Available online at http://scik.org
Adv. Inequal. Appl. 2017, 2017:8
ISSN: 2050-7461

ON YANG MEANS

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Abstract. Two bivariate means introduced by Z.-H. Yang in [16] are studied. New representation formulas for the means under discussion are obtained. Several inequalities involving means under discussion are established. Most of the results derived in this paper are obtained with the aid of the Schwab-Borchardt mean.

Keywords: Yang means; Schwab-Borchardt mean; inequalities.

2000 AMS Subject Classification: 26E60, 26D07

1. Introduction

In this section we give a brief overview of known results which pertain to the main results of this paper.

In what follows letters a and b will stand for two positive and unequal numbers.

Recently Z.-H Yang [16] introduced two bivariate means denoted by $V(a,b) \equiv V$ and $U(a,b) \equiv U$. They are defined as follows:

(1) $$V(a,b) = \frac{a-b}{\sqrt{2}\sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)}.$$

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Received January 4, 2017

$$U(a,b)=\frac{a-b}{\sqrt{2}\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)}. \tag{2}$$

2. Definitions of some means of two variables

In what follows the letters x and y will stand for the positive and unequal numbers.

The power mean of order t of x and y will be denoted by A_t. Recall that (see [2])

$$A_t \equiv A_t(x,y)=\begin{cases}\left(\frac{x^t+y^t}{2}\right)^{1/t} & \text{if } t\neq 0,\\ \sqrt{xy} & \text{if } t=0.\end{cases} \tag{3}$$

The unweighted square root mean $Q(x,y)$, arithmetic mean $A(x,y)$ and the geometric mean $G(x,y)$ of x and y are the power means of orders 2, 1 and 0, respectively.

For the sake of presentation we include definitions of several bivariate means used in this paper.

We recall now definitions of the first and the second Seiffert means, denoted respectively by P and T

$$P=A\frac{v}{\sin^{-1}v}, \quad T=A\frac{v}{\tan^{-1}v}, \tag{4}$$

(see [13], [14]). where

$$v=\frac{x-y}{x+y}. \tag{5}$$

Clearly $0<|v|<1$. Other two means used here are the logarithmic mean L and the Neuman-Sándor mean M (cf. [10])

$$\begin{aligned} L\equiv L(x,y)&=\frac{x-y}{\ln x-\ln y}=A\frac{v}{\tanh^{-1}v},\\ M\equiv M(x,y)&=A\frac{v}{\sinh^{-1}v}. \end{aligned} \tag{6}$$

It is well-known that these means satisfy the chain of inequalities

$$G<L<P<A<M<T<Q.$$

The most important mean used in this paper is the Schwab-Borchardt mean $SB(x,y) \equiv SB$ which is defined as follows (see [1]), [3])

(7) $$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\cos^{-1}(x/y)} & \text{if } x<y, \\ \dfrac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)} & \text{if } y<x. \end{cases}$$

Mean SB is non-symmetric, homogeneous of degree 1 and strictly increasing in each variable.

First of all we will give new formulas for means SB. We have [9]

(8) $$SB(x,y) \equiv SB = \begin{cases} y\dfrac{\sin r}{r} = x\dfrac{\tan r}{r} & \text{if } x<y, \\ y\dfrac{\sinh s}{s} = x\dfrac{\tanh s}{s} & \text{if } y<x, \end{cases}$$

where

(9) $$\cos r = x/y \qquad \text{if} \qquad x<y \qquad \text{and} \qquad \cosh s = x/y \qquad \text{if} \qquad x>y.$$

Clearly

(10) $$0<r\le r_0, \qquad \text{where} \qquad r_0 = \max\{\cos^{-1}(x/y) : 0<x<y\}$$

and

(11) $$0<s\le s_0, \qquad \text{where} \qquad s_0 = \max\{\cosh^{-1}(x/y) : x>y>0\}$$

Several inequalities involving Schwab - Borchardt mean have been obtained in [10]

(12) $$(xy^2)^{1/3} < (ySB(y,x))^{1/2} < SB(x,y) < (y+SB(y,x))/2 < \frac{x+2y}{3},$$

where the third inequality is valid only if $x>y$. For the later use let us record the following inequality [10]:

(13) $$SB(x,y) < SB(y,A).$$

More inequalities for the Schwab - Borchardt mean can be found in [5] and [11].

