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TWO SCHUR-CONVEX FUNCTIONS RELATED TO THE GENERALIZED INTEGRAL QUASIARITHMETIC MEANS

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Abstract. The Schur-convexity of two functions which related to the generalized integral quasiarithmetic means are researched, and two new inequalities are established. As applications, some refinements of Hadamard-type inequalities for convex functions and log-convex function are obtained.

Keywords: Schur-convex function; inequality; convex function; log-convex function; Hadamard's inequality; quasiarithmetic means.

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1. Introduction

Throughout the paper we assume that the set of n-dimensional row vector on real number field by $\mathbb{R}^n$, and $\mathbb{R}^n_+ = \{\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n : x_i > 0, i = 1,\dots,n\}$. In particular, $\mathbb{R}^1$ and $\mathbb{R}^1_+$ denoted by $\mathbb{R}$ and $\mathbb{R}_+$ respectively.

Let f be a convex function defined on the interval $I \subseteq \mathbb{R} \to \mathbb{R}$ and the real numbers $a, b \in I$ with $a < b$. Then

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$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

is known as the Hadamard's inequality for convex function [1]. For some recent results which generalize, improve, and extend this classical inequality, see [2-8].

When $f, -g$ both are convex functions satisfying $\int_a^b g(x)dx > 0$ and $f(\frac{a+b}{2}) \ge 0$, S.-J. Yang in [5] generalized (1) as

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \frac{\frac{1}{b-a}\int_a^b f(x)dx}{\frac{1}{b-a}\int_a^b g(x)dx}. \tag{2}$$

To go further in exploring (2), Lan He in [8] define two mappings L and F by

$L : [a,b]\times[a,b] \to \mathbb{R}$,

$$L(x,y;f,g) = \left[\int_x^y f(t)dt - (y-x)f\left(\frac{x+y}{2}\right)\right]\left[(y-x)g\left(\frac{x+y}{2}\right) - \int_x^y g(t)dt\right]$$

and

$F : [a,b]\times[a,b] \to \mathbb{R}$,

$$F(x,y;f,g) = g\left(\frac{x+y}{2}\right)\int_x^y f(t)dt - f\left(\frac{x+y}{2}\right)\int_x^y g(t)dt.$$

Huan-nan Shi in [9] studied the Schur-convexity of $L(x,y;f,g)$ and $F(x,y;f,g)$ with variables (x,y) in $[a,b]\times[a,b] \subseteq \mathbb{R}^2$, obtained the following results.

Theorem A *Let f and $-g$ both be convex function on $[a,b]$. Then $L(x,y;f,g)$ is Schur-convex on $[a,b]\times[a,b] \subseteq \mathbb{R}^2$.*

Theorem B *Let f and $-g$ both be nonnegative convex function on $[a,b]$. Then $F(x,y;f,g)$ is Schur-convex on $[a,b]\times[a,b] \subseteq \mathbb{R}^2$.*

And then Shi established the refinement of the inequality of (2).

Theorem C *Let f and $-g$ both be convex function on $[a,b] \subseteq \mathbb{R}$. If $\int_b^a g(x)dx > 0$ and $f\left(\frac{a+b}{2}\right) \ge 0$, then*

$$\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \le \frac{\int_a^b f(t)dt - \int_{ta+(1-t)b}^{tb+(1-t)a} f(t)dt}{\int_a^b g(t)dt - \int_{ta+(1-t)b}^{tb+(1-t)a} g(t)dt} \le \frac{\int_a^b f(t)dt}{\int_a^b g(t)dt}, \tag{3}$$

where $\frac{1}{2} \le t < 1$ or $0 \le t \le \frac{1}{2}$.

Vera Čuljak et al in [10] discovered the following property of Schur-convexity of the generalized integral quasiarithmetic means.

Theorem D *Let f be a real Lebesgue integrable function defined on the interval $I \subseteq \mathbb{R}$, with range J. Let k be a real continuous strictly monotone function on J. Then, for the generalized integral quasiarithmetic mean of function f defined as*

$$M_k(f;a,b) = \begin{cases} k^{-1}\left(\frac{1}{b-a}\int_a^b (k\circ f)(t)dt\right), & a \neq b; \\ f(a), & a = b. \end{cases} \tag{4}$$

the following hold:

(i) $M_k(f;x,y)$ is Schur-convex on I^2 if $k\circ f$ is convex on I and k is increasing on J or if $k\circ f$ is concave on I and k is decreasing on J;

(ii) $M_k(f;x,y)$ is Schur-concave on I^2 if $k\circ f$ is convex on I and k is decreasing on J or if $k\circ f$ is concave on I and k is increasing on J.

