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FURTHER REFINEMENT OF RESULTS ABOUT MIXED SYMMETRIC MEANS AND CAUCHY MEANS

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Abstract. Recently, Horváth introduced a new method to refine the well known discrete Jensen's inequality (see [2]). He also gave a parameter dependant refinement of the discrete Jensen's inequality (see [3]). We apply the new exponential convexity method as illustrated in [7], to the functionals obtained from the refinement results of [2] and [3]. In this way we are able to generalize the results given in [4] as well as given in [1].

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1. Introduction and Preliminary Results

We start with the notations given in [2]:

Let X be a set. The power set of X is denoted by $P(X)$. $|X|$ means the number of

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elements in X. For every nonnegative integer m, let

$$P_m(X) := \{Y \subset X \mid |Y| = m\}.$$

We need the following hypotheses:

(H$_1$) Let $S_1, \ldots, S_n$ be finite, pairwise disjoint and nonempty sets, let

$$S := \bigcup_{j=1}^{n} S_j,$$

and let c be a function from S into $\mathbb{R}$ such that

$$c(s) > 0, \quad s \in S, \quad \text{and} \quad \sum_{s \in S_j} c(s) = 1, \quad j = 1, \ldots, n.$$

Let the function $\tau : S \to \{1, \ldots, n\}$ be defined by

$$\tau(s) := j, \quad \text{if} \quad s \in S_j.$$

(H$_2$) Suppose $\mathcal{A} \subset P(S)$ is a partition of S into pairwise disjoint and nonempty sets. Let

$$k := \max\{|A| \mid A \in \mathcal{A}\},$$

and let

$$\mathcal{A}_l := \{A \in \mathcal{A} \mid |A| = l\}, \quad l = 1, \ldots, k.$$

Then $\mathcal{A}_l$ $(l = 1, \ldots, k-1)$ may be the empty set, and $|S| = \sum\limits_{l=1}^{k} l\,|\mathcal{A}_l|$. The empty sum is taken to be zero.

In the sequel, we also require the following hypotheses:

(H$_3$) Let $J \subset \mathbb{R}$ be an interval, $\mathbf{x} := (x_1, ..., x_n) \in J^n$, and $\mathbf{p} := (p_1, ..., p_n)$ be a positive n-tuple such that $\sum\limits_{i=1}^{n} p_i = 1$.

(H$_4$) Let $f : J \to \mathbb{R}$ be a convex function.

(H$_5$) Let h, $g : J \to \mathbb{R}$ be continuous and strictly monotone functions.

Theorem 1.1. [2] *Assume (H$_1$)-(H$_4$). Then*

$$f\left(\sum_{j=1}^{n} p_j x_j\right) \le N_k \le N_{k-1} \le \ldots \le N_2 \le N_1 = \sum_{j=1}^{n} p_j f(x_j), \tag{1}$$

where

(2)
$$N_k := \sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)f\left(\frac{\sum_{s\in A}c(s)p_{\tau(s)}x_{\tau(s)}}{\sum_{s\in A}c(s)p_{\tau(s)}}\right)\right)\right),$$

and for every $1\le m\le k-1$ *the number* N_{k-m} *is given by*

(3)
$$N_{k-m} := \sum_{l=1}^{m}\left(\sum_{A\in\mathcal{A}_l}\left(\sum_{s\in A}c(s)p_{\tau(s)}f(x_{\tau(s)})\right)\right)+\sum_{l=m+1}^{k}\left(\frac{m!}{(l-1)\dots(l-m)}\right.$$
$$\left.\cdot\sum_{A\in\mathcal{A}_l}\left(\sum_{B\in P_{l-m}(A)}\left(\left(\sum_{s\in B}c(s)p_{\tau(s)}\right)f\left(\frac{\sum_{s\in B}c(s)p_{\tau(s)}x_{\tau(s)}}{\sum_{s\in B}c(s)p_{\tau(s)}}\right)\right)\right)\right).$$

Assume (H_1), (H_2) and

(H_6) Let $\mathbf{x} := (x_1, ..., x_n)$ and $\mathbf{p} := (p_1, ..., p_n)$ be positive n-tuples such that $\sum_{i=1}^{n}p_i = 1$.

We define the power means of order $r\in\mathbb{R}$ corresponding to $A\in\mathcal{A}_l$ $(l=1,\dots,k)$ as follows:

(4)
$$M_r(\mathbf{x},\mathbf{p},A) := \begin{cases}\left(\frac{\sum_{s\in A}c(s)p_{\tau(s)}x^r_{\tau(s)}}{\sum_{s\in A}c(s)p_{\tau(s)}}\right)^{\frac{1}{r}}; & r\neq 0,\\ \left(\prod_{s\in A}x_{\tau(s)}^{c(s)p_{\tau(s)}}\right)^{\frac{1}{\sum_{s\in A}c(s)p_{\tau(s)}}}; & r=0.\end{cases}$$

We also use the means

$$M_r := \begin{cases}\left(\sum_{i=1}^{n}p_i x_i^r\right)^{\frac{1}{r}}, & r\neq 0,\\ \prod_{i=1}^{n}x_i^{p_i}, & r=0.\end{cases}$$

Assume (H_1), (H_2) and (H_6), and let γ, $\eta\in\mathbb{R}$. We define the mixed symmetric means corresponding to (2) and (3) as follows:

$$M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k)$$
$$:= \begin{cases}\left(\sum_{l=1}^{k}\left(\sum_{A\in\mathcal{A}_l}\left(\left(\sum_{s\in A}c(s)p_{\tau(s)}\right)M^{\eta}_{\gamma}(\mathbf{x},\mathbf{p},A)\right)\right)\right)^{\frac{1}{\eta}}; & \eta\neq 0,\\ \prod_{l=1}^{k}\left(\prod_{A\in\mathcal{A}_l}\left((M_\gamma(\mathbf{x},\mathbf{p},A))^{\sum_{s\in A}c(s)p_{\tau(s)}}\right)\right); & \eta=0,\end{cases}$$

