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COMMON FIXED POINT THEOREM FOR SIX SELFMAPS OF A COMPLETE G-METRIC SPACE

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Abstract: In the present paper, we prove a common fixed point theorem for six weakly compatible selfmaps of a complete G-metric space. As an illustration, we give an example.

Keywords: G-metric space; weakly compatible mappings; fixed point; associated sequence of a point relative to six self maps; implicit relation.

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1. Introduction

Generally fixed point theorems are proved for selfmaps of metric spaces. Fixed point Theorems on metric spaces have important theoretical and practical applications. In 1963 Gahler [1,2] introduced the notion of 2-metric spaces while Dhage[3] initiated the notion of D-metric spaces in 1984. Subsequently several researchers have proved that most of their claims made are not valid. As a probable modification to D-metric spaces Shaban Sedghi, Nabi Shobe and Haiyan

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Zhou [4] introduced D* metric spaces. In 2006, Zead Mustafa and Brailey Sims [5,6] initiated G-metric spaces. Of these two generalizations, the G-metric space evinced interest in many researchers.

Sessa [7] introduced the concept of weakly commuting mappings as a generalization of commuting maps. This was further generalized by G,Jungck [8,9] in 1986 as compatible mappings. In 1996 Jungck and Rhoades [10] introduced the notion of weakly compatible mappings.

The purpose of this paper is to prove a common fixed point theorem for six weakly compatible selfmaps of a complete G-metric space.

2. Preliminaries

Definition 2.1: [6] Let *X* be a non-empty set and $G: X^3 \to [0, \infty)$ be a function satisfying:

(G1) $G(x, y, z) = 0$ if $x = y = z$

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$

(G3) $G(x, x, y) < G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$

(G4) $G(x, y, z) = G(\sigma(x, y, z))$ for all $x, y, z \in X$, where $\sigma(x, y, z)$ is a permutation of the set $\{x, y, z\}$ and

(G5) $G(x, y, z) < G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$

Then *G* is called a *G* - metric on *X* and the pair (X, G) is called a *G* - metric Space.

Example 2.2: Let *(X, d)* be a metric space. Define $G_m^d : X^3 \to [0, \infty)$ by

$G_m^d(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for $x, y, z \in X$ .Then (*X*, G_m^d) is a *G*-metric Space.

Lemma 2.3: [6] If (X, G) is a *G*-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$

Definition 2.4: Let (X,G) be a G-metric Space. A sequence $\{x_n\}$ in X is said to be G-convergent if there is a $x_0 \in X$ such that to each $\varepsilon > 0$ there is a natural number N for which $G(x_n, x_n, x_0) < \varepsilon$ for all $n \geq N$.

Lemma 2.5: [6] Let (X,G) be a G-metric Space, then for a sequence $\{x_n\} \subseteq X$ and point $x \in X$ the following are equivalent.

(i) $\{x_n\}$ is G- convergent to x.

(ii) $d_G(x_n, x) \to 0$ as $n \to \infty$ (that is $\{x_n\}$ converges to x relative to the metric d_G)

(iii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$

(iv) $G(x_n, x, x) \to 0$ as $n \to \infty$

(v) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$

Definition 2.6: [6] Let (X,G) be a G-metric space, then a sequence $\{x_n\} \subseteq X$ is said to be G-Cauchy if for each $\varepsilon > 0$, there exists a natural number N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$.

Note that every G-convergent sequence in a G-metric space (X,G) is G-Cauchy.

Definition 2.7: [6] A G-metric space (X,G) is said to be G-complete if every G -Cauchy sequence in (X,G) is G-convergent in (X,G)

Definition 2.8: Let f and g are self maps of a G-metric space (X,G) such that $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. Then the functions f and g are said to be compatible.

Definition 2.9: [11] Suppose f and g are self maps of a G-metric space (X,G). The pair f and g is said to be weakly compatible if $G(fgx, gfx, gfx) = 0$ whenever $G(fx, gx, gx) = 0$

Definition 2.10: A function $\phi:(\mathbb{R}^+)^4 \to \mathbb{R}^+$ which is continuous and increasing in each coordinate with $\phi(t,t,t,t) < t$ for every $t \in \mathbb{R}^+$ is called an Implicit relation.
The set all implicit relations is denoted by Φ

