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WEIGHTED (0;0,2)-INTERPOLATION ON THE UNIT CIRCLE

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Abstract. In this paper, we study the explicit representation of Pál-type weighted $(0;0,2)$-interpolation on the unit circle on two pairwise disjoint sets of nodes obtained by projecting vertically the zeros of $\left(1-\mathrm{x}^2\right)\,\mathrm{P_n}\left(\mathrm{x}\right)$ and $\left(1-\mathrm{x}^2\right)\,\mathrm{P}''_{\mathrm{n}}\left(\mathrm{x}\right)$ respectively onto the unit circle, where $\mathrm{P_n}\left(\mathrm{x}\right)$ stands for n^{th} Legendre polynomial.

Keywords: Legendre polynomial; weighted interpolation; Pál-type interpolation; existence; explicit forms.

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1. Introduction

The problem of Lacunary interpolation was initiated by P. Turán [12] on the zeros of $\left(1-x^2\right)P'_{n-1}\left(x\right)$, where $P_{n-1}\left(x\right)$ is the Legendre polynomial of degree $(n-1)$. The problem of $(0,2)$ interpolation on the roots of unity was first studied by O. Kiš [9]. Later on several mathematicians have studied such kind of interpolatory polynomials on the different set of nodes viz. real line, unit circle etc. L.G.Pál [11] considered two sets of nodes with one additional condition,where the function values are prescribed on one set and the derivative on other one and the functional value at the additional point and he obtained a unique polynomial of degree $2n-1$. In 1990,

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M. R. Akhlaghi and A. Sharma [1] considered Pál-type interpolation problems on the zeros of $\prod_n(x)$ and $\prod_n'(x)$ and obtained the fundamental polynomials. After that J.Szabadós and A.K.Varma [8] considered convergence of $(0,2)$ interpolation process on the zeros of $\prod_n(x)$ and obtained existence,explicit representation and convergence theorem . In 2010, M.Lénárd [10] considered the Pál-type interpolation on different kind of interpolation conditions on the real line. H.P. Dikshit [2] considered the existence of Pál -type interpolation on the non-uniformly distributed nodes on the unit circle. In 2011, K. K. Mathur and author[1] [3] established convergence of weighted $(0,2)$ interpolation on unit circle. In 2012, she [4] considered $(0;0,2)$ interpolation on projected nodes of $\left(1-x^2\right)P_n(x)$ and $P_n'(x)$ on the unit circle.Later on she [5] considered the $(0,2;0)$ interpolation on the same set of nodes. After that she (with M. Shukla) [6] considered weighted Pál-type $(0;1)$ interpolation on the two disjoint set of nodes ,which are obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ respectively onto the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for the Jacobi polynomial. Recently, authors [7] considered weighted $(0,2)$ interpolation on the nodes obtained by projecting vertically the zeros of $\left(1-x^2\right)P_n'(x)$ onto the unit circle obtained the existence and established a convergence theorem for the same.

These have motivated us to consider $(0;0,2)$ interpolation on two pairwise disjoint set of nodes onto the unit circle, in which the Lagrange data is prescribed on the first set of nodes where as Lacunary data on the other one .We obtained regularity and explicit forms of interpolatory polynomials.

In this paper, we consider two pairwise disjoint set of the nodes Z_n and G_n, where Lagrange data on Z_n and weighted (0,2) data on G_n are prescribed , where Z_n and G_n are given as,

$$Z_n = \left\{ \begin{array}{c} z_0 = 1, \quad z_{2n+1} = -1, \\ z_k = cos\theta_k + i\, sin\theta_k, \\ z_{n+k} = \overline{z_k}\,, \quad k = 1(1)n \end{array} \right\} \tag{1}$$

$$G_n = \left\{ \begin{array}{c} t_0 = 1, \quad t_{2n-3} = -1, \\ t_k = cos\varphi_k + i\, sin\varphi_k, \\ t_{n+k} = \overline{t_k}\,, \quad k = 1(1)n-2 \end{array} \right\} \tag{2}$$

In section 2,we give some Preliminaries, in section 3, we describe the problem and regularity, in section 4, we present the explicit forms of Pál-type weighted $(0;0,2)$-interpolation onto the unit circle.