and for $1 \leq m \leq k-1$

$$M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k-m) :=$$
$$\left(\begin{array}{l} \sum\limits_{l=1}^{m} \left(\sum\limits_{A\in\mathcal{A}_l} \left(\sum\limits_{s\in A} c(s)p_{\tau(s)}x^{\eta}_{\tau(s)} \right) \right) + \\ \sum\limits_{l=m+1}^{k} \left(\frac{m!}{(l-1)\dots(l-m)} \sum\limits_{A\in\mathcal{A}_l} \left(\sum\limits_{B\in P_{l-m}(A)} \left(\left(\sum\limits_{s\in B} c(s)p_{\tau(s)} \right) M^{\eta}_{\gamma}(\mathbf{x},\mathbf{p},B) \right) \right) \right) \end{array} \right)^{\frac{1}{\eta}},$$

if $\eta \neq 0$ and for $\eta = 0$, we have

$$M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k-m) := \prod_{l=1}^{m} \left(\prod_{A\in\mathcal{A}_l} \left(\prod_{s\in A} x_{\tau(s)}^{c(s)p_{\tau(s)}} \right) \right) \times$$
$$\prod_{l=m+1}^{k} \left(\left(\prod_{A\in\mathcal{A}_l} \left(\prod_{B\in P_{l-m}(A)} (M_{\gamma}(\mathbf{x},\mathbf{p},B))^{\left(\sum\limits_{s\in B} c(s)p_{\tau(s)}\right)} \right) \right)^{\frac{m!}{(l-1)\dots(l-m)}} \right).$$

The monotonicity of these mixed symmetric means is a consequence of Theorem 1.1.

Corollary 1.2. *Assume (H$_1$), (H$_2$) and (H$_6$). Let η, $\gamma \in \mathbb{R}$ such that $\eta \leq \gamma$. Then*

$$M_{\eta} \leq {M_{\gamma,\eta}}^1(\mathbf{x},\mathbf{p},k) \leq {M_{\gamma,\eta}}^1(\mathbf{x},\mathbf{p},k-1) \leq \dots \leq {M_{\gamma,\eta}}^1(\mathbf{x},\mathbf{p},1) = M_{\gamma}, \tag{5}$$

and

$$M_{\eta} = M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},1) \leq \dots. \leq M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k-1) \leq M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k) \leq M_{\gamma}. \tag{6}$$

Proof. Assume η, $\gamma \neq 0$. To obtain (5) we apply Theorem 1.1 for the function $f(x) = x^{\frac{\gamma}{\eta}}$ $(x > 0)$ and the n-tuples $(x_1^{\eta}, \dots, x_n^{\eta})$ in (1) and then raising the power $\frac{1}{\gamma}$. (6) can be proved in a similar way by using $f(x) = x^{\frac{\eta}{\gamma}}$ $(x > 0)$ and $(x_1^{\gamma}, \dots, x_n^{\gamma})$ and raising the power $\frac{1}{\eta}$.
When $\eta = 0$ or $\gamma = 0$, we get the required results by taking limit. □

Assume (H$_1$-H$_3$) and (H$_5$). Then we define the generalized means with respect to (2) and (3) as follows:

$$M^1_{h,g}(\mathbf{x},\mathbf{p},k) :=$$
$$h^{-1}\left(\sum_{l=1}^{k} \left(\sum_{A\in\mathcal{A}_l} \left(\left(\sum_{s\in A} c(s)p_{\tau(s)} \right) h\circ g^{-1} \left(\frac{\sum\limits_{s\in A} c(s)p_{\tau(s)}g(x_{\tau(s)})}{\sum\limits_{s\in A} c(s)p_{\tau(s)}} \right) \right) \right) \right),$$

and for $1 \leq m \leq k-1$

$$M^1_{h,g}(\mathbf{x},\mathbf{p},k-m) :=$$
$$h^{-1}\left(\begin{array}{c}\sum\limits_{l=1}^{m}\left(\sum\limits_{A\in\mathcal{A}_l}\left(\sum\limits_{s\in A}c(s)p_{\tau(s)}h(x_{\tau(s)})\right)\right)+\sum\limits_{l=m+1}^{k}\left(\frac{m!}{(l-1)\ldots(l-m)}\right.\\ \times\sum\limits_{A\in\mathcal{A}_l}\left(\sum\limits_{B\in P_{l-m}(A)}\left(\left(\sum\limits_{s\in B}c(s)p_{\tau(s)}\right)h\circ g^{-1}\left(\frac{\sum\limits_{s\in B}c(s)p_{\tau(s)}g(x_{\tau(s)})}{\sum\limits_{s\in B}c(s)p_{\tau(s)}}\right)\right)\right)\left.\right)\end{array}\right).$$

We also need the quasi-arithmetic mean. Assume (H$_3$), and let $q : J \to \mathbb{R}$ be a continuous and strictly monotone function:

$$M_q := q^{-1}\left(\sum_{i=1}^{n} p_i q(x_i)\right).$$

The monotonicity of the generalized means is given in the next corollary.

Corollary 1.3. *Assume (H$_1$-H$_3$) and (H$_5$). Then*

$$M_g \leq M^1_{h,g}(\mathbf{x},\mathbf{p},k) \leq M^1_{h,g}(\mathbf{x},\mathbf{p},k-1) \leq \ldots \leq M^1_{h,g}(\mathbf{x},\mathbf{p},1) = M_h, \tag{7}$$

if either $h \circ g^{-1}$ is convex and h is strictly increasing or $h \circ g^{-1}$ is concave and h is strictly decreasing;

$$M_g = M^1_{g,h}(\mathbf{x},\mathbf{p},1) \leq \ldots \leq M^1_{g,h}(\mathbf{x},\mathbf{p},k-1) \leq M^1_{g,h}(\mathbf{x},\mathbf{p},k) \leq M_h, \tag{8}$$

if either $g \circ h^{-1}$ is convex and g is strictly decreasing or $g \circ h^{-1}$ is concave and g is strictly increasing.

Proof. First, we can apply Theorem 1.1 to the function $h \circ g^{-1}$ and the n-tuples $(g(x_1), \ldots, g(x_n))$, then we can apply h^{-1} to the inequality coming from (1). This gives (7). A similar argument gives (8): $g \circ h^{-1}$, $(h(x_1), \ldots, h(x_n))$ and g^{-1}can be used. □

We illustrate the means defined above by a concrete example based on an example from [2]. Further interesting means can be derived from the other examples in [2].