Definition 2.11: Suppose f, g, h, R, S and T be self maps of a G-metric space such that $f(X) \subseteq R(X), g(X) \subseteq S(X)$ and $h(X) \subseteq T(X)$. For x_0 in X, If $\{x_n\}$ is a sequence in X such that $fx_{3n} = Rx_{3n+1}, gx_{3n+1} = Sx_{3n+2}, hx_{3n+2} = Tx_{3n+3},\ n \geq 0.$ Then $\{x_n\}$ is called an associated sequence of x_0 relative to selfmaps f, g, h, R, S and T

3. Main results

Theorem 3.1. *Let* f, g, h, R, S *and* T *be self maps of a complete* G*-metric space* (X, G) *with following conditions*

(i) $f(X) \subseteq R(X), g(X) \subseteq S(X), h(X) \subseteq T(X)$ *and*

(ii) one of $f(X)$, $g(X)$ *and* $h(X)$ *is closed subset of* X

(iii) $G(fx, gy, hz) \leq q\phi\big(G(Tx, Ry, Sz), G(Tx, Ry, gy), G(Ry, Sz, hz), G(Sz, Tx, fx)\big)$ *for every* $x, y, z \in X$ *some* $0 < q < \frac{1}{2}$ *and* $\phi \in \Phi$

(iv) The pairs $(f, T), (g, R)$ *and* (h, S) *are weakly compatible*

Then f, g, h, R, S *and* T *have a unique common fixed point in* X .

Proof. Let $x_0 \in X$ be an arbitrary point. Then we can construct a sequence $\{x_n\}$ in X such that

$$y_{3n} = fx_{3n} = Rx_{3n+1}, \quad y_{3n+1} = gx_{3n+1} = Sx_{3n+2},\ y_{3n+2} = hx_{3n+2} = Tx_{3n+3}. \text{ for } n = 0, 1, 2.......$$

Let $G_m = G(y_m, y_{m+1}, y_{m+2})$

If $m = 3n$ then we have

$$\begin{aligned} G_{3n} &= G(y_{3n}, y_{3n+1}, y_{3n+2}) \\ &= G(fx_{3n}, gx_{3n+1}, hx_{3n+2}) \\ &\le q\phi\left(G(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), G(Tx_{3n}, Rx_{3n+1}, gx_{3n+1}), G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tx_{3n}, fx_{3n})\right) \\ &\le q\phi\left(G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n-1}, y_{3n}, y_{3n+1}), G(y_{3n}, y_{3n+1}, y_{3n+2}), G(y_{3n+1}, y_{3n-1}, y_{3n})\right) \\ &= q\phi\left(G_{3n-1}, G_{3n-1}, G_{3n}, G_{3n-1}\right) \end{aligned}$$

we now prove that $G_{3n} \le G_{3n-1}$ for every $n \in \mathbb{N}$

If $G_{3n} > G_{3n-1}$ for some $n \in \mathbb{N}$ by above inequality we have $G_{3n} < qG_{3n}$ which is a contradiction since $0 < q < \frac{1}{2}$

Similarly, we can prove that $G_{3n+1} \le G_{3n}$ and $G_{3n+2} \le G_{3n+1}$

Hence $G_n \le G_{n-1}$ for all $n \ge 1$

This gives

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}) &< qG(y_{n-1}, y_n, y_{n+1}) \\ &< q^2 G(y_{n-2}, y_{n-1}, y_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &< q^n G(y_0, y_1, y_2) \end{aligned}$$

We have $G(y_n, y_n, y_{n+1}) \le qG(y_n, y_{n+1}, y_{n+2}) < q^n G(y_0, y_1, y_2)$

We now claim that $\{y_n\}$ is Cauchy sequence.

For every $m, n \in N$ with $m > n$ we have

$$\begin{aligned} G(y_n, y_m, y_m) &< G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_m, y_m) \\ &\le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + \ldots\ldots + G(y_{m-1}, y_m, y_m) \\ &\le 2[G(y_{n+1}, y_n, y_n) + G(y_{n+2}, y_{n+1}, y_{n+1}) + \ldots\ldots + G(y_m, y_{m-1}, y_{m-1})] \\ &= 2[G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + \ldots\ldots + G(y_{m-1}, y_{m-1}, y_m)] \\ &< 2[q^n G(y_0, y_1, y_2) + q^{n+1} G(y_0, y_1, y_2,) \ldots\ldots + q^{m-1} G(y_0, y_1, y_2)] \\ &= 2[q^n + q^{n+1} + \ldots\ldots. + q^{m-1}] G(y_0, y_1, y_2) \\ &< 2.\frac{q^n}{1-q} G(y_0, y_1, y_2) \to 0 \text{ as } n \to \infty \end{aligned}$$