2. Preliminaries

In this section, we shall give some well-known results, which we shall use.
The differential equation satisfied by $P_n(x)$ is

$$\left(1-x^2\right)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0 \tag{3}$$

$$W(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n\left(\frac{1+z^2}{2z}\right)z^n \tag{4}$$

$$R(z)=\left(z^2-1\right)\ W(z) \tag{5}$$

$$H(z)=\prod_{k=1}^{2n-4}(z-t_k)=K_n^*P_n''\left(\frac{1+z^2}{2z}\right)z^{n-2} \tag{6}$$

$$N(z)=\left(z^2-1\right)\ H(z) \tag{7}$$

We shall require the fundamental polynomial of Lagrange interpolation based on the zeros of $R(z)$, $H(z)$ and $N(z)$ are respectively given as:

$$L_k(z)=\frac{R(z)}{R'(z_k)\ (z-z_k)},k=0\,(1)\,2n+1 \tag{8}$$

$$l_k(z)=\frac{H(z)}{H'(t_k)\ (z-t_k)},\ k=1\,(1)\,2n-4 \tag{9}$$

$$l_{1k}(z)=\frac{N(z)}{N'(t_k)\ (z-t_k)},k=0\,(1)\,2n-3 \tag{10}$$

$$J_k(z) = \int_0^z t \ \ l_k(t)\, dt \tag{11}$$

$$J_n(z) = \int_0^z H(t)\, dt \tag{12}$$

which satisfies,

$$J_n(-z) = - \ J_n(z) \tag{13}$$

We will also use the following results

$$W'(z_k) = \frac{1}{2} K_n \left(z_k^2 - 1\right) z_k^{n-2} P_n'(x_k) \tag{14}$$

$$W''(z_k) = K_n \left[(n-1)\left(z_k^2 - 1\right) - 1\right] z_k^{n-3} P_n'(x_k) \tag{15}$$

$$W'(t_k) = \frac{1}{2} K_n \frac{n\left\{(n+3)\left(t_k^2 - 1\right) + 4\right\}}{\left(t_k^2 + 1\right)} t_k^{n-1} P_n(x_k^*), \tag{16}$$

$$W''(t_k) = \frac{1}{2} K_n \frac{n(n-1)\left\{(n-1)\left(t_k^2 - 1\right) - 1\right\}}{\left(t_k^2 + 1\right)} t_k^{n-2} P_n(x_k^*), \tag{17}$$

$$\mathrm{R}'(\mathrm{z_k}) = \left(\mathrm{z_k^2} - 1\right) \mathrm{W}'(\mathrm{z_k}) \tag{18}$$

$$R''(z_k) = 4 z_k W'(z_k) + \left(z_k^2 - 1\right) W''(z_k) \tag{19}$$

$$\mathrm{R}'(\mathrm{t_k}) = \left(\mathrm{t_k^2} - 1\right) \mathrm{W}'(\mathrm{t_k}) + 2 t_k W(t_k) \tag{20}$$

$$R''(t_k) = 4 t_k W'(t_k) + \left(t_k^2 - 1\right) W''(t_k) + 2W(t_k) \tag{21}$$

$$H'(t_k) = \frac{K_n^*}{2} \left(t_k^2 - 1\right) t_k^{n-4} P_n'''(x_k^*), \tag{22}$$

$$H''(t_k) = K_n^* \ \left\{(n-5)\left(t_k^2 - 1\right) - 5\right\} t_k^{n-5} P_n'''(x_k^*), \tag{23}$$

$$\mathrm{N}'(\mathrm{z_k}) = \left(\mathrm{z_k^2} - 1\right) \mathrm{H}'(\mathrm{z_k}) + 2 z_k \ H(z_k) \tag{24}$$

$$N''(z_k) = 4 z_k H'(z_k) + \left(z_k^2 - 1\right) H''(z_k) + 2H(z_k) \tag{25}$$

$$\mathrm{N}'(\mathrm{t_k}) = \left(\mathrm{t_k^2} - 1\right)\mathrm{H}'(\mathrm{t_k}) \tag{26}$$