Example 1.4. *Let $n \geq 1$ and $k \geq 1$ be fixed integers, and let $I_k \subset \{1, \ldots, n\}^k$ such that*

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n,$$

where $\alpha_{I_k,i}$ *means the number of occurrences of* i *in all the sequences* $\mathbf{i}^k := (i_1, \ldots, i_k)$ *from* I_k*. For* $j = 1, \ldots, n$ *we introduce the sets*

$$S_j := \{((i_1, \ldots, i_k), l) \mid (i_1, \ldots, i_k) \in I_k, \quad 1 \le l \le k, \quad i_l = j\}.$$

Let c *be a positive function on* $S := \bigcup_{j=1}^{n} S_j$ *such that*

$$\sum_{((i_1,\ldots,i_k),l)\in S_j} c((i_1, \ldots, i_k), l) = 1, \quad j = 1, \ldots, n.$$

The condition (H_2*) is fulfilled if*

$$\mathcal{A} := \{\{((i_1, \ldots, i_k), l) \mid l = 1, \ldots, k\} \mid (i_1, \ldots, i_k) \in I_k\}.$$

In this case $\mathcal{A}_k = \mathcal{A}$ *and* $\mathcal{A}_l = \varnothing$ $(l = 1, \ldots, k-1)$*. If (*H_3*) and (*H_4*) also hold, then the numbers* N_{k-m} $(0 \le m \le k-1)$ *are given as follows:*

$$N_k := \sum_{(i_1,\ldots,i_k)\in I_k} \left(\left(\sum_{l=1}^{k} c((i_1, \ldots, i_k), l) p_{i_l} \right) f \left(\frac{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l} x_{i_l}}{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l}} \right) \right), \tag{9}$$

and for every $1 \le m \le k-1$

$$N_{k-m} := \frac{m!}{(k-1)\ldots(k-m)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{1\le l_1<\ldots<l_{k-m}\le k} \left(\left(\sum_{j=1}^{k-m} c((i_1, \ldots, i_k), l_j) p_{i_{l_j}} \right) \times f \left(\frac{\sum_{l=1}^{k-m} c((i_1, \ldots, i_k), l_j) p_{i_{l_j}} x_{i_{l_j}}}{\sum_{l=1}^{k-m} c((i_1, \ldots, i_k), l_j) p_{i_{l_j}}} \right) \right) \right). \tag{10}$$

(a) Assume (H_6*). For* $1 \le m \le k-1$ *let*

$$J_{k-m} := \left\{ (l_1, \ldots, l_{k-m}) \in \{1, \ldots, k\}^{k-m} \mid 1 \le l_1 < \ldots < l_{k-m} \le k \right\}.$$

We give the analogue of the power means defined in (4). For $r \in \mathbb{R}$ *and* $\mathbf{i}^k \in I_k$

$$M_r(\mathbf{x}, \mathbf{p}, \mathbf{i}^k) := \begin{cases} \left(\frac{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l} x_{i_l}^r}{\sum_{l=1}^{k} c((i_1,\ldots,i_k),l) p_{i_l}} \right)^{\frac{1}{r}}; & r \ne 0, \\ \left(\prod_{l=1}^{k} x_{i_l}^{c((i_1,\ldots,i_k),l) p_{i_l}} \right)^{\frac{1}{\sum_{l=1}^{k-m} c((i_1,\ldots,i_k),l) p_{i_l}}}; & r = 0, \end{cases}$$

and for $r \in \mathbb{R}$, $\mathbf{i}^k \in I_k$ and $\mathbf{l}^{k-m} \in J_{k-m}$

$$M_r(\mathbf{x},\mathbf{p},\mathbf{i}^k,\mathbf{l}^{k-m}) := \begin{cases} \left(\dfrac{\sum\limits_{j=1}^{k-m} c((i_1,\ldots,i_k),l_j)p_{i_{l_j}} x_{i_{l_j}}^r}{\sum\limits_{j=1}^{k-m} c((i_1,\ldots,i_k),l_j)p_{i_{l_j}}}\right)^{\frac{1}{r}}; & r \neq 0, \\ \left(\prod\limits_{j=1}^{k-m} x_{i_{l_j}}^{c((i_1,\ldots,i_k),l_j)p_{i_{l_j}}}\right)^{\frac{1}{\sum\limits_{j=1}^{k-m} c((i_1,\ldots,i_k),l)p_{i_l}}}; & r = 0, \end{cases}$$

Now let γ, $\eta \in \mathbb{R}$. The mixed symmetric means corresponding to (9) and (10) can be written as

$$M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k) = \begin{cases} \left(\sum\limits_{(i_1,\ldots,i_k)\in I_k} \left(\sum\limits_{l=1}^{k} c\left((i_1,\ldots,i_k),l\right)p_{i_l}\right) M^\eta_\gamma(\mathbf{x},\mathbf{p},\mathbf{i}^k)\right)^{\frac{1}{\eta}}; & \eta \neq 0, \\ \prod\limits_{(i_1,\ldots,i_k)\in I_k} \left(\left(M_\gamma(\mathbf{x},\mathbf{p},\mathbf{i}^k)\right)^{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l)p_{i_l}}\right); & \eta = 0, \end{cases}$$

and for $1 \leq m \leq k-1$

$$M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k-m) = \left(\frac{m!}{(k-1)\ldots(k-m)} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{1\leq l_1<\ldots<l_{k-m}\leq k} \left(\left(\sum_{j=1}^{k-m} c\left((i_1,\ldots,i_k),l_j\right)p_{i_{l_j}}\right) M^\eta_\gamma(\mathbf{x},\mathbf{p},\mathbf{i}^k,\mathbf{l}^{k-m})\right)\right)\right)^{\frac{1}{\eta}},$$

if $\eta \neq 0$, and for $\eta = 0$, we have

$$M^1_{\eta,\gamma}(\mathbf{x},\mathbf{p},k-m) = \left(\prod_{(i_1,\ldots,i_k)\in I_k} \left(\prod_{1\leq l_1<\ldots<l_{k-m}\leq k} \left(M_\gamma(\mathbf{x},\mathbf{p},\mathbf{i}^k,\mathbf{l}^{k-m})\right)^{\left(\sum\limits_{j=1}^{k-m} c((i_1,\ldots,i_k),l_j)p_{i_{l_j}}\right)}\right)\right)^{\frac{m!}{(k-1)\ldots(k-m)}}.$$

