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ON YANG MEANS II

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Abstract. Several inequalities involving bivariate means introduced by Z.-H. Yang in [15] are established. Also, lower and upper bounds for the means under discussion are obtained. Bounding quantities are expressed in terms of the geometric and quadratic means. Results presented in this paper are obtained with the aid of the Schwab-Borchardt mean.

Keywords: Yang means; Schwab-Borchardt mean; bivariate means; inequalities; lower and upper bounds.

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1. Introduction

Recently Z.-H Yang [15] introduced two bivariate means denoted in the sequel by V and U. For the sake of presentation we include below explicit formulas for these means. This paper is a continuation of a research initiated in [9] and is organized as follows. Definitions of other bivariate means utilized in this work are given in Section 2. List of those means include two Seiffert means, logarithmic mean, Neuman-Sándor mean and the Schwab-Borchardt mean SB. The latter plays a crucial role in our presentation. Several known inequalities satisfied by mean

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SB are given in Section 3. New inequalities involving means U and V are established in Section 4.

Throughout the sequel the letters a and b will stand for two positive and unequal numbers. The Yang means are defined as follows:

(1) $$V(a,b) = \frac{a-b}{\sqrt{2}\sinh^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)}.$$

(2) $$U(a,b) = \frac{a-b}{\sqrt{2}\tan^{-1}\left(\frac{a-b}{\sqrt{2ab}}\right)}.$$

2. Bivariate means used in this paper

In what follows the letters x and y will stand for the nonnegative and unequal numbers.

The power mean of order t of x and y will be denoted by A_t. Recall that (see [2])

(3) $$A_t \equiv A_t(x,y) = \begin{cases} \left(\frac{x^t+y^t}{2}\right)^{1/t} & \text{if } t \neq 0, \\ \sqrt{xy} & \text{if } t = 0. \end{cases}$$

The unweighted square root mean $Q(x,y)$, arithmetic mean $A(x,y)$ and the geometric mean $G(x,y)$ of x and y are the power means of orders 2, 1 and 0, respectively.

For the sake of presentation we include definitions of several bivariate means used in this paper.

We recall now definitions of the first and the second Seiffert means which are denoted respectively by P and T

(4) $$P = A\frac{v}{\sin^{-1} v}, \quad T = A\frac{v}{\tan^{-1} v},$$

(see [13], [14]). where

(5) $$v = \frac{x-y}{x+y}.$$

Clearly $0 < |v| \le 1$. Other two means used here are the logarithmic mean L and the Neuman-Sándor mean M (cf. [11])

$$L \equiv L(x,y) = \frac{x-y}{\ln x - \ln y} = A\frac{v}{\tanh^{-1} v}, \qquad M \equiv M(x,y) = A\frac{v}{\sinh^{-1} v}. \tag{6}$$

It is well-known (see [11]) that these means satisfy the chain of inequalities

$$G < L < P < A < M < T < Q.$$

The most important mean used in this paper is the Schwab-Borchardt mean $SB(x,y) \equiv SB$ which is defined as follows (see [1]), [3])

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\cos^{-1}(x/y)} & \text{if } 0 \le x < y, \\ \dfrac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)} & \text{if } y < x. \end{cases} \tag{7}$$

Mean SB is non-symmetric, homogeneous of degree 1 and strictly increasing in each variable. This mean is well defined when the first variable is equal to 0.

We will give new formulas for means SB. We have [10]

$$SB(x,y) \equiv SB = \begin{cases} y\dfrac{\sin r}{r} = x\dfrac{\tan r}{r} & \text{if } 0 \le x < y, \\ y\dfrac{\sinh s}{s} = x\dfrac{\tanh s}{s} & \text{if } y < x, \end{cases} \tag{8}$$

where

$$\cos r = x/y \quad \text{if} \quad x < y \quad \text{and} \quad \cosh s = x/y \quad \text{if} \quad x > y. \tag{9}$$

