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ALTERNATIVE PROOFS OF THE GENERALIZED REVERSE YOUNG INEQUALITIES

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Abstract. We give alternative proofs of the generalized reverse Young inequalities shown in our previous paper [S. Furuichi, M.B. Ghaemi and N. Gharakhanlu, Generalized reverse Young and Heinz inequalities, Bull. Malays. Math. Sci. Soc. (2017). doi:10.1007/s40840-017-0483-y].

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1. Main results

We start from the famous formula

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{r\to 0}(1+rx)^{1/r}.$$

In this article, we consider the inverse function of r-exponential function $\exp_r(x) \equiv (1+rx)^{1/r}$, namely r-logarithmic function defined by $\ln_r x \equiv \frac{x^r-1}{r}$ for $x \geq 0$ and a real number $r \neq 0$.

Lemma 1.1. $\ln_r x$ is a monotone increasing function in r.

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Proof. In the inequality $\log t \le t-1$ for $t>0$, we set $t = x^{-r}$, we obtain the following

$$\frac{\partial \ln_r x}{\partial r} = \frac{x^r\left(\log x^r - 1 + x^{-r}\right)}{r^2} \ge 0.$$

□

Lemma 1.1. implies the following lemma.

Lemma 1.2. Let r, v, t be real numbers with $r \neq 0$ and $t > 0$.

(i) For $0 < r < v - \frac{1}{2}$ or $v - \frac{1}{2} < 0 < r$, we have

$$\left(v - \frac{1}{2}\right)\frac{t^r - 1}{r} \le t^{v-\frac{1}{2}} - 1. \tag{1}$$

(ii) For $0 < r < v$ or $v < 0 < r$, we have

$$v\frac{t^r - 1}{r} \le t^v - 1. \tag{2}$$

Applying (i) and (ii) of Lemma , we can derive respectively [1, Theorem 1] and [1, Theorem 3] without using the supplemental Young's inequality given in [1, Lemma 5] which used to prove [1, Theorem 1] and [1, Theorem 3].

Theorem 1.1. ([1, Theorem 1]) Let v, t be real numbers with $t > 0$, and $n \in \mathbb{N}$ with $n \ge 2$.

(i) For $v \notin \left[\frac{1}{2}, \frac{2^{n-1}+1}{2^n}\right]$, we have

$$(1-v) + vt \le t^v + (1-v)\left(1-\sqrt{t}\right)^2 + (2v-1)\sqrt{t}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{t} - 1\right)^2 \tag{3}$$

(ii) For $v \notin \left[\frac{2^{n-1}-1}{2^n}, \frac{1}{2}\right]$, we have

$$(1-v)t + v \le t^{1-v} + v\left(\sqrt{t}-1\right)^2 + (1-2v)\sqrt{t}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{t} - 1\right)^2 \tag{4}$$

Proof.

(i) Direct calculations imply

$$(1-v) + vt - (1-v)\left(1-\sqrt{t}\right)^2 - (2v-1)\sqrt{t}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{t} - 1\right)^2$$

$$= \sqrt{t} + \sqrt{t}\left(v - \frac{1}{2}\right)2^n\left(\sqrt[2^n]{t} - 1\right). \tag{5}$$

Thus the inequality (3) is equivalent to the inequality

$$\left(v - \frac{1}{2}\right)2^n\left(\sqrt[2^n]{t} - 1\right) \le t^{v-\frac{1}{2}} - 1. \tag{6}$$

This inequality is true by (i) of Lemma 1.2. with $r = \frac{1}{2^n}$, since the conditions $0 < r < v - \frac{1}{2}$ or $v - \frac{1}{2} < 0 < r$ are satisfied in the case of $v \notin \left[\frac{1}{2}, \frac{2^{n-1}+1}{2^n}\right]$ in (i) of Lemma 1.2.