3. New formulas for means U and V

The goal of this section is to demonstrate that the means U and V admit simple representations in terms of the Schwab-Borchardt mean. The main result of this section is contained in the following:

Theorem 1. *Let $Q \equiv Q(a,b)$ and let $G \equiv G(a,b)$. Then*

$$U(a,b) = SB(G,Q) \tag{14}$$

and

$$V(a,b) = SB(Q,G). \tag{15}$$

Proof. We shall prove first formula (14). Using (9) with $x = G$ and $y = Q$ we get

$$\cos r = \sqrt{\frac{2ab}{a^2+b^2}}.$$

This yields

$$\tan r = \frac{a-b}{\sqrt{2ab}}.$$

To derive the desired result we use (8) to obtain

$$SB(G,Q) = G\frac{\tan r}{r} = \frac{a-b}{\sqrt{2}\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)} = U.$$

Formula (15) can be established in a similar manner. We use (9) first to obtain

$$\cosh s = \frac{Q}{G} = \sqrt{\frac{a^2+b^2}{2ab}}.$$

Hence

$$\sinh s = \sqrt{\cosh^2 s - 1} = \frac{a-b}{\sqrt{2ab}}.$$

Application of (8) yields

$$SB(Q,G) = G\frac{\sinh s}{s} = \frac{a-b}{\sqrt{2}\sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)} = V.$$

The proof is complete. □

4. Inequalities involving means U and V

All bivariate means used in this section are mean values of a and b. Our first result reads as follows:

Theorem 2. *The chain of inequalities*

$$L < V < P < U < M < T \tag{16}$$

is valid.

Proof. It has been pointed out in [10] that

$$P = SB(G,A), \quad T = SB(A,Q). \tag{17}$$

The first inequality in (16) can be written as $S(A,G) < SB(Q,G)$ which is an obvious consequence of the monotonicity of SB in its variables. For the proof of the second inequality in (16) we use inequality (13) with $x = Q$ and $y = G$ to obtain $SB(Q,G) < SB(G,A)$. Application of (15) and (17) yields the asserted inequality. The third inequality in (16) is the same as $SB(G,A) < SB(G,Q)$ (see (17) and (14)). The last inequality follows immediately from the monotonicity of SB. For the proof of the fourth inequality in the chain (16) we utilize (13) with $x = Q$ and $y = G$ and next apply (14) and (17). The last inequality in (16) is established in [10]. The proof is complete. □

Our next result reads as follows:

Theorem 3. *Yang means U and V satisfy the following inequalities*

$$(GQ^2)^{1/3} < (QV)^{1/2} < U < \frac{G+2Q}{3} \tag{18}$$

and

$$(QG^2)^{1/3} < (GU)^{1/2} < V < \frac{G+U}{2} < \frac{Q+2G}{3}. \tag{19}$$

Proof. Chain of inequalities (18) follow immediately from (12). We let $x = G$ and $y = Q$ and next we utilize formulas (14) and (15). Inequalities (19) can be established in a similar way. This time we let $x = Q$ and $y = G$. We omit further details. This completes the proof. □

The power means bounds for the mean U have been established in [17].

5. Wilker and Huygens-type inequalities

In recent years many researchers invastigated inequalities introduced by Wilker (see [15]) and Huygens cf. [4]. Various generalizations of those inequalities have been published (see, e.g., [6, 7, 12]). The goal of this section is to obtain the Wilker and Huygens-type inequalities for the Yang means.

Throughout the sequel the letters u and v will stand for two positive numbers which satisfy the following conditions

$$\min(u,v) < 1 < \max(u,v), \tag{20}$$

$$1 < u^{\alpha} v^{\beta}, \tag{21}$$

and

$$1 < \frac{\alpha}{\alpha+\beta}\frac{1}{u} + \frac{\beta}{\alpha+\beta}\frac{1}{v}, \tag{22}$$

where the last two inequalities must be satisfied for some positive numbers α and β.