(b) Assume (H_3) and (H_5). The generalized means with respect to (9) and (10) can be written as

$$M^1_{h,g}(\mathbf{x},\mathbf{p},k) := h^{-1}\left(\sum_{(i_1,\ldots,i_k)\in I_k} \left(\left(\sum_{l=1}^{k} c\left((i_1,\ldots,i_k),l\right)p_{i_l}\right) h\circ g^{-1}\left(\frac{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l)p_{i_l} g(x_{i_l})}{\sum\limits_{l=1}^{k} c((i_1,\ldots,i_k),l)p_{i_l}}\right)\right)\right),$$

and for $1 \leq m \leq k-1$

$$M_{h,g}^{1}(\mathbf{x},\mathbf{p},k-m):=$$

$$h^{-1}\left(\begin{array}{c}\frac{m!}{(l-1)\dots(l-m)}\sum\limits_{(i_1,\dots,i_k)\in I_k}\\ \left(\sum\limits_{1\le l_1<\dots<l_{k-m}\le k}\left(\left(\sum\limits_{j=1}^{k-m}c\left((i_1,\dots,i_k),l_j\right)p_{i_{l_j}}\right)h\circ g^{-1}\left(\frac{\sum\limits_{j=1}^{k-m}c\big((i_1,\dots,i_k),l_j\big)p_{i_{l_j}}g(x_{i_{l_j}})}{\sum\limits_{j=1}^{k-m}c\big((i_1,\dots,i_k),l_j\big)p_{i_{l_j}}}\right)\right)\right)\end{array}\right).$$

Remark 1.5. *By choosing*

$$c\left((i_1,\dots,i_k),l\right)=\frac{1}{|S_j|}=\frac{1}{\alpha_{I_k,j}}\quad \textit{if}\quad \left((i_1,\dots,i_k),l\right)\in S_j$$

in the previous example, we have the means defined in [4] *(1.14), (1.15), (1.20) and (1.21) as special cases. Thus we have some extensions of these means.*

Remark 1.6. *Results about mixed symmetric means and generalized means similar to Corollary 1.2 and Corollary 1.3 can be given for Example 1.4 as a special case.*

The following result is given in [3].

Theorem 1.7. *Assume (H_3) and (H_4) and let $\lambda\ge 1$ be a real number. We introduce the sets*

$$S_k:=\left\{(i_1,\dots,i_n)\in\mathbb{N}^n\mid\sum_{j=1}^{n}i_j=k\right\},\quad k\in\mathbb{N},$$

and for $k\in\mathbb{N}$ define the numbers

$$\begin{aligned}&C_k(\lambda)=C_k(x_1,\dots,x_n;p_1,\dots,p_n;\lambda)\\&:=\tfrac{1}{(n+\lambda-1)^k}\sum_{(i_1,\dots,i_n)\in S_k}\tfrac{k!}{i_1!\dots i_n!}\left(\sum_{j=1}^{n}\lambda^{i_j}p_j\right)f\left(\frac{\sum\limits_{j=1}^{n}\lambda^{i_j}p_jx_j}{\sum\limits_{j=1}^{n}\lambda^{i_j}p_j}\right).\end{aligned}$$

Then

$$f\left(\sum_{j=1}^{n}p_jx_j\right)=C_0(\lambda)\le C_1(\lambda)\le\dots\le C_k(\lambda)\le\dots\le\sum_{j=1}^{n}p_jf(x_j).$$

Remark 1.8. *The results about mixed symmetric means and generalized means corresponding to Theorem 1.7 are given in* [3].

Remark 1.9. *Under the conditions (H_1)-(H_3) we define*

$$\Upsilon^1(f) := \Upsilon^1_{m,l}(\mathbf{x}, \mathbf{p}, f) := N_m - N_l;\ 1 \leq m < l \leq k,$$
$$\Upsilon^2(f) := \Upsilon^2_l(\mathbf{x}, \mathbf{p}, f) := N_l - f\left(\sum_{j=1}^{n} p_j x_j\right);\ 1 \leq l \leq k,$$

and under the conditions (H_3) *we define*

$$\Upsilon^3(f) := \Upsilon^3_{m,l}(\mathbf{x}, \mathbf{p}, f) := C_m(\lambda) - C_l(\lambda);\ 0 \leq l < m \leq k, \quad k \in \mathbb{N},$$
$$\Upsilon^4(f) := \Upsilon^4_l(\mathbf{x}, \mathbf{p}, f) := \sum_{j=1}^{n} p_j f(x_j) - C_l(\lambda);\ 0 \leq l \leq k, \quad k \in \mathbb{N},$$

where $f : J \to \mathbb{R}$ *is a function.*

It is easy to see that the functionals $f \to \Upsilon^i(f)$ *are linear,* $i = 1, ..., 4$, *and Theorem 1.1 and Theorem 1.7 imply that*

$$\Upsilon^i(f) \geq 0, \quad i = 1, ..., 4$$

if $f : J \to \mathbb{R}$ *is a convex function.*

In [1] the log-convexity and in [6] exponential convexity is proved for some functionals obtained from the interpolations of the discrete Jensen's inequality. The results in [1] are without weights but in [6] are with weights. In [4], a more general class of twice differentiable convex functions is used to construct the exponential convexity of some more general functionals. In [7] a notion of n-exponential convexity is introduced as a generalizations of log-convexity.

In this paper, we use a new method given in [7], to prove the n-exponential convexity and exponential convexity of the functionals $f \to \Upsilon^i(f)$ for $i = 1, ..., 4$, together with the Cauchy type mean value theorems. In this way, our results are more general than the corresponding results in [4].