Proving that $\{y_n\}$ is a Cauchy sequence and since X is complete, there exists a z in X such

That $\lim_{n\to\infty} y_n = z.$ this implies

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_{3n} = \lim_{n\to\infty} gx_{3n+1} = \lim_{n\to\infty} hx_{3n+2} = \lim_{n\to\infty} Rx_{3n+1} = \lim_{n\to\infty} Sx_{3n+2} = \lim_{n\to\infty} Tx_{3n+3} = z$$

Suppose $h(X)$ be a closed subset of X . Hence there exists $u \in X$ such that $Tu = z$

We shall prove that $fu = z$. If $fu \neq z$ then $G(fu, z, z) > 0$

By (iii) of the Theorem 3.1 we have

$$G(fu, gx_{3n+1}, hx_{3n+2}) \leq q\phi\left(G(Tu, Rx_{3n+1}, Sx_{3n+2}), G(Tu, Rx_{3n+1}, gx_{3n+1}), G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tu, fu)\right)$$

on letting $n \to \infty$ we obtain

$$\begin{aligned} G(fu, z, z) &\leq q\phi\left(G(Tu, z, z), G(Tu, z, z), G(z, z, z), G(z, Tu, fu)\right) \\ &= q\phi\left(G(z, z, z), G(z, z, z), G(z, z, z), G(z, z, fu)\right) \end{aligned}$$

If $G(fu, z, z) > 0$ then we have $G(fu, z, z) < qG(fu, z, z)$

Which leads to a contradiction since $0 < q < \frac{1}{2}$,hence $G(fu, z, z) = 0$ implies $fu = z$

Since the pair (f, T) is weakly compatible, then we have $fTu = Tfu$. This gives $fz = Tz$

Now we show that $fz = z$

If $fz \neq z$ then by (iii) of the Theorem 3.1 we have

$$G(fz, gx_{3n+1}, hx_{3n+2}) \leq q\phi\left(G(Tz, Rx_{3n+1}, Sx_{3n+2}), G(Tz, Rx_{3n+1}, gx_{3n+1}), G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tz, fz)\right)$$

On letting $n \to \infty$ and using that fact $fz = Tz$, we get

$$G(fz, z, z) \leq q\phi\left(G(fz, z, z), G(fz, z, z), G(z, z, z), G(z, fz, fz)\right)$$

Since $G(z, fz, fz) \leq 2G(fz, z, z)$ and ϕ is increasing in each co-ordinate then

$$G(fz, z, z) \leq q\phi\left(2G(fz, z, z), 2G(fz, z, z), 2G(fz, z, z), 2G(fz, z, z)\right) < 2qG(fz, z, z)$$

Which is a contradiction since $0 < q < \frac{1}{2}$ and hence $fz = z$

Showing that $fz = Tz = z$

Since $fz = z$ and $f(X) \subseteq R(X)$, then there exists $v \in X$ such that $Rv = z$

Now we shall prove that $gv = z$

If $gv \neq z$ then $G(z, gv, z) > 0$. Now by (iii) of the Theorem 3.1 we have

$$\begin{aligned} G(z, gv, hx_{3n+2}) &= G(fz, gv, hx_{3n+2}) \\ &\leq q\phi\big(G(Tz, Rv, Sx_{3n+2}), G(Tz, Rv, gv), G(Rv, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tz, fz)\big) \end{aligned}$$

on letting $n \to \infty$ we have

$$\begin{aligned} G(z, gv, z) &\leq q\phi\big(G(z,z,z), G(z,z,gv), G(z,z,z), G(z,z,z)\big) \\ &= q\phi\big(0, G(z,z,gv), 0, 0\big) \\ &\leq q\phi\big(G(z,gv,z), G(z,,gv,z), G(z,gv,z), G(z,gv,z)\big) \\ &< qG(z,z,gz) \end{aligned}$$

Which is a contradiction since $0 < q < \dfrac{1}{2}$ and hence $gv = z$

Since the pair (g, R) is weakly compatible then we have $gRv = Rgv$. Hence $gz = Rz$