$$N''(t_k) = 4t_k H'(t_k) + \left(t_k^2 - 1\right) H''(t_k) \tag{27}$$

3. The Problem and Regularity

Let $\{z_k\}_0^{2n+1}$ and $\{t_k\}_0^{2n-3}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $\left(1-\mathrm{x}^2\right)\mathrm{P_n}(\mathrm{x})$ and $\left(1-\mathrm{x}^2\right)\mathrm{P_n''}(\mathrm{x})$ onto the unit circle respectively, where $\mathrm{P_n}(\mathrm{x})$ stands for n^{th} Legendre polynomial. Here we are interested to determine the polynamial $R_n(z)$ of degree $\leq 6n-5$ satisfying the conditions:

$$\begin{cases} R_n(z_k) = \alpha_k, & k = 0(1)2n+1 \\ R_n(t_k) = \beta_k, & k = 1(1)2n-4 \\ \left[p(z) R_n(z)\right]''_{z=t_k} = \gamma_k, & k = 0(1)2n-3 \end{cases} \tag{28}$$

where, $p(z)$ is weight function such that $\left[p(z) W(z) N(z)\right]''_{z=t_k} = 0$, and α_k , β_k and γ_k are arbitrary complex constants.

Theorem 3.1: $\mathrm{R_n}(z)$ is regular on Z_n and G_n.

Proof : It is sufficient, if we show, the unique solution of (28) is $\mathrm{R_n}(z) \equiv 0$, when all data $\alpha_k = \beta_k = \gamma_k = 0$.

In this case, we have

$$\mathrm{R_n}(z) = W(z) N(z) q(z),$$

where, $q(z)$ is a polynomial of degree $\leq 2n-3$.

Obviously, $R_n(z_k) = 0$ for $k = 0(1)2n+1$,

and also $R_n(t_k) = 0$ for $k = 1(1)2n-4$, from

$$\left[p(z) R_n(z)\right]''_{z=t_k} = 0$$

we get,

$$q'(t_k) = 0$$

Therefore , we have

$$q'(z) = aH(z)$$

$$q(z) = aJ_n(z) + b$$

where $J_n(z)$ is given in (12).

Now for $z = 1$ & -1, we get

$$a = b = 0$$

Therefore, $\mathrm{R_n}(z) \equiv 0$

Hence the theorem follows.

4. Explicit Representation Of Interpolatory Polynomials

We shall write $R_n(z)$ satisfying (28) as:

$$R_n(z) = \sum_{k=0}^{2n+1} \alpha_k A_k(z) + \sum_{k=1}^{2n-4} \beta_k B_k(z) + \sum_{k=0}^{2n-3} \gamma_k C_k(z) \tag{29}$$

where $A_k(z)$, $B_k(z)$ and $C_k(z)$ are unique polynomial ,each of degree at most $6n-5$ satisfying the conditions :

For $k = 0(1)2n+1$

$$\begin{cases} A_k(z_j) = \delta_{jk}, & j = 0(1)2n+1 \\ A_k(t_j) = 0, & j = 1(1)2n-4 \\ [p(z)A_k(z)]''_{z=t_j} = 0, & j = 0(1)2n-3 \end{cases} \tag{30}$$

For $k = 1(1)2n-4$

$$\begin{cases} B_k(z_j) = 0, & j = 0(1)2n+1 \\ B_k(t_j) = \delta_{jk} & j = 1(1)2n-4 \\ [p(z)B_k(z)]''_{z=t_j} = 0, & j = 0(1)2n-3 \end{cases} \tag{31}$$

For $k = 0(1)2n-3$

$$\begin{cases} C_k(z_j) = 0, & j = 0(1)2n+1 \\ C_k(t_j) = 0, & j = 1(1)2n-4 \\ [p(z)C_k(z)]''_{z=t_j} = \delta_{jk}, & j = 0(1)2n-3 \end{cases} \tag{32}$$

Theorem 4.1: For $k=0\,(1)\,2n-3$, we have

$$C_k(z)=\frac{1}{2t_k\,R(t_k)\;p(t_k)H'(t_k)}\,R(z)\;H(z)\;\int_0^z t\;l_k(t)\;dt \tag{33}$$

Proof: Let,

$$C_k(z)=\;c_k\,R(z)\;H(z)\;J_k(z) \tag{34}$$

where $J_k(z)$ is given in (11).