The notion of *n-exponentially convex* function and the following properties of *exponentially convex* function defined on an interval $I \subset \mathbb{R}$, are given in [7].

Definition 1. A function $g : I \to \mathbb{R}$ is called *n-exponentially convex* in the Jensen sense if

$$\sum_{i,j=1}^{n} a_i a_j g\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for every $a_i \in \mathbb{R}$ and every $x_i \in I$, $i = 1, 2, ..., n$.

A function $g : I \to \mathbb{R}$ is *n-exponentially convex* if it is *n-exponentially convex* in the Jensen sense and continuous on I.

Remark 1.10. *From the definition it is clear that* 1*-exponentially convex functions in the Jensen sense are in fact the nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are m-exponentially convex in the Jensen sense for every $m \in \mathbb{N}$, $m \leq n$.*

Proposition 1.11. *If $g : I \to \mathbb{R}$ is an n-exponentially convex function, then for every $x_i \in I$, $i = 1, 2, ..., n$ and for all $m \in \mathbb{N}$, $m \leq n$ the matrix $\left[g\left(\frac{x_i+x_j}{2}\right) \right]_{i,j=1}^{m}$ is a positive semi-definite matrix. Particularly,*

$$\det \left[g\left(\frac{x_i + x_j}{2}\right) \right]_{i,j=1}^{m} \geq 0$$

for all $m \in \mathbb{N}$, $m = 1, 2, ..., n$.

Definition 2. A function $g : I \to \mathbb{R}$ is exponentially convex in the Jensen sense, if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.
A function $g : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 1.12. *It is easy to see that a positive function $g : I \to \mathbb{R}$ is* log*-convex in the Jensen sense if and only if it is* 2*-exponentially convex in the Jensen sense, that is*

$$a_1^2 g(x) + 2a_1 a_2 g\left(\frac{x + y}{2}\right) + a_2^2 g(y) \geq 0$$

holds for every $a_1, a_2 \in \mathbb{R}$ and $x, y \in I$.
Similarly, if g is 2*-exponentially convex, then g is* log*-convex. Conversely, if g is* log*-convex and continuous, then g is* 2*-exponentially convex.*

Divided differences are fertile to study functions having different degree of smoothness.

Definition 3. The second order divided difference of a function $g : I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$[y_i; g] = g(y_i), \quad i = 0, 1, 2$$

$$[y_i, y_{i+1}; g] = \frac{g(y_{i+1}) - g(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1$$

$$[y_0, y_1, y_2; g] = \frac{[y_1, y_2; g] - [y_0, y_1; g]}{y_2 - y_0}. \tag{11}$$

Remark 1.13. *The value $[y_0, y_1, y_2; g]$ is independent of the order of the points y_0, y_1, and y_2. By taking limits this definition may be extended to include the cases in which any two or all three points coincide as follows: $\forall\ y_0,\ y_1,\ y_2 \in I$ such that $y_2 \neq y_0$*

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; g] = [y_0, y_0, y_2; g] = \frac{g(y_2) - g(y_0) - g^{'}(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}$$

provided that g' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$ in (11), we get

$$[y_0, y_0, y_0; g] \ = \lim_{y_i \to y_0} [y_0, y_1, y_2; g] = \frac{g^{''}(y_0)}{2} \text{ for } i = 1, 2$$

provided that $g^{''}$ exist on I.

2. Main Results

Theorem 2.1. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Υ be a linear functional on the vector space of real functions defined on I such that $\Upsilon(f) \geq 0$ for every convex function f on I. Then $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J. If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous, then it is n-exponentially convex on J.*

Proof. Let $t_k, t_l \in J$, $t_{kl} := \frac{t_k + t_l}{2}$ and $b_k, b_l \in \mathbb{R}$ for $k, l = 1, 2, ..., n$, and define the function ω on I by

$$\omega := \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}.$$

Since the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; \omega] = \sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] \geq 0,$$

which implies that ω is a convex function on I. Therefore we have $\Upsilon(\omega) \geq 0$, which yields by the linearity of Υ, that

$$\sum_{k,l=1}^{n} b_k b_l \Upsilon(\phi_{t_{kl}}) \geq 0.$$

We conclude that the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an n-exponentially convex function in the Jensen sense on J.

If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous on J, then it is n-exponentially convex on J by definition. □

As a consequence of the above theorem we can give the following corollaries.

Corollary 2.2. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t \mid t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Υ be a linear functional on the vector space of real functions defined on I such that $\Upsilon(f) \geq 0$ for every convex function on I. Then $t \to \Upsilon(\phi_t)$ $(t \in J)$ is an exponentially convex function in the Jensen sense on J. If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is continuous, then it is exponentially convex on J.*

Corollary 2.3. *Assume $J \subset \mathbb{R}$ is an interval, and assume $\Lambda = \{\phi_t : t \in J\}$ is a family of functions defined on an interval $I \subset \mathbb{R}$, such that the function $t \to [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$. Let Υ be a linear functional on the vector space of real functions defined on I such that $\Upsilon(f) \geq 0$ for every convex function on I. Then the following two statements hold:*

(i) *If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is positive and continuous, then it is 2-exponentially convex on J, and thus* log-*convex.*

(ii) *If the function $t \to \Upsilon(\phi_t)$ $(t \in J)$ is positive and differentiable, then for every $s, t, u, v \in J$, such that $s \leq u$ and $t \leq v$, we have*

$$\mathfrak{u}_{s,t}(\Upsilon, \Lambda) \leq \mathfrak{u}_{u,v}(\Upsilon, \Lambda) \tag{12}$$

where

$$\mathfrak{u}_{s,t}(\Upsilon,\Lambda) := \begin{cases} \left(\frac{\Upsilon(\phi_s)}{\Upsilon(\phi_t)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(\frac{\frac{d}{ds}\Upsilon(\phi_s)}{\Upsilon(\phi_s)}\right), & s = t \end{cases} \tag{13}$$

for $\phi_s, \phi_t \in \Lambda$.

Proof. (i) See Remark 1.12 and Theorem 2.1.