In recent years, Schur-convexity of various functions connected to the Hermite-Hadamard inequality has invoked the interest of many researchers and numerous papers have been dedicated to the investigation of it, see [9-13].

In this paper, comparing (2) with (4), we studied the Schur-convexity of the following two functions:

$$H_{p,q}(f,g;a,b) = \begin{cases} \frac{M_p(f;a,b)}{M_q(g;a,b)}, & a \neq b; \\ \frac{f(a)}{g(a)}, & a = b. \end{cases} \tag{5}$$

and

$$L_{p,q}(f;g;a,b) = \begin{cases} \left[M_p(f;a,b) - f(\frac{a+b}{2})\right]\cdot\left[g(\frac{a+b}{2}) - M_q(g;a,b)\right], & a \neq b; \\ 0, & a = b. \end{cases} \tag{6}$$

2. Preliminaries

We need the following definitions and lemmas.

Definition 1. [14],[15] *Let* $\mathbf{x} = (x_1,\ldots,x_n)$ *and* $\mathbf{y} = (y_1,\ldots,y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x}$ *is said to be majorized by* $\mathbf{y}$ *(in symbols* $\mathbf{x} \prec \mathbf{y}$*) if* $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ *for* $k = 1, 2, \ldots, n-1$ *and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$*, where* $x_{[1]} \geq \cdots \geq x_{[n]}$ *and* $y_{[1]} \geq \cdots \geq y_{[n]}$ *are of* $\mathbf{x}$ *and* $\mathbf{y}$ *in a descending order.*

(ii) *Let* $\Omega \subseteq \mathbb{R}^n$*. The function* φ*:* $\Omega \to \mathbb{R}$ *be said to be a Schur-convex function on* Ω *if* $\mathbf{x} \prec \mathbf{y}$ *on* Ω *implies* $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$*.* φ *is said to be a Schur-concave function on* Ω *if and only if* $-\varphi$ *is Schur-convex.*

Lemma 1.[14],[15] *Let* $\Omega \subseteq \mathbb{R}^n$ *be a symmetric set and with a nonempty interior* Ω^0*,* $\varphi : \Omega \to \mathbb{R}$ *be a continuous on* Ω *and differentiable in* Ω^0*. Then* φ *is the* $Schur-convex(Schur-concave)function$*, if and only if* φ *is symmetric on* Ω *and*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (\leq 0) \tag{7}$$

holds for any $\mathbf{x} = (x_1, \cdots, x_n) \in \Omega^0$*.*

Lemma 2.[16] *Let* $a \leq b, u(t) = ta + (1-t)b, v(t) = tb + (1-t)a$*. If* $\frac{1}{2} \leq t \leq 1$ *or* $0 \leq t \leq \frac{1}{2}$*, then*

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (u(t), v(t)) \prec (a, b). \tag{8}$$

Lemma 3.

(i) *If* $\varphi(x)$ *is a convex function defined on the convex set* $A \subseteq \mathbb{R}$ *and if* $h : \mathbb{R} \to \mathbb{R}$ *is an increasing convex function, then the function* $\psi : \mathbb{R} \to \mathbb{R}$ *defined by* $\psi(x) = h(\varphi(x))$ *is convex on* A*.*

(ii) *If* $\varphi(x)$ *is a concave function defined on the convex set* $A \subseteq \mathbb{R}$ *and if* $h : \mathbb{R} \to \mathbb{R}$ *is an increasing concave function, then the function* $\psi : \mathbb{R} \to \mathbb{R}$ *defined by* $\psi(x) = h(\varphi(x))$ *is concave on* A*.*

Proof. We only give the proof of Lemma 3 (i) in detail. Similar argument leads to the proof of Lemma 3 (ii). If $x, y \in A$, then for all $\alpha \in [0, 1]$,

$$\begin{aligned}\psi(\alpha x+(1-\alpha)y) &= h(\varphi(\alpha x+(1-\alpha)y))\\ &\le h(\alpha\varphi(x)+(1-\alpha)\varphi(y))\\ &\le \alpha h(\varphi(x)+(1-\alpha)h(\varphi(y))\\ &= \alpha\psi(x)+(1-\alpha)\psi(y).\end{aligned}$$