Obviously, $C_k(z_j)=0$, for each $j=0\,(1)\,2n+1$.

and $C_k(t_j)=0$, for each $j=1\,(1)\,2\mathrm{n}-4$.

One can see that $[p(z)C_k(z)]''_{z=t_j}=0$, for $j\neq k$ and for $j=k$, we get

$$c_k=\frac{1}{2t_k\,R(t_k)\;p(t_k)H'(t_k)} \tag{35}$$

using (35) in (34) we get (33).

Theorem 4.2: For $k=1\,(1)\,2n-4$, we have

(36)

$$B_k(z)=\frac{l_{1k}^2\;(z)\,W(z)}{W(t_k)}-\;\frac{W(z)\,N(z)}{H'(t_k)\,R(t_k)}\left\{\int_0^z\frac{[l_{1k}{}'(t)-l_{1k}{}'(t_k)\,l_{1k}(t)]}{(t-t_k)}\;dt+\;b_k\int_0^z t\;l_k(t)\right\}$$

where,

$$b_k=\frac{1}{t_k}\left[\frac{p''(t_k)}{2p(t_k)}+\frac{W''(t_k)}{2W(t_k)}+\frac{p'(t_k)\,W'(t_k)}{p(t_k)\,W(t_k)}+4l_k'(t_k)\left\{\frac{p'(t_k)}{p(t_k)}+\frac{W'(t_k)}{W(t_k)}\right\}\right] \tag{37}$$

Proof: Let

$$B_k(z)=\;\frac{l_{1k}^2\;(z)\,W(z)}{W(t_k)}+\frac{W(z)\,N(z)}{H'(t_k)\;R(t_k)}\,\{S_k(z)+b_kJ_k(z)\} \tag{38}$$

where $J_k(z)$ is given in (11).

Obviously,

$$B_k(z_j)\;=\;\;0\,,\;\text{for}\;\;\;j=0\,(1)\,2n+1$$

One can check that

$$B_k\left(t_j\right) = \delta_{jk}\text{ , for } \quad j = 1\,(1)\,2n-4$$

Further from $\left[\,p\,(z)\,B_k\,(z)\right]''_{z=t_j} = 0$, for $j \neq k$, we get,

$$S_k\,(z) = -\int_0^z \frac{\left[l'_{1k}\,(t) - l'_{1k}\,(t_k)\,l_{1k}(t)\right]}{(t-t_k)}\,dt \tag{39}$$

again for $j = k$, we get (37).

using (37) and (39) in (38) ,we get (36).

Theorem 4.3: For $k = 0\,(1)\,2n+1$, we have

(40)
$$A_k\,(z) = \frac{L_k\,(z)\,N\,(z)\,H\,(z)}{N\,(z_k)\,H\,(z_k)} - \frac{N(z)W\,(z)}{H^2\,(z_k)\,R'\,(z_k)\,N\,(z_k)}\int_0^z \left(t^2-1\right)\frac{[H'\,(t)\;H(z_k) - H'\,(z_k)\;H(t)]}{(\mathbf{t}-z_k)}\,dt$$

Proof: Let ,

$$A_k\,(z) = \frac{L_k(z)\;N(z)\;H\,(z)}{N(z_k)\;H\,(z_k)} + \frac{N(z)W\,(z)}{H^{\,2}\,(z_k)\;\;R'\,(z_k)\;\;N\,(z_k)}M_k\,(z) \tag{41}$$

Obviously, $A_k\left(z_j\right) = \delta_{jk}$, for $j = 0\,(1)\,2n+1$,

and $A_k\left(t_j\right) = 0$, for $j = 1\,(1)\,2n-4$.

one can check that, $\left[\,p\,(z)\,A_k\,(z)\right]''_{z=t_j} = 0$, for $j = k$.

Further from $\left[\,p\,(z)\,A_k\,(z)\right]''_{z=t_j} = 0$, for $j \neq k$,we get

$$M_k\,(z) = -\int_0^z \left(\mathbf{t}^2-1\right)\frac{\left[H'\,(t)\,H(z_k) - H'\,(z_k)\,H(t)\right]}{(\mathbf{t}-z_k)}\,dt \tag{42}$$

Using (42) in (41) we get (40).

Conflict of Interests

The author declares that there is no conflict of interests.

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