We now show that $gz = z$. If $gz \neq z$, then by (iii) of the Theorem 3.1 we have

$$G(fz, gz, hx_{3n+2}) \leq q\phi\big(G(Tz, Rz, Sx_{3n+2}), G(Tz, Rz, gz), G(Rz, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tz, fz)\big)$$

on letting $n \to \infty$ we get

$$G(fz, gz, z) \leq q\phi\big(G(Tz, Rz, z), G(Tz, Rz, gz), G(Rz, z, z), G(z, Tz, fz)\big)$$

$$\begin{aligned} G(z, gz, z) &\leq q\phi\big(G(z,gz,z), G(z,gz,gz), G(gz,z,z), G(z,z,z)\big) \\ &\leq q\phi\big(2G(z,gz,z), 2G(z,gz,z), 2G(z,gz,z), G(z,gz,z)\big) \\ &< 2qG(z,gz,z) \end{aligned}$$

Which is a contradiction since $0 < q < \dfrac{1}{2}$, and hence $gz = z$

Therefore $gz = Rz = z$

Since $gz = z$ and $g(X) \subseteq S(X)$, then there exists $w \in X$ such that $Sw = z$

Now we prove that $hw = z$

If $hw \neq z$, then $G(z, z, hw) > 0$. Now by (iii) of the Theorem 3.1 we have

$$G(z,z,hw) = G(fz,gz,hw) \le q\phi\left(G(Tz,Rz,Sw),G(Tz,Rz,gz),G(Rz,Sw,hw),G(Sw,Tz,fz)\right)$$
$$= q\phi\left(G(z,z,z),G(z,z,z),G(z,z,hw),G(z,z,z)\right)$$
$$\le q\phi\left(G(z,z,hw),G(z,z,hw),G(z,z,hw),G(z,z,hw)\right)$$
$$< qG(z,z,hw)$$

Which is a contradiction since $0 < q < \frac{1}{2}$ and hence $hw = z$

Since, the pair (h,S) is weakly compatible then we have $hSw = Shw$ implies $hz = Sz$.

If $hz \ne z$ then from (iii) of the Theorem 3.1 we have

$$G(z,z,hz) = G(fz,gz,hz) \le q\phi\left(G(Tz,Rz,Sz),G(Tz,Rz,gz),G(Rz,Sz,hz),G(Sz,Tz,fz)\right)$$
$$= q\phi\left(G(z,z,hz),G(z,z,z),G(z,hz,hz),G(hz,z,z)\right)$$
$$\le q\phi\left(G(z,z,hz),0,2G(hz,z,z),G(hz,z,z)\right)$$
$$= q\phi\left(2G(z,z,hz),2G(z,z,hz),2G(z,z,hz),2G(z,z,hz)\right)$$
$$< 2qG(z,z,hz)$$

Which is a contradiction since $0 < q < \frac{1}{2}$ and hence $hz = z$

Proving that $hz = Sz = z$

Hence z is a common fixed point of f,g,h,R,S and T

The proof is similar when $g(X)$ or $h(X)$ closed subset of X with appropriate changes.

Now we prove the uniqueness of common fixed point**.** If possible let z' be another common fixed point of f,g,h,R,S and T .

Then from (iii) of the Theorem 3.1 we have

$$G(z,z',z') = G(fz,gz',hz')$$
$$\le q\phi\left(G(Tz,Rz',Sz'),G(Tz,Rz',gz'),G(Rz',Sz',hz'),G(Sz',Tz,fz)\right)$$
$$= q\phi\left(G(z,z',z'),G(z,z',z'),G(z',z',z'),G(z',z,z)\right)$$
$$\le q\phi\left(G(z,z',z'),G(z,z',z'),0,2G(z,z',z')\right)$$
$$\le q\phi\left(2G(z,z',z'),2G(z,z',z'),2G(z,z',z'),2G(z,z',z')\right)$$
$$< 2qG(z,z',z')$$

Which is a contradiction since $0 < q < \frac{1}{2}$ and hence $z = z'$

Showing that z is a unique common fixed point of f,g,h,R,S and T .

As an illustration we have the following example.

Example 3.2: Let $X = [0,1]$ with $G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}$ for $x, y, z \in X$.

Then G is a G- metric on X.