In the proofs of the main results of this section we will utilize the following result [8]:

Theorem A. Let $\lambda > 0$ and $\mu > 0$. If $u < 1 < v$, then

$$1 < \frac{\lambda}{\lambda+\mu}u^{p} + \frac{\mu}{\lambda+\mu}v^{q} \tag{23}$$

if either

$$q > 0 \qquad \text{and} \qquad p \le q\frac{\alpha\mu}{\beta\lambda} \tag{24}$$

or if

$$p \le q \le -1 \qquad \text{and} \qquad \beta\lambda \ge \alpha\mu. \tag{25}$$

If $v < 1 < u$, then the inequality (23) holds true if either

$$p > 0 \qquad \text{and} \qquad q \le p\frac{\beta\lambda}{\alpha\mu} \tag{26}$$

or if

$$q \le p \le -1 \qquad \text{and} \qquad \alpha\mu \ge \beta\lambda. \tag{27}$$

We are in a position to prove the main result of this section.

Theorem 4. *Let $\lambda > 0$, $\mu > 0$. Then the inequality*

$$1 < \frac{\lambda}{\lambda+\mu}(\frac{U}{Q})^p + \frac{\mu}{\lambda+\mu}(\frac{U}{G})^q \tag{28}$$

is satisfied if either

$$q > 0 \qquad \textit{and} \qquad p\lambda \le 2q\mu \tag{29}$$

or if

$$p \le q \le -1 \qquad \textit{and} \qquad 2\mu \le \lambda. \tag{30}$$

Also, the inequality

$$1 < \frac{\lambda}{\lambda+\mu}(\frac{V}{Q})^p + \frac{\mu}{\lambda+\mu}(\frac{V}{G})^q \tag{31}$$

is valid if either

$$q > 0 \qquad \textit{and} \qquad 2p\lambda \le q\mu \tag{32}$$

or if

$$p \le q \le -1 \qquad \textit{and} \qquad \mu \le 2\lambda. \tag{33}$$

Proof. We shall prove first that the inequality (28) holds true if conditions (29) and (30) are satisfied. Since U is the mean value of G and Q,

$$G < U < Q.$$

This yields

$$\frac{U}{Q} < 1 < \frac{U}{G}.$$

Letting

$$u = \frac{U}{Q} \qquad \text{and} \qquad v = \frac{U}{G}$$

we see that the separation condition $u < 1 < v$ is satisfied. It follows from (18) that $(GQ^2)^{1/3} < U$. This inequality is equivalent to the following one

$$1 < u^{2/3}v^{1/3}.$$

Comparison with (21) yields

$$\alpha = \frac{2}{3} \qquad \text{and} \qquad \beta = \frac{1}{3}. \tag{34}$$

Utilizing inequality which connects the third and fourth members of (18) we see that with u, v, α and β as above the inequality (22) holds true. Thus all assumptions of Theorem A are satisfied. This yields inequality (28) with conditions of validity (29) and (30). The second assertion of this theorem can be established in a similar fashion. We leave it to the interested reader. The proof is completed. □

Corollary 5. *Inequality (28) can also be written as*

$$1 < \frac{\lambda}{\lambda+\mu}(\frac{\sin r}{r})^p + \frac{\mu}{\lambda+\mu}(\frac{\tan r}{r})^q, \tag{35}$$

where

$$\cos r = \frac{G}{Q} = \sqrt{\frac{2ab}{a^2+b^2}},$$

where λ, μ, p, q must to satisfy either conditions (29) or (30). Also, inequality (31) is equivalent to

$$1 < \frac{\lambda}{\lambda+\mu}(\frac{\tanh s}{s})^p + \frac{\mu}{\lambda+\mu}(\frac{\sinh s}{s})^q, \tag{36}$$

where

$$\cosh s = \frac{Q}{G} = \sqrt{\frac{a^2+b^2}{2ab}}$$

where λ, μ, p, q must to satisfy either conditions (32) or (33).

Proof. For the proof of (35) we use (8) and (14) to obtain

$$U = SB(G,Q) = Q\frac{\sin r}{r} = G\frac{\tan r}{r}.$$

This yields

$$\frac{U}{Q} = \frac{\sin r}{r} \qquad \text{and} \qquad \frac{U}{G} = \frac{\tan r}{r}.$$

Applying the last two formulas to (28) we obtain the assertion. The proof of (35) goes along the lines used above. We omit further details. The proof is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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