(ii) Exchanging $1-v$ with v in (i) of Lemma 1.2., we have

$$\left(\frac{1}{2} - v\right)\frac{t^r - 1}{r} \le t^{\frac{1}{2}-v} - 1 \tag{7}$$

for $v < \frac{1}{2} - r$ or $\frac{1}{2} < v$. Exchanging $1-v$ with v in the inequality (6), we have

$$\left(\frac{1}{2} - v\right)2^n\left(\sqrt[2^n]{t} - 1\right) \le t^{\frac{1}{2}-v} - 1. \tag{8}$$

This inequality is true by the inequality (7) with $r = \frac{1}{2^n}$, since the conditions $v < \frac{1}{2} - r$ or $\frac{1}{2} < v$ are satisfied in the case of $v \notin \left[\frac{2^{n-1}-1}{2^n}, \frac{1}{2}\right]$ in (i) of Lemma 1.2.

□

Theorem 1.2. ([1, Theorem 3]) Let v, t be real numbers with $t > 0$, and $n \in \mathbb{N}$.

(i) For $v \notin \left[0, \frac{1}{2^n}\right]$, we have

$$(1-v) + vt \le t^v + v\sum_{k=1}^{n} 2^{k-1}\left(1 - \sqrt[2^k]{t}\right)^2 \tag{9}$$

(ii) For $v \notin \left[\frac{2^n-1}{2^n}, 1\right]$, we have

$$(1-v)t + v \le t^{1-v} + (1-v)\sum_{k=1}^{n} 2^{k-1}\left(1 - \sqrt[2^k]{t}\right)^2 \tag{10}$$

Proof.

(i) Direct calculations imply

$$(1-v) + vt - v\sum_{k=1}^{n} 2^{k-1}\left(1 - \sqrt[2^k]{t}\right)^2 = v2^n\left(\sqrt[2^n]{t} - 1\right) + 1 \tag{11}$$

so that the inequality (9) is equivalent to the inequality

$$v2^n\left(\sqrt[2^n]{t} - 1\right) \le t^v - 1 \tag{12}$$

This inequality is true by (ii) of Lemma 1.2. with $r = \frac{1}{2^n}$, since the conditions $0 < r < v$ or $v < 0 < r$ are satisfied in the case of $v \notin \left[0, \frac{1}{2^n}\right]$ in (ii) of Lemma 1.2.

(ii) Exchanging $1-v$ with v in (ii) of Lemma 1.2., we have

$$(1-v)\frac{t^r-1}{r} \le t^{1-v}-1 \tag{13}$$

for $0<r<1-v$ or $1-v<0<r$. Exchanging $1-v$ with v in the inequality (12), we also have

$$(1-v)2^n\left(\sqrt[2^n]{t}-1\right) \le t^{1-v}-1 \tag{14}$$

This inequality is true by the inequality (13) with $r=\frac{1}{2^n}$, since the conditions $0<r<1-v$ or $1-v<0<r$ are satisfied in the case of $v\notin\left[\frac{2^n-1}{2^n},1\right]$ in (ii) of Lemma 1.2.

□

By theory of Kubo-Ando [2], we have the following corollary from Lemma 1.2.

Corollary 1.3. Let r,v,t be real numbers with $r\neq 0$ and $t>0$. For $\alpha\in\mathbb{R}$, a positive definite matrix A and a positive semidefinite matrix B, we define $A\natural_\alpha B\equiv A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}$. Then we have the following matrix inequalities.

(i) For $0<r<v-\frac{1}{2}$ or $v-\frac{1}{2}<0<r$, we have

$$\left(v-\frac{1}{2}\right)\frac{A\natural_r B-A}{r} \le A\natural_{v-\frac{1}{2}}B-A. \tag{15}$$

(ii) For $0<r<v$ or $v<0<r$, we have

$$v\frac{A\natural_r B-A}{r} \le A\natural_v B-A. \tag{16}$$

2. Additional results

The methods in previous section are applicable to obtain the inequalities in the following propositions.