Define $f : X \to X, g : X \to X, h : X \to X, T : X \to X, R : X \to X, S : X \to X$ by

$$f(x) = g(x) = \begin{cases} \frac{1}{3} \; if \;\; x = 0 \\ \frac{1}{2} \; if \;\; x \in (0,1] \end{cases} \quad and \;\; h(x) = \begin{cases} \frac{1}{5} \; if \;\; x = 0 \\ \frac{1}{2} \; if \;\; x \in (0,1] \end{cases}$$

$$R(x) = S(x) = \tfrac{x+1}{3} \; if \;\; x \in [0,1] \; and \, T(x) = x \; if \; x \in [0,1]$$

$$f(X) = g(X) = \{\tfrac{1}{3}, \tfrac{1}{2}\} \quad h(X) = \{\tfrac{1}{5}, \tfrac{1}{2}\} \; R(X) = S(X) = [\tfrac{1}{3}, \tfrac{1}{2}] \quad T(X) = [0,1]$$

Clearly $f(X) \subseteq R(X), g(X) \subseteq S(X)$ and $h(X) \subseteq T(X)$

Also $f(X), g(X), h(X)$ are closed subsets of X

The pairs (f, T), (g, R), and (h, S) are commute at their coincident point $\frac{1}{2}$ and hence they are weakly compatible

We now prove the mappings satisfying the condition (iii) of the Theorem 3.1

Case (i): If $x = y = z = 0$, then

$$G(fx, gy, hz) = \tfrac{2}{15}, G(Tx, Ry, Sz) = \tfrac{1}{3}, G(Tx, Ry, gy) = \tfrac{1}{3}, G(Ry, Sz, hz) = \tfrac{2}{15}, G(Sz, Tx, fx) = \tfrac{1}{3}$$

Therefore, the condition (iii) of the Theorem 3.1 holds if $\dfrac{2}{15} \le q\phi\left(\dfrac{2}{15}, \dfrac{1}{3}, \dfrac{2}{15}, \dfrac{1}{3}\right) < q\dfrac{1}{3}$

This is possible by choosing $q > 0$ such that $\dfrac{2}{5} < q < \dfrac{1}{2}$

Proving that the condition (iii) of the Theorem 3.1 satisfied in this case

Case (ii): If $x = y = 0$, and $z \in (0,1]$ then

$$G(fx, gy, hz) = \tfrac{1}{6}, G(Tx, Ry, Sz) = \tfrac{2}{3}, G(Tx, Ry, gy) = \tfrac{1}{3}, G(Ry, Sz, hz) \le \tfrac{1}{3}, \;\; G(Sz, Tx, fx) \le \tfrac{2}{3}$$

$$\frac{1}{6} \le q\phi\left(\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{2}{3}\right) < q\frac{2}{3}$$

Hence the condition (iii) of the Theorem 3.1 holds with q satisfying $\frac{1}{4} < q < \frac{1}{2}$

Case (iii): If $x = z = 0,$ and $y \in (0,1]$ then

$$G(fx, gy, hz) = \frac{3}{10},\ G(Tx, Ry, Sz) \le \frac{2}{3},\ G(Tx, Ry, gy) \le \frac{2}{3},\ G(Ry, Sz, hz) \le \frac{7}{15},\ G(Sz, Tx, fx) = \frac{2}{15}$$

$$\frac{3}{10} \le q\phi\left(\frac{2}{3},\frac{2}{3},\frac{7}{15},\frac{2}{15}\right) < q\frac{2}{3}$$

Hence the condition (iii) of the Theorem 3.1 holds with q satisfying $\frac{9}{20} < q < \frac{1}{2}$

Case (iv): If $y = z = 0,$ and $x \in (0,1]$ then

$$G(fx, gy, hz) = \frac{3}{10},\ G(Tx, Ry, Sz) \le \frac{2}{3},\ G(Tx, Ry, gy) \le \frac{2}{3},\ G(Ry, Sz, hz) = \frac{2}{15},\ G(Sz, Tx, fx) \le \frac{2}{3}$$

$$G(fx, gy, hz) \le q\phi\big(G(Tx, Ry, Sz), G(Tx, Ry, gy), G(Ry, Sz, hz), G(Sz, Tx, fx)\big)$$

$$\frac{3}{10} \le q\phi\left(\frac{2}{3},\frac{2}{3},\frac{2}{15},\frac{2}{3}\right) < q\frac{2}{3}$$

Hence the condition (iii) of the Theorem 3.1 hold with $q > 0$ satisfying $\frac{9}{20} < q < \frac{1}{2}$