Clearly

$$0 < r \le r_0, \quad \text{where} \quad r_0 = \max\{\cos^{-1}(x/y) : 0 \le x < y\} \tag{10}$$

and

$$0 < s \le s_0, \quad \text{where} \quad s_0 = \max\{\cosh^{-1}(x/y) : x > y > 0\} \tag{11}$$

It follows from (8) and (9) that

$$SB(0,y)=\frac{2y}{\pi}. \tag{12}$$

The important fact that the Yang means can be represented in terms of the mean *SB* has been established in [9]:

$$U=SB(G,Q) \tag{13}$$

and

$$V=SB(Q,G). \tag{14}$$

It has been demonstrated in [9] that

$$L<V<P<U<M<T. \tag{15}$$

Means L,P,M and T also admit representation in terms of *SB*. It is known (see [11]) that

$$L=SB(A,G), \qquad P=SB(G,A), \qquad M=SB(Q,A), \qquad T=SB(A,Q). \tag{16}$$

We close this section with formulas for two other means which will be also utilized in the sequel:

$$N(x,y)=\frac{1}{2}\Big(x+\frac{y^2}{SB(x,y)}\Big) \tag{17}$$

(see [6]) and

$$R(x,y)=ye^{x/SB(x,y)-1} \tag{18}$$

(see [7, 8]).

3. Inequalities involving means *SB*, *N* and *R*

Proofs of main results presented in the next section relay on certain inequalities presented below.

We begin with inequalities for the Schwab-Borchardt mean. The following one

$$SB(x_1,y_1)SB(x_2,y_2) < SB(x_1,y_2)SB(x_2,y_1), \tag{19}$$

where

$$0 \le x_1 < x_2 \qquad \text{and} \qquad 0 < y_1 < y_2 \tag{20}$$

follows from a result obtained in [4]. In particular, if $x_1 = 0$ and $x_2 = x > 0$, then (19) becomes

$$y_1 SB(x,y_2) < y_2 SB(x,y_1). \tag{21}$$

Another inequality involving a product of two Schwab-Borchardt means appears in [11]:

$$SB(x_1,y_1)SB(x_2,y_2) < SB^2(A_2(x_1,x_2),A_2(y_1,y_2)). \tag{22}$$

For more inequalities involving Schwab-Borchardt mean see [12, 5] and the references therein.

The next three inequalities involve means SB, N and R.

If $0 < x < y$, then

$$R(x,y) < SB(x,y) < N(x,y). \tag{23}$$

If $x > y > 0$, then

$$SB(x,y) < N(x,y) < R(x,y). \tag{24}$$

The last two inequalities have been established in [7]. The next inequality (see [8, 11]) reads as follows

$$SB(y,x) < \frac{2x+y}{3} < R(x,y) < SB(x,y). \tag{25}$$

Here the third inequality holds true provided $x < y$. We close this section with a two-sided inequality [8]:

$$\left(\frac{SB(x,y)}{y}\right)^{\alpha} < \frac{R(x,y)}{y} < \left(\frac{SB(x,y)}{y}\right)^{\beta} \tag{26}$$

which is valid provided $x < y$ and numbers α and β satisfy the following conditions

$$a \ge \log(\pi/2) = 2.214\ldots \qquad \text{and} \qquad \beta \le 2. \tag{27}$$

If $x > y$, then the inequality (26) holds true if

$$\alpha \le 1 \qquad \text{and} \qquad \beta \ge 2. \tag{28}$$

4. Main results

Our first result reads as follows:

Theorem 1. *The following inequalities are valid*

$$PT < AU, \qquad PQ < UM, \qquad GQ < UV, \qquad LQ < TV, \tag{29}$$

$$GM < AV, \qquad AU < QP \tag{30}$$

and

$$UV < A^2 \tag{31}$$

are valid.

A second inequality in (30) has been obtained in [15]. Below we offer a simple proof of this result.