(ii) From the definition of a convex function ψ on J, we have the following inequality (see [8, page 2])

$$\frac{\psi(s) - \psi(t)}{s - t} \leq \frac{\psi(u) - \psi(v)}{u - v}, \tag{14}$$

$\forall\, s,t,u,v \in J$ such that $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$.

By (i), $s \to \Upsilon(\phi_s)$, $s \in J$ is log-convex, and hence (14) shows with $\psi(s) = \log \Upsilon(\phi_s)$, $s \in J$ that

$$\frac{\log \Upsilon(\phi_s) - \log \Upsilon(\phi_t)}{s-t} \leq \frac{\log \Upsilon(\phi_u) - \log \Upsilon(\phi_v)}{u-v} \tag{15}$$

for $s \leq u$, $t \leq v$, $s \neq t$, $u \neq v$, which is equivalent to (12). For $s = t$ or $u = v$ (12) follows from (15) by taking limit.

□

Remark 2.4. *Note that the results from Theorem 2.1, Corollary 2.2, Corollary 2.3 are valid when two of the points* $y_0, y_1, y_2 \in I$ *coincide, say* $y_1 = y_0$, *for a family of differentiable functions* ϕ_t *such that the function* $t \to [y_0, y_1, y_2; \phi_t]$ *is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and moreover, they are are also valid when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 1.13 and suitable characterization of convexity.*

The following result is related to the first condition of Theorem 2.1.

Theorem 2.5. *Assume* $J \subset \mathbb{R}$ *is an interval, and assume* $\Lambda = \{\phi_t \mid t \in J\}$ *is a family of twice differentiable functions defined on an interval* $I \subset \mathbb{R}$ *such that the function* $t \mapsto \phi_t''(x)$

$(t \in J)$ is exponentially convex for every fixed $x \in I$. Then the function $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ is exponentially convex in the Jensen sense for any three points y_0, y_1, $y_2 \in I$.

Proof. Let $t_k, t_l \in J$, $t_{kl} := \frac{t_k+t_l}{2}$ and $b_k, b_l \in \mathbb{R}$ for $k, l = 1, 2, ..., n$, and fix $x \in I$. Then

$$\left(\sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}\right)''(x) = \sum_{k,l=1}^{n} b_k b_l \phi''_{t_{kl}}(x) \geq 0.$$

It follows that the function

$$\sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}$$

is convex, and hence

$$\sum_{k,l=1}^{n} b_k b_l [y_0, y_1, y_2; \phi_{t_{kl}}] = [y_0, y_1, y_2; \sum_{k,l=1}^{n} b_k b_l \phi_{t_{kl}}] \geq 0$$

for every three mutually different points $y_0, y_1, y_2 \in I$. This implies the exponential convexity of $t \mapsto [y_0, y_1, y_2; \phi_t]$ $(t \in J)$ in the Jensen sense. □

Remark 2.6. *It comes from either the conditions of Theorem 2.5 or the proof of this theorem that the functions ϕ_t, $t \in J$ are convex.*

Now we formulate mean value theorems.

Theorem 2.7. *Let Υ be a linear functional on the vector space of real functions defined on $[a, b] \subset \mathbb{R}$ such that $\Upsilon(f) \geq 0$ for every convex function f on $[a, b]$. Let $g \in C^2[a, b]$. Then there exists $\xi \in [a, b]$ such that*

$$\Upsilon(g) = \frac{1}{2} g''(\xi) \Upsilon\left(x^2\right).$$

Proof. Since $g \in C^2[a, b]$, there exist the real numbers $m = \min_{x \in [a,b]} g''(x)$ and $M = \max_{x \in [a,b]} g''(x)$. It is easy to show that the functions ϕ_1 and ϕ_2 defined on $[a, b]$ by

$$\phi_1(x) = \frac{M}{2} x^2 - g(x),$$

and

$$\phi_2(x) = g(x) - \frac{m}{2} x^2,$$

are convex.

By applying the functional Υ to the functions ϕ_1 and ϕ_2, we have the properties of Υ that

$$\Upsilon\left(\frac{M}{2}x^2 - g(x)\right) \geq 0,$$

$$\Rightarrow \Upsilon(g) \leq \frac{M}{2}\Upsilon\left(x^2\right), \tag{16}$$

and

$$\Upsilon\left(g(x) - \frac{m}{2}x^2\right) \geqslant 0$$

$$\Rightarrow \frac{m}{2}\Upsilon\left(x^2\right) \leq \Upsilon(g). \tag{17}$$

From (16) and from (17), we get

$$\frac{m}{2}\Upsilon\left(x^2\right) \leq \Upsilon(g) \leq \frac{M}{2}\Upsilon\left(x^2\right).$$

If $\Upsilon(x^2) = 0$, then nothing to prove. If $\Upsilon(x^2) \neq 0$, then

$$m \leq \frac{2\Upsilon(g)}{\Upsilon(x^2)} \leq M.$$

Hence we have

$$\Upsilon(g) = \frac{1}{2}g''(\xi)\,\Upsilon\left(x^2\right).$$

□

Theorem 2.8. *Let Υ be a linear functional on the vector space of real functions defined on $[a,b] \subset \mathbb{R}$ such that $\Upsilon(f) \geq 0$ for every convex function f on $[a,b]$. Let $g, h \in C^2[a,b]$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)},$$

provided that $\Upsilon(h) \neq 0$.

Proof. Define $L \in C^2[a,b]$ by

$$L := c_1 g - c_2 h,$$

where

$$c_1 := \Upsilon(h)$$

and

$$c_2 := \Upsilon(g).$$

Now using Theorem 2.7 for the function L, we have

$$\left(c_1 \frac{g''(\xi)}{2} - c_2 \frac{h''(\xi)}{2}\right) \Upsilon\left(x^2\right) = 0. \tag{18}$$

Since $\Upsilon(h) \neq 0$, Theorem 2.7 implies that $\Upsilon(x^2) \neq 0$, and therefore (18) gives

$$\frac{\Upsilon(g)}{\Upsilon(h)} = \frac{g''(\xi)}{h''(\xi)}.$$

□

3. Applications to Cauchy Means

In this section we apply the main results to generate new Cauchy means. We mention that the functionals Υ^i, $i = 1, \ldots, 4$ defined in Remark 1.9 are linear on the vector space of real functions defined on the inteval $J \subset \mathbb{R}$, and $\Upsilon^i(f) \geq 0$ for every convex function on J.