Case (v): If $x = 0, y \in (0,1]$ and $z \in (0,1]$ then

$$G(fx, gy, hz) = \frac{1}{6},\ G(Tx, Ry, Sz) \le \frac{2}{3},\ G(Tx, Ry, gy) \le \frac{2}{3},\ G(Ry, Sz, hz) \le \frac{1}{3},\ G(Sz, Tx, fx) \le \frac{2}{3}$$

$$G(fx, gy, hz) \le q\phi\big(G(Tx, Ry, Sz), G(Tx, Ry, gy), G(Ry, Sz, hz), G(Sz, Tx, fx)\big)$$

$$\frac{1}{6} \le q\phi\left(\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\right) < q\frac{2}{3}$$

Hence the condition (iii) of the Theorem 3.1 hold with $q > 0$ satisfying $\frac{1}{4} < q < \frac{1}{2}$

Case (vi): If $y = 0, x \in (0,1]$ and $z \in (0,1]$ then

$$G(fx,gy,hz)=\frac{1}{6},\ G(Tx,Ry,Sz)\le\frac{2}{3},\ G(Tx,Ry,gy)\le\frac{2}{3},\ G(Ry,Sz,hz)\le\frac{1}{3},\ G(Sz,Tx,fx)=\frac{2}{3}$$

$$G(fx,gy,hz)\le q\phi\left(G(Tx,Ry,Sz),G(Tx,Ry,gy),G(Ry,Sz,hz),G(Sz,Tx,fx)\right)$$

$$\frac{1}{6}\le q\phi\left(\frac{2}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3}\right)<q\frac{2}{3}$$

Hence the condition (iii) of the Theorem 3.1 hold with $q>0$ satisfying $\frac{1}{4}<q<\frac{1}{2}$

Case (vii): If $z=0, x\in(0,1]$ and $y\in(0,1]$ then

$$G(fx,gy,hz)=\frac{3}{10},\ G(Tx,Ry,Sz)\le\frac{4}{5},\ G(Tx,Ry,gy)\le\frac{2}{3},\ G(Ry,Sz,hz)\le\frac{7}{15},\ G(Sz,Tx,fx)\le\frac{2}{3}$$

$$G(fx,gy,hz)\le q\phi\left(G(Tx,Ry,Sz),G(Tx,Ry,gy),G(Ry,Sz,hz),G(Sz,Tx,fx)\right)$$

$$\frac{3}{10}\le q\phi\left(\frac{4}{5},\frac{2}{3},\frac{7}{15},\frac{2}{3}\right)<q\frac{4}{5}$$

Hence the condition (iii) of the Theorem 3.1 hold with $q>0$ satisfying $\frac{3}{8}<q<\frac{1}{2}$

Case (viii): If $x=y\ne 0,$ and $z\ne 0$ then $G(fx,gy,hz)=0$

$$G(fx,gy,hz)=0\le q\phi\left(G(Tx,Ry,Sz),G(Tx,Ry,gy),G(Ry,Sz,hz),G(Sz,Tx,fx)\right)$$

Hence the condition (iii) of the Theorem 3.1 hold with $q>0$ satisfying $0<q<\frac{1}{2}$

From above all cases if we choose $q>0$ such that $\frac{9}{20}\le q<\frac{1}{2}$ then the condition (iii) of the Theorem 3.1 holds

From the above all cases all the conditions of the Theorem 3.1 hold

Hence the selfmaps f,h,g,R,S and T have a unique common fixed point in X

Moreover, $\frac{1}{2}$ is the unique fixed point for all mappings f,h,g,R,S and T.

Corollary3.3: *Let f,g,h,R,S and T be self maps of a complete G-metric space (X,G) with following conditions*

(i) $f(X)\subseteq R(X), g(X)\subseteq S(X), h(X)\subseteq T(X)$.

(ii) one of $f(X)$, $g(X)$ *and* $h(X)$ *is closed subset of* X

(ii) $G(fx, gy, hz) \leq q\phi\big(G(Tx, Ry, Sz), G(Tx, Ry, gy), G(Ry, Sz, hz), G(Sz, Tx, fx)\big)$ *for every* $x, y, z \in X$ *some* $0 < q < \frac{1}{2}$ *and* $\phi \in \Phi$

(iii) $fT = Tf$, $gR = Rg$ *and* $hS = Sh$

Then f, g, h, R, S *and* T *have a unique common fixed point in* X.

Proof: from the fact that the commutativity implies the weakly compatibility of a pair of selfmaps, proof of this corollary follows from the Theorem 3.1

Conflict of Interests

The authors declare that there is no conflict of interests.

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