Here the first inequality uses the monotonicity of h together with the convexity of φ; the second inequality uses the convexity of h. □

3. Main results

Our main results are as follows:

Theorem 1. *Let f and g be a real Lebesgue integrable function defined on the interval $I \subseteq \mathbb{R}$, with range J_1 and J_2, respectively, p and q be a real continuous strictly increasing function on J_1 and J_2, respectively, and let $M_p(f;a,b) \geq 0$, $M_q(g;a,b) > 0$ and $g\left(\frac{a+b}{2}\right) \neq 0$.*

(i) *if $p\circ f$ is convex on I, $q\circ g$ is concave on I, then $H_{p,q}(f,g;a,b)$ is Schur-convex on I^2. And then for $a<b$, we have*

$$\frac{M_p(f;a,b)}{M_q(g;a,b)} \geq \frac{M_p(f;ta+(1-t)b,tb+(1-t)a)}{M_q(g;ta+(1-t)b,tb+(1-t)a)} \geq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}, \tag{9}$$

where $\frac{1}{2}\leq t\leq 1$ or $0\leq t\leq \frac{1}{2}$.

(ii) *if $p\circ f$ is concave on I, $q\circ g$ is convex on I, then $H_{p,q}(f,g;a,b)$ is Schur-concave on I^2. And then the inequality chains (7) reverse hold.*

Proof. (i) It is clear that $H_{p,q}(f,g;a,b)$ is symmetric with a,b. Without loss of generality, we may assume $b\geq a$. Directly calculating yields

$$\frac{\partial H_{p,q}}{\partial a} = \frac{1}{M_q^2(g;a,b)}\left(\frac{\partial M_p}{\partial a}M_q - \frac{\partial M_q}{\partial a}M_p\right),$$

$$\frac{\partial H_{p,q}}{\partial b} = \frac{1}{M_q^2(g;a,b)}\left(\frac{\partial M_p}{\partial b}M_q - \frac{\partial M_q}{\partial b}M_p\right),$$

and then

$$\begin{aligned}\Delta :&= (b-a)\left(\frac{\partial H_{p,q}}{\partial b}-\frac{\partial H_{p,q}}{\partial a}\right)\\ &= \frac{M_q}{M_q^2(g;a,b)}(b-a)\left(\frac{\partial M_p}{\partial b}-\frac{\partial M_p}{\partial a}\right)-\frac{M_p}{M_q^2(g;a,b)}(b-a)\left(\frac{\partial M_q}{\partial b}-\frac{\partial M_q}{\partial a}\right)\end{aligned}$$

From Theorem D and Lemma 1, it follows that

$$(b-a)\left(\frac{\partial M_p}{\partial b}-\frac{\partial M_p}{\partial a}\right)\geq 0,\ (b-a)\left(\frac{\partial M_q}{\partial b}-\frac{\partial M_q}{\partial a}\right)\leq 0,$$

so $\Delta \geq 0$, from Lemma 1, it follows that $H_{p,q}(f,g;a,b)$ is Schur-convex on I^2. And then from Lemma 2, we have

$$H_{p,q}(f,g;a,b)\geq H_{p,q}(f,g;ta+(1-t)b,tb+(1-t)a)\geq H_{p,q}\left(f,g;\frac{a+b}{2},\frac{a+b}{2}\right),$$

that is the inequalities (7) hold.

By the same arguments, we can carry out the proof of the proposition (ii).

This completes the proof.

Theorem 2. *Let f and g be a real Lebesgue integrable non negative function defined on the interval $I\subseteq\mathbb{R}$, with range J_1 and J_2, respectively, and let $M_p(f;a,b)\geq 0$, $M_q(g;a,b)>0$ and $g\left(\frac{a+b}{2}\right)\neq 0$. If p,q is a real continuous strictly increasing function on J_1 and J_2, respectively, and $p\circ f$ is convex on I, $q\circ g$ is concave on I, then $L_{p,q}(f,g;a,b)$ is Schur-convex on I^2. And then the following inequality chains hold.*

$$\frac{M_p(f;a,b)}{M_q(g;a,b)}\geq\frac{M_p(f;a,b)}{2M_q(g;a,b)}+\frac{f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)}\geq\frac{f\left(\frac{a+b}{2}\right)}{2M_q(g;a,b)}+\frac{M_p(f;a,b)}{2g\left(\frac{a+b}{2}\right)}\geq\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \tag{10}$$