Example 3.1. *Assume (H_1)-(H_3) with $J = \mathbb{R}$, and consider the linear functionals Υ^i defined in Remark 1.9.*
We consider the class of convex functions

$$\Lambda_1 := \{\phi_t : \mathbb{R} \to [0, \infty) \mid t \in \mathbb{R}\},$$

where

$$\phi_t(x) := \begin{cases} \frac{1}{t^2} e^{tx}; & t \neq 0, \\ \frac{1}{2} x^2; & t = 0. \end{cases}$$

Then $t \mapsto \phi_t''(x)$ ($t \in \mathbb{R}$) is exponentially convex for every fixed $x \in \mathbb{R}$ (see [5]), thus by Theorem 2.5, the function $t \mapsto [y_0, y_1, y_2; \phi_t]$, $t \in \mathbb{R}$ is exponentially convex in the Jensen sense for every three mutually different points $y_0, y_1, y_2 \in \mathbb{R}$.

Now fix $1 \leq i \leq 4$. By applying Corollary 2.2 with $\Lambda = \Lambda_1$, we get the exponential convexity of $t \mapsto \Upsilon^i(\phi_t)$ ($t \in \mathbb{R}$) in the Jensen sense. This mapping is also differentiable, therefore exponentially convex, and the expression in (13) has the form

$$\mathfrak{u}_{s,t}(\Upsilon^i,\Lambda_1)=\begin{cases}\left(\frac{\Upsilon^i(\phi_s)}{\Upsilon^i(\phi_t)}\right)^{\frac{1}{s-t}}, \ s\neq t,\\ \exp\left(\frac{\Upsilon^i(id\,\phi_s)}{\Upsilon^i(\phi_s)}-\frac{2}{s}\right), \ s=t\neq 0,\\ \exp\left(\frac{\Upsilon^i(id\,\phi_0)}{3\Upsilon^i(\phi_0)}\right), \ s=t=0,\end{cases}$$

where "id" means the identity function on $\mathbb{R}$.

From (12) we have the monotonicity of the functions $\mathfrak{u}_{s,t}(\Upsilon^i,\Lambda_1)$ *in both parameters.*

Suppose $\Upsilon^i(\phi_t)>0$ $(t\in\mathbb{R})$, $a:=min\{x_1,...,x_n\}$, $b:=max\{x_1,...,x_n\}$, *and let*

$$\mathfrak{M}_{s,t}(\Upsilon^i,\Lambda_1):=\log\mathfrak{u}_{s,t}(\Upsilon^i,\Lambda_1); \quad s,t\in\mathbb{R}.$$

Then from Theorem 2.8 we have

$$a\leq\mathfrak{M}_{s,t}(\Upsilon^i,\Lambda_1)\leq b,$$

and thus $\mathfrak{M}_{s,t}(\Upsilon^i,\Lambda_1)$ $(s,t\in\mathbb{R})$ *are means. The monotonicity of these means is followed by (12).*

Example 3.2. *Assume* (H_1)*-*(H_3) *with* $J=(0,\infty)$, *and consider the linear functionals* Υ^i *defined in Remark 1.9.*

We consider the class of convex functions

$$\Lambda_2=\{\psi_t:(0,\infty)\to\mathbb{R} \mid t\in\mathbb{R}\},$$

where

$$\psi_t(x):=\begin{cases}\frac{x^t}{t(t-1)}; \ t\neq 0,1,\\ -\log x; \ t=0,\\ x\log x; \ t=1.\end{cases}$$

Then $t\mapsto\psi_t''(x)=x^{t-2}=e^{(t-2)\log x}$ $(t\in\mathbb{R})$ *is exponentially convex for every fixed* $x\in(0,\infty)$.

Now fix $1\leq i\leq 4$. *By similar arguments as given in Example 3.1 we get the exponential convexity of* $t\mapsto\Upsilon^i(\psi_t)$ $(t\in\mathbb{R})$ *in the Jensen sense. This mapping is differentiable too, therefore exponentially convex. It is easy to calculate that (13) can be written as*

$$\mathfrak{u}_{s,t}(\mathbf{x},\mathbf{p},\Upsilon^i,\Lambda_2)=\begin{cases}\left(\frac{\Upsilon^i(\psi_s)}{\Upsilon^i(\psi_t)}\right)^{\frac{1}{s-t}};\ s\neq t,\\ \exp\left(\frac{1-2s}{s(s-1)}-\frac{\Upsilon^i(\psi_s\psi_0)}{\Upsilon^i(\psi_s)}\right);\ s=t\neq 0,1,\\ \exp\left(1-\frac{\Upsilon^i(\psi_0^2)}{2\Upsilon^i(\psi_0)}\right);\ s=t=0,\\ \exp\left(-1-\frac{\Upsilon^i(\psi_0\psi_1)}{2\Upsilon^i(\psi_1)}\right);\ s=t=1.\end{cases}$$

Suppose $\Upsilon^i(\psi_t)>0$ $(t\in\mathbb{R})$, *and let* $a:=min\{x_1,...,x_n\}$, $b:=max\{x_1,...,x_n\}$. *By Theorem 2.8, we can check that*

$$a\leq\mathfrak{u}_{s,t}(\mathbf{x},\mathbf{p},\Upsilon^i,\Lambda_2)\leq b;\quad s,t\in\mathbb{R}. \tag{19}$$

The means $\mathfrak{u}_{s,t}(\mathbf{x},\mathbf{p},\Upsilon^i,\Lambda_2)$ $(s,t\in\mathbb{R})$ *are continuous, symmetric and monotone in both parameters (by use of (12)).*