Proposition 2.1. Let v,t be real numbers with $t>0$, and $n\in\mathbb{N}$.

(i) For $v\in\left[0,\frac{1}{2^n}\right]$, we have

$$(1-v)+vt \ge t^v + v\sum_{k=1}^{n}2^{k-1}\left(1-\sqrt[2^k]{t}\right)^2 \tag{17}$$

If $\alpha\in[0,1]$, $A\sharp_\alpha B\equiv A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^\alpha A^{1/2}$ is called α-weighted geometric mean.

(ii) For $v \in \left[\frac{2^n-1}{2^n}, 1\right]$, we have

$$(1-v)t + v \geq t^{1-v} + (1-v)\sum_{k=1}^{n} 2^{k-1}\left(1 - \sqrt[2^k]{t}\right)^2 \tag{18}$$

Proof.

(i) By Lemma 1.1., we have

$$v\frac{t^r-1}{r} \geq t^r - 1, \quad (0 \leq v \leq r)$$

which implies

$$v2^n\left(t^{\frac{1}{2^n}} - 1\right) \geq t^v - 1, \quad \text{for} \quad v \in \left[0, \frac{1}{2^n}\right] \tag{19}$$

by putting $r = \frac{1}{2^n}$. Since we have the identity (11), the inequality (17) is equivalent to the inequality (19).

(ii) Exchanging $1-v$ with v, the inequality (19) becomes

$$(1-v)2^n\left(t^{\frac{1}{2^n}} - 1\right) \geq t^{1-v} - 1, \quad \text{for} \quad v \in \left[\frac{2^n-1}{2^n}, 1\right]. \tag{20}$$

Then the inequality (17) is also changed to the inequality (18), which is true by the inequality (20).

□

Proposition 2.2. Let v, t be real numbers with $t > 0$, and $n \in \mathbb{N}$ with $n \geq 2$.

(i) For $v \in \left[\frac{1}{2}, \frac{2^{n-1}+1}{2^n}\right]$, we have

$$(1-v) + vt \geq t^v + (1-v)\left(1-\sqrt{t}\right)^2 + (2v-1)\sqrt{t}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{t} - 1\right)^2 \tag{21}$$

(ii) For $v \in \left[\frac{2^{n-1}-1}{2^n}, \frac{1}{2}\right]$, we have

$$(1-v)t + v \geq t^{1-v} + v\left(\sqrt{t}-1\right)^2 + (1-2v)\sqrt{t}\sum_{k=2}^{n} 2^{k-2}\left(\sqrt[2^k]{t} - 1\right)^2 \tag{22}$$

Proof.

(i) By Lemma 1.1., we have

$$\left(v - \frac{1}{2}\right)\frac{t^r-1}{r} \geq t^{v-\frac{1}{2}} - 1, \quad \left(0 \leq v - \frac{1}{2} \leq r\right),$$

which implies

$$\left(v-\frac{1}{2}\right)2^n\left(t^{\frac{1}{2^n}}-1\right)\geq t^{v-\frac{1}{2}}-1,\quad \text{for}\quad v\in\left[\frac{1}{2},\frac{2^{n-1}+1}{2^n}\right] \tag{23}$$

by putting $r=\frac{1}{2^n}$. Since we have the identity (5), the inequality (21) is equivalent to the inequality (23).

(ii) Exchanging $1-v$ with v, the inequality (23) becomes

$$\left(\frac{1}{2}-v\right)2^n\left(t^{\frac{1}{2^n}}-1\right)\geq t^{\frac{1}{2}-v}-1,\quad \text{for}\quad v\in\left[\frac{2^{n-1}-1}{2^n},\frac{1}{2}\right]. \tag{24}$$

Then the inequality (21) is also changed to the inequality (22), which is true by the inequality (24).

□

Conflict of Interests

The authors declare that there is no conflict of interests.

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