Proof. It is clear that $L_{p,q}(f,g;a,b)$ is symmetric with a,b. Without loss of generality, we may assume $b\geq a$. Directly calculating yields

$$\begin{aligned}\frac{\partial L_{p,q}}{\partial a}&=\left[\frac{\partial M_p}{\partial a}-\frac{1}{2}f'\left(\frac{a+b}{2}\right)\right]\cdot\left[g\left(\frac{a+b}{2}\right)-M_q(g;a,b)\right]\\ &\quad+\left[\frac{1}{2}g'\left(\frac{a+b}{2}\right)-\frac{\partial M_q}{\partial a}\right]\cdot\left[M_p(f;a,b)-f\left(\frac{a+b}{2}\right)\right],\end{aligned}$$

$$\frac{\partial L_{p,q}}{\partial b} = \left[\frac{\partial M_p}{\partial b} - \frac{1}{2}f'\left(\frac{a+b}{2}\right)\right] \cdot \left[g\left(\frac{a+b}{2}\right) - M_q(g;a,b)\right] + \left[\frac{1}{2}g'\left(\frac{a+b}{2}\right) - \frac{\partial M_q}{\partial b}\right] \cdot \left[M_p(f;a,b) - f\left(\frac{a+b}{2}\right)\right],$$

and then

$$\begin{aligned}\Delta := (b-a)\left(\frac{\partial L_{p,q}}{\partial b} - \frac{\partial L_{p,q}}{\partial a}\right) &= \left[g\left(\frac{a+b}{2}\right) - M_q(g;a,b)\right](b-a)\left(\frac{\partial M_p}{\partial b} - \frac{\partial M_p}{\partial a}\right) \\ &- \left[M_p(f;a,b) - f\left(\frac{a+b}{2}\right)\right](b-a)\left(\frac{\partial M_q}{\partial b} - \frac{\partial M_q}{\partial a}\right).\end{aligned}$$

From Theorem D and Lemma 1, it follows that

$$(b-a)\left(\frac{\partial M_p}{\partial b} - \frac{\partial M_p}{\partial a}\right) \geq 0,\ (b-a)\left(\frac{\partial M_q}{\partial b} - \frac{\partial M_q}{\partial a}\right) \leq 0.$$

Since $\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \prec (a,b)$, from (i) and (ii) in Theorem D, we have $g\left(\frac{a+b}{2}\right) \geq M_q(g;a,b)$ and $M_p(f;a,b) \geq f\left(\frac{a+b}{2}\right)$, respectively, so $\Delta \geq 0$, from Lemma 1, it follows that $L_{p,q}(f,g;a,b)$ is Schur-convex on I^2.

And then, we have

$$L_{p,q}(f,g;a,b) \geq L_{p,q}\left(f,g;\frac{a+b}{2},\frac{a+b}{2}\right) = 0,$$

namely

$$\left[M_p(f;a,b) - f(\frac{a+b}{2})\right] \cdot \left[g(\frac{a+b}{2}) - M_q(g;a,b)\right] \geq 0,$$

it is equivalent to

$$g(\frac{a+b}{2})M_p(f;a,b) + f(\frac{a+b}{2})M_q(g;a,b) \geq f(\frac{a+b}{2})g(\frac{a+b}{2}) + M_p(f;a,b)M_q(g;a,b). \tag{11}$$

Dividing each term of the inequalities (11) by $2M_q(g;a,b)g(\frac{a+b}{2})$, we get second inequality in (8).

From the inequalities (7), it is easy to see that

$$g\left(\frac{a+b}{2}\right)M_p(f;a,b) - f\left(\frac{a+b}{2}\right)M_q(g;a,b) \geq 0. \tag{12}$$

Dividing each term of the inequalities (12) by $M_q(g;a,b)$, we obtain

$$2g\left(\frac{a+b}{2}\right)\frac{M_p(f;a,b)}{M_q(g;a,b)}-g\left(\frac{a+b}{2}\right)\frac{M_p(f;a,b)}{M_q(g;a,b)}-f\left(\frac{a+b}{2}\right)\geq 0, \tag{13}$$

further, dividing each term of the inequalities (13) by $2g\left(\frac{a+b}{2}\right)$, we get first inequality in (8).

