\Delta\lambda_n| = \mathrm{O}(1).$$

Lemma 1.4. *Under the conditions of theorem 1.2,*

$$\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}| = \mathrm{O}(a_{nn}), \tag{1.9}$$

$$\sum_{n=v+1}^{m+1}|\Delta_v\hat{a}_{nv}| = \mathrm{O}(a_{vv}), \tag{1.10}$$

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| = \mathrm{O}(1). \tag{1.11}$$

For the proof, see [3].

2. Proof of Theorem 1.2.

Let T_n denote the nth term of A-transform of the series $\sum (a_n\lambda_n)/na_{nn}$, then

$$T_n - T_{n-1} = \sum_{v=1}^{n} a_v \frac{\hat{a}_{nv}\lambda_v}{va_{vv}}$$

$$= \sum_{v=1}^{n-1} s_v \Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{va_{vv}}\right) + \frac{s_n\,\lambda_n}{n}$$

$$= \sum_{v=1}^{n-1}\left(\frac{\Delta_v\hat{a}_{nv}\,\lambda_v s_v}{va_{vv}} + \frac{\hat{a}_{n,v+1}\,\lambda_v s_v}{v(v+1)a_{vv}} + \frac{\hat{a}_{n,v+1}}{v+1}\Delta\left(\frac{1}{a_{vv}}\right)\lambda_v s_v + \frac{\hat{a}_{n,v+1}\Delta\lambda_v s_v}{(v+1)a_{v+1,v+1}}\right) + \frac{\lambda_n s_n}{n}$$

$$= T_{n1} + T_{n2} + T_{n3} + T_{n4} + T_{n5}\,.$$

In order to prove the theorem , by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1}|T_{nv}|^k < \infty,\ \ v = 1,2,3,4,5.$$

Applying Holder's inequality, (ii), (iii), Lemma 1.3, and (ix),

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n1}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\frac{\Delta_v\hat{a}_{nv}\lambda_v s_v}{va_{vv}}\right|^k$$

$$= \mathrm{O}(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_v||s_v|\right)^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}||\lambda_v|^k\,|s_v|^k\left(\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}|\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_v \hat{a}_{nv}||\lambda_v|^k\,|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}|\lambda_v|^k|s_v|^k\sum_{n=v+1}^{m+1}|\Delta_v \hat{a}_{nv}|$$

$$= O(1)\sum_{v=1}^{m}a_{vv}\,|\lambda_v|^k|s_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_v||s_v|^k}{vX_v^{k-1}}\left(|\lambda_v|\,X_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_v||s_v|^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=0}^{v}\frac{|s_r|^k}{rX_r^{k-1}} + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{|s_v|^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|\,X_v + O(1)|\lambda_m|\,X_m$$

$$= O(1),$$

by using (1.9), Lemma 1.3, (1.10), (ix), and Holder's inequality.

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n2}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=0}^{n-1}\frac{\hat{a}_{n,v+1}\lambda_v s_v}{v(v+1)a_{vv}}\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}||\lambda_v||s_v|\right)^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}||\lambda_v|^k |s_v|^k \left(\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}|\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1} (na_{nn})^{k-1}\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}||\lambda_v|^k |s_v|^k$$

$$= O(1)\sum_{v=1}^{m} a_{vv}|\lambda_v|^k |s_v|^k \sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|$$

$$= O(1)\sum_{v=1}^{m} a_{vv}|\lambda_v|^k |s_v|^k, \quad \text{by using (v), (1.11), and Holder's inequality,}$$

$$= O(1),$$

as in the case of T_{n1}.

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}}{v+1}\Delta\left(\frac{1}{a_{vv}}\right)\lambda_v s_v\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1} a_{vv}|\hat{a}_{n,v+1}||\lambda_v||s_v|\right)^k, \text{ by using (iv),}$$

$$= O(1),$$

as in the case of T_{n2}.

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n4}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v s_v}{(v+1)a_{v+1,v+1}}\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,v+1}||\Delta\lambda_v||s_v|X_v}{X_v}\right)^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\sum_{v=1}^{n-1}\frac{|\hat{a}_{n,v+1}|^k|\Delta\lambda_v||s_v|^k}{X_v^{k-1}}\left(\sum_{v=0}^{n-1}|\Delta\lambda_v|X_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\frac{|\Delta\lambda_v||s_v|^k}{X_v^{k-1}}\sum_{n=v+1}^{m+1} n^{k-1}|\hat{a}_{n,v+1}|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{v|\Delta\lambda_v||s_v|^k}{vX_v^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta\left(v|\Delta\lambda_v|\right)\sum_{r=1}^{v}\frac{|s_r|^k}{rX_r^{k-1}} + O(1)\,m|\Delta\lambda_m|X_m$$

$$= O(1)\sum_{v=1}|\Delta\lambda_v|X_v + O(1)\sum_{v=1}^{m} v|\Delta^2\lambda_v|X_v + O(1)\,m|\Delta\lambda_m|X_m,$$

using Holder's inequality, Lemma 1.3, (vi), (ix), and (viii).

Finally, using Lemma 1.3,

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n5}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\frac{\lambda_n s_n}{n}\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} \frac{|\lambda_n||s_n|^k\left(|\lambda_n| X_n\right)^{k-1}}{n X_n^{k-1}}$$

$$= O(1)\sum_{n=1}^{m+1} \frac{|\lambda_n||s_n|^k}{n X_n^{k-1}}, \quad \text{using Lemma 1.3,}$$

$$= O(1),$$

as in the case of T_{n1}. The proof is complete.

3. Corollary

Corollary 3.1. *Let*

(i) $np_n = O(P_n)$, $P_n = O(np_n)$,

(ii) $\Delta(P_n / p_n) = O(1)$,

(iii) $\sum_{n=v+1}^{\infty} n^{k-1}\left(\frac{p_n}{P_n P_{n-1}}\right)^k = O\left(1/P_v^k\right)$.

If (X_n) *is a positive nondecreasing sequence and the sequence* (λ_n) *is satisfy conditions* (vii)-(ix) *of theorem* 1.2, *then the series* $\sum(a_n P_n \lambda_n)/np_n$ *is summable* $|\bar{N}, p_n|_k$, $k \geq 1$.

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