Let $s,t,r\in\mathbb{R}$ *such that* $r\neq 0$. *By the substitutions* $s\to\frac{s}{r}$, $t\to\frac{t}{r}$, $(x_1,\dots,x_n)\to(x_1^r,\dots,x_n^r)$ *in (19), we get*

$$\bar{a}\leq\mathfrak{u}_{s/r,t/r}(\mathbf{x}^r,\mathbf{p},\Upsilon^i,\Lambda_2)\leq\bar{b},$$

where $\bar{a}:=min\{x_1^r,\dots,x_n^r\}$ *and* $\bar{b}:=max\{x_1^r,\dots,x_n^r\}$. *Thus new means can be defined with three parameters:*

$$\mathfrak{u}_{s,t,r}(\mathbf{x},\mathbf{p},\Upsilon^i,\Lambda_2):=\begin{cases}\left(\mathfrak{u}_{s/r,t/r}(\mathbf{x}^r,\mathbf{p},\Upsilon^i,\Lambda_2)\right)^{\frac{1}{r}};\quad r\neq 0,\\ \mathfrak{u}_{s,t}(\log\mathbf{x},\mathbf{p},\Upsilon^i,\Lambda_1);\qquad r=0,\end{cases}$$

where $\log\mathbf{x}=(\log x_1,...,\log x_n)$.

The monotonicity of these three parameter means is followed by the monotonicity and continuity of the two parameter means.

Example 3.3. *Assume* (H_1)*-*(H_3) *with* $J=(0,\infty)$, *and consider the linear functionals* Υ^i *defined in Remark 1.9.*

We consider the class of convex functions

$$\Lambda_3=\{\eta_t:(0,\infty)\to(0,\infty)\mid t\in(0,\infty)\},$$

where

$$\eta_t(x) := \begin{cases} \frac{t^{-x}}{\log^2 t}; \; t \neq 1, \\ \frac{x^2}{2}; \quad t = 1. \end{cases}$$

$t \mapsto \psi_t''(x)$ $(t \in (0,\infty))$ *is exponentially convex for every fixed* $x \in (0,\infty)$*, being the restriction of the Laplace transform of a nonnegative function (see* [5] *or* [9] *page 210).*
Now fix $1 \leq i \leq 4$*. We can get the exponential convexity of* $t \mapsto \Upsilon^i(\psi_t)$ $(t \in \mathbb{R})$ *as in Example 3.1. For the class* Λ_3*, (13) has the form*

$$\mathfrak{u}_{s,t}(\Upsilon^i, \Lambda_3) = \begin{cases} \left(\frac{\Upsilon^i(\eta_s)}{\Upsilon^i(\eta_t)}\right)^{\frac{1}{s-t}}; \; s \neq t, \\ \exp\left(-\frac{2}{s\log s} - \frac{\Upsilon^i(id\eta_s)}{s\Upsilon^i(\eta_s)}\right); \; s = t \neq 1, \\ \exp\left(-\frac{\Upsilon^i(id\eta_1)}{3\Upsilon^i(\eta_1)}\right); \; s = t = 1. \end{cases}$$

The monotonicity of $\mathfrak{u}_{s,t}(\Upsilon^i, \Lambda_3)$ $(s,t \in (0,\infty))$ *comes from (12).*
Suppose $\Upsilon^i(\eta_t) > 0$ $(t \in (0,\infty))$*, and let* $a := min\{x_1, ..., x_n\}$*,* $b := max\{x_1, ..., x_n\}$*, and define*

$$\mathfrak{M}_{s,t}(\Upsilon^i, \Lambda_3) := -L(s,t) \log \mathfrak{u}_{s,t}(\Upsilon^i, \Lambda_3), \quad s,t \in (0,\infty),$$

where $L(s,t)$ *is the well known logarithmic mean*

$$L(s,t) := \begin{cases} \frac{s-t}{\log s - \log t}; \; s \neq t, \\ t; \qquad s = t. \end{cases}$$

From Theorem 2.8 we have

$$a \leq \mathfrak{M}_{s,t}(\Upsilon^i, \Lambda_3) \leq b, \quad s,t \in (0,\infty),$$

and therefore we get means.

Example 3.4. *Assume (*H_1*)-(*H_3*) with* $J = (0,\infty)$*, and consider the linear functionals* Υ^i *defined in Remark 1.9.*
We consider the class of convex functions

$$\Lambda_4 = \{\gamma_t : (0,\infty) \to (0,\infty) \mid t \in (0,\infty)\},$$

where

$$\gamma_t(x) := \frac{e^{-x\sqrt{t}}}{t}.$$

$t \mapsto \psi_t''(x) = e^{-x\sqrt{t}}$, $t \in (0,\infty)$ *is exponentially convex for every fixed* $x \in (0,\infty)$, *being the restriction of the Laplace transform of a non-negative function (see* [5] *or* [9] *page 214).*

Now fix $1 \leq i \leq 4$. *As before* $t \mapsto \Upsilon^i(\psi_t)$ $(t \in \mathbb{R})$ *is exponentially convex and differentiable. For the class* Λ_4, *(13) becomes*

$$\mathfrak{u}_{s,t}(\Upsilon^i, \Lambda_4) = \begin{cases} \left(\frac{\Upsilon^i(\gamma_s)}{\Upsilon^i(\gamma_t)}\right)^{\frac{1}{s-t}}; & s \neq t, \\ \exp\left(-\frac{1}{t} - \frac{\Upsilon^i(id\gamma_t)}{2\sqrt{t}\Upsilon^i(\gamma_t)}\right); & s = t, \end{cases}$$

where id means the identity function on $(0,\infty)$. *The monotonicity of* $\mathfrak{u}_{s,t}(\Upsilon^i, \Lambda_4)$ $(s, t \in (0,\infty))$ *is followed by (12).*

Suppose $\Upsilon^i(\eta_t) > 0$ $(t \in (0,\infty))$, *let* $a := min\{x_1, ..., x_n\}$, $b := max\{x_1, ..., x_n\}$, *and define*

$$\mathfrak{M}_{s,t}(\Upsilon^i, \Lambda_4) := -(\sqrt{s} + \sqrt{t}) \log \mathfrak{u}_{s,t}(\Upsilon^i, \Lambda_4), \quad s, t \in (0,\infty).$$

Then Theorem 2.8 yields that

$$a \leq \mathfrak{M}_{s,t}(\Upsilon^i, \Lambda_4) \leq b,$$

thus we have new means.

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