From Theorem D, it follows that

$$M_p(f;a,b)\geq M_p\left(f;\frac{a+b}{2},\frac{a+b}{2}\right)$$

and

$$M_q(g;a,b)\leq M_q\left(g;\frac{a+b}{2},\frac{a+b}{2}\right),$$

namely

$$M_p(f;a,b)-f\left(\frac{a+b}{2}\right)\geq 0$$

and

$$g\left(\frac{a+b}{2}\right)-M_q(g;a,b)\geq 0,$$

and then, we have

$$g\left(\frac{a+b}{2}\right)\left[f\left(\frac{a+b}{2}\right)\left(g\left(\frac{a+b}{2}\right)-M_q(g;a,b)\right)+M_q(g;a,b)\left(M_p(f;a,b)-f\left(\frac{a+b}{2}\right)\right)\right]\geq 0,$$

this is

$$\begin{aligned}&\left(g\left(\frac{a+b}{2}\right)\right)^2 f\left(\frac{a+b}{2}\right)+g\left(\frac{a+b}{2}\right)M_p(f;a,b)M_q(g;a,b)\\ \geq&2g\left(\frac{a+b}{2}\right)f\left(\frac{a+b}{2}\right)M_q(g;a,b).\end{aligned} \tag{14}$$

Dividing each term of the inequalities (14) by $2\left(g\left(\frac{a+b}{2}\right)\right)^2 M_q(g;a,b)$, we get third inequality in (8).

This completes the proof.

3. Applications

Theorem 3. *Let f and g be non negative integrable function on $I=[a,b]\subseteq\mathbb{R}_+$, satisfying $\frac{1}{b-a}\int_a^b(g(t))^s dt>0$ and $g\left(\frac{a+b}{2}\right)>0$, for $r\geq 1$ and $0<s\leq 1$. If f is convex and g is concave*

on I, then

$$\frac{\left(\frac{1}{b-a}\int_a^b (f(t))^r dt\right)^{\frac{1}{r}}}{\left(\frac{1}{b-a}\int_a^b (g(t))^s dt\right)^{\frac{1}{s}}} \geq \frac{\left(\frac{1}{b-a}\int_{tb+(1-t)a}^{ta+(1-t)b} (f(t))^r dt\right)^{\frac{1}{r}}}{\left(\frac{1}{b-a}\int_{tb+(1-t)a}^{ta+(1-t)b} (g(t))^s dt\right)^{\frac{1}{s}}} \geq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}. \tag{15}$$

where $\frac{1}{2} \leq t < 1$ *or* $0 \leq t \leq \frac{1}{2}$.

If f is concave and g is convex, then the inequality chains (15) *reverse hold.*

Proof. For $r \geq 1$ and $0 < s \leq 1$, taking $p(x) = x^r$ and $q(x) = x^s$, then p and q is strictly increasing convex and concave on $\mathbb{R}_+$, respectively, and then from Lemma 3, it follows that $f \circ p$ is convex on $[a,b]$ and $g \circ q$ is concave on $[a,b]$, and then by Theorem 1, it is deduced that inequalities (15) hold.

The proof of Theorem 3 is completed. □

By a similar proof of Theorem 1, from Theorem 2, we can obtain the following Theorem.

Theorem 4. *Let f and g be non negative integrable function on* $I = [a,b] \subseteq \mathbb{R}_+$*, satisfying* $\frac{1}{b-a}\int_a^b (g(t))^s dt > 0$ *and* $g\left(\frac{a+b}{2}\right) > 0$*, for* $r \geq 1$ *and* $0 < s \leq 1$*. If f is convex and g is concave on I, then*

$$\begin{aligned} \frac{\left(\frac{1}{b-a}\int_a^b (f(t))^r dt\right)^{\frac{1}{r}}}{\left(\frac{1}{b-a}\int_a^b (g(t))^s dt\right)^{\frac{1}{s}}} &\geq \frac{\left(\frac{1}{b-a}\int_a^b (f(t))^r dt\right)^{\frac{1}{r}}}{2\left(\frac{1}{b-a}\int_a^b (g(t))^s dt\right)^{\frac{1}{s}}} + \frac{f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)} \\ &\geq \frac{f\left(\frac{a+b}{2}\right)}{2\left(\frac{1}{b-a}\int_a^b (g(t))^s dt\right)^{\frac{1}{s}}} + \frac{\left(\frac{1}{b-a}\int_a^b (f(t))^r dt\right)^{\frac{1}{r}}}{2g\left(\frac{a+b}{2}\right)} \geq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}. \end{aligned} \tag{16}$$

Remark 1. *It is obvious that inequalities* (15) *and* (16) *are strengthening and extension of the inequality* (2).