Proof. Inequalities in (29) are established with the aid of formula (19) and formulas (13), (14) and (16). To obtain the first one we let in (19) $x_1 = G$, $x_2 = A$, $y_1 = A$ and $y_2 = Q$. To obtain the second one we let $x_1 = G$, $x_2 = Q$, $y_1 = A$ and $y_2 = Q$. Third inequality in the chain (29) is obtained in a similar fashion by letting $x_1 = G$, $x_2 = Q$, $y_1 = G$ and $y_2 = Q$. Finally with $x_1 = A$, $x_2 = Q$, $y_1 = G$ and $y_2 = Q$ we obtain the fourth inequality in (29). Letting in (21) $x = Q$, $y_1 = G$ and $y_2 = A$ and next employing formulas (14) and (16) we obtain the first inequality in (30). Second inequality in (30) can be established in a similar manner. With $x = G$, $y_1 = A$ and $y_2 = Q$ inequality (21) yields, with the aid of (13) and (16) the assertion. In the proof of (31) we will utilize (22), (13), (14) and (16) with $x_1 = G$, $x_2 = Q$, $y_1 = Q$ and $y_2 = G$ to obtain

$$UV < SB^2(A_2(G,Q), A_2(Q,G)) = A_2^2(G,Q) = A^2.$$

The proof is complete. □

We shall establish now the following:

Theorem 2. *Yang means satisfy the following inequalities*

$$Qe^{G/U-1} < U < \frac{1}{2}\left(G + \frac{Q^2}{U}\right), \tag{32}$$

$$V < \frac{1}{2}\left(Q + \frac{G^2}{V}\right) < Ge^{Q/V-1} \tag{33}$$

$$V < \frac{2G+Q}{3} < Qe^{G/U-1} < U, \tag{34}$$

$$\left(\frac{U}{Q}\right)^{\alpha} < e^{G/U-1} < \left(\frac{U}{Q}\right)^{\beta}, \tag{35}$$

where α and β are the same as in (27). Also, the inequality

$$\left(\frac{V}{G}\right)^{\alpha} < e^{Q/V-1} < \left(\frac{V}{G}\right)^{\beta} \tag{36}$$

is valid where now

$$\alpha \le 1 \qquad \textit{and} \qquad \beta \ge 1.$$

Proof. For the proof of (32) we use (23) with $x = G$ and $y = Q$ and also utilize formulas (17) and (16) to obtain the desired result. Inequality (33) can be established in a similar manner. We employ (24) with $x = Q$ and $y = G$ to obtain the assertion. Making use of (25) with $x = G$ and $y = Q$ we obtain, using (13), (14) and (17), inequality (34). The remaining two inequalities (35) and (36) follow from (26). The former is obtained letting in (26) $x = G$ and $y = Q$ while the latter one is a special case of (26) provided $x = Q$ and $y = G$. This completes the proof. □

We close this section with the following:

Theorem 3. *Let*

$$W = \frac{G+Q}{2}. \tag{37}$$

Then

$$(QW^2)^{1/3} < U < \sqrt{W}\,\frac{\sqrt{W}+2\sqrt{Q}}{3} \tag{38}$$

and

$$(GW^2)^{1/3} < V < \sqrt{W}\frac{\sqrt{W}+2\sqrt{G}}{3}. \tag{39}$$

Proof. We shall utilize the invariance property of the Schwab-Borchardt mean, cf. [1]:

$$SB(x,y) = SB(A, \sqrt{Ay}), \tag{40}$$

where A stands for the arithmetic mean of x and y. Also, we shall apply the two-sided inequality

$$(xy^2)^{1/3} < SB(x,y) < \frac{x+2y}{3} \tag{41}$$

(see [11]). For the proof of (38) we employ (13), (37) and (40) with $x = G$ and $y = Q$ to obtain

$$U = SB(G,Q) = SB(W, \sqrt{WQ}).$$

Making use of (41) we obtain the asserted result. Inequality (39) can be established in an analogous way. We let $x = Q$ and $y = G$ and proceed as in the proof of (38). We omit further details. □

Power means bounds for the Yang means are established in [16].

Conflict of Interests

The author declares that there is no conflict of interests.

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