Theorem 5. *Let f and g be positive integrable function on* $I = [a,b] \subseteq \mathbb{R}_+$*, satisfying* $g\left(\frac{a+b}{2}\right) > 0$*. If* $f(x)$ *be log-convex function, and* $g''(x) \leq 0, x \in I$*, then*

$$\frac{\exp\{\frac{1}{b-a}\int_a^b \log f(t)dt\}}{\exp\{\frac{1}{b-a}\int_a^b \log g(t)dt\}} \geq \frac{\exp\{\frac{1}{b-a}\int_{tb+(1-t)a}^{ta+(1-t)b} \log f(t)dt\}}{\exp\{\frac{1}{b-a}\int_{tb+(1-t)a}^{ta+(1-t)b} \log g(t)dt\}} \geq \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}. \tag{17}$$

where $\frac{1}{2} \leq t < 1$ *or* $0 \leq t \leq \frac{1}{2}$.

Proof. Taking $p(x)=q(x)=\log x$, since $g^{"}(x)\le 0$, and then $(\log g(x))^{"}=\frac{g(x)g^{''}(x)-(g^{'}(x))^2}{(g(x))^2}\le 0$, this is $\log g(x)$ is concave. $f(x)$ is a log-convex function, namely, $\log f(x)$ is convex. So from Theorem 1, it is deduced that inequalities (17) hold. □

Similar to the proof of Theorem 5, by the theorem 2, we can prove the following theorem.

Theorem 6. *Let f and g be positive integrable function on $I=[a,b]\subseteq\mathbb{R}_+$, satisfying $g\left(\frac{a+b}{2}\right)>0$. If $f(x)$ is a log-convex function, and $g^{"}(x)\le 0, x\in I$, then*

$$\begin{aligned}\frac{\exp\{\frac{1}{b-a}\int_a^b\log f(t)dt\}}{\exp\{\frac{1}{b-a}\int_a^b\log g(t)dt\}}&\ge\frac{\exp\{\frac{1}{b-a}\int_a^b\log f(t)dt\}}{2\exp\{\frac{1}{b-a}\int_a^b\log g(t)dt\}}+\frac{f\left(\frac{a+b}{2}\right)}{2g\left(\frac{a+b}{2}\right)}\\&\ge\frac{f\left(\frac{a+b}{2}\right)}{2\exp\{\frac{1}{b-a}\int_a^b\log g(t)dt\}}+\frac{\exp\{\frac{1}{b-a}\int_a^b\log f(t)dt\}}{2g\left(\frac{a+b}{2}\right)}\ge\frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)}.\end{aligned}\tag{18}$$

In particular, taking $g(x)=e, x\in[a,b]$, from Theorem 5, we have the following corollary.

Corollary 1. *Let f be positive integrable function on $I=[a,b]\subseteq\mathbb{R}_+$. If $f(x)$ is a log-convex function, then*

$$\exp\{\frac{1}{b-a}\int_a^b\log f(t)dt\}\ge\exp\{\frac{1}{b-a}\int_{tb+(1-t)a}^{ta+(1-t)b}\log f(t)dt\}\ge f\left(\frac{a+b}{2}\right).\tag{19}$$

where $\frac{1}{2}\le t<1$ or $0\le t\le\frac{1}{2}$.

Remark 2. *In [17], Dragomir and Mond proved that the following inequalities of Hermite-Hadamard type hold for log-convex functions:*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\le\exp\{\frac{1}{b-a}\int_a^b\log f(t)dt\}\\&\le\frac{1}{b-a}\int_a^b\sqrt{f(t)f(a+b-t)}dt\\&\le\frac{1}{b-a}\int_a^b\log f(t)dt\\&\le\frac{f(a)-f(b)}{\log f(a)-\log f(b)}\\&\le\frac{f(a)+f(b)}{2}.\end{aligned}\tag{20}$$

The inequality chain (19) is a refinement of the first inequality in [20].

Conflict of Interests

The authors declare that there is no conflict of interests.

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