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UNIQUE COMMON FIXED POINTS FOR A INFINITE FAMILY OF MAPPINGS WITH IMPLICIT CONTRACTIVE CONDITIONS OF INTEGRAL TYPES ON 2-METRIC SPACES

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Abstract. In this paper, we discuss the existence problems of common fixed points for a infinite family of mappings satisfying implicit $\mathscr{A}$-contractive condition and ψ-φ-contractive condition with integral types on 2-metric spaces.

Keywords: Common fixed point; altering distance function; $\mathscr{A}$-contraction; implicit.

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1. Introduction and preliminaries

It is well known that the Banach contraction principle is a fundamental result in fixed point theory. After this classical result, many fixed point results have been developed, see[1-5] and others. In 2002, Branciari[6] obtained the following integral version of the Banach contraction principle as follows:

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Theorem 1.1 Let (X,d) be a complete metric space, $\lambda \in (0,1)$ and $T : X \to X$ be a mapping such that for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \phi(t)dt \leq \lambda \int_0^{d(x,y)} \phi(t)dt,$$

where $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable mapping which is summable(i.e., with finite integral) on each compact subset of $[0,\infty) := \mathbb{R}^+$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt > 0$. Then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Later, Some generalizations of Theorem 1.1 have been appeared, see[7-12] and others.

G*ä*hler[13, 14] investigated the concept of 2-metric spaces, and many authors discussed and obtained fixed point or common fixed point theorems on 2-metric spaces, see [15-21] and others. These results greatly improve and generalize the Banach fixed point theory. But, few authors discussed the problems of (common) fixed points for mappings with contractive conditions of integral type on 2-metric spaces. Hence, the purpose of this paper is to consider several real functions and discuss the existence problems of common fixed points for a family of mappings with implicit contractive conditions of integral types on 2-metric spaces.

Definition 1.1([13-14]) A 2-metric space (X,d) consists of a nonempty set X and a function $d : X \times X \times X \to [0,+\infty)$ such that

(i) for distant elements $x, y \in X$, there exists an $u \in X$ such that $d(x,y,u) \neq 0$;

(ii) $d(x,y,z) = 0$ if and only if at least two elements in $\{x,y,z\}$ are equal;

(iii) $d(x,y,z) = d(u,v,w)$, where $\{u,v,w\}$ is any permutation of $\{x,y,z\}$;

(iv) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

Definition 1.2([13-14,18]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space (X,d) is said to be a Cauchy sequence, if for each $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n, x_m, a) < \varepsilon$ for all $a \in X$ and $n, m > N$.

Definition 1.3([13-14,18]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$, if for each $a \in X$, $\lim_{n\to+\infty} d(x_n, x, a) = 0$. And we write that $x_n \to x$ and call x the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.4([13-14,18]) A 2-metric space (X,d) is said to be complete, if every Cauchy sequence in X is convergent.

Now, we give some notations and well-known lemmas.

Let $\phi \in \Phi \Longleftrightarrow \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable mapping which is summable(i.e., with finite integral) on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \phi(t)dt > 0$.

Lemma 1.1([22]) Let $\{x_n\}$ be a sequence in 2-metric space (X,d) such that $\lim_{n\to\infty} d(x_n,x_{n+1},a) = 0$ for all $a \in X$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $a \in X$ and $\varepsilon > 0$ such that for each $i \in \mathbb{N}$ there exist $m(i), n(i) \in \mathbb{N}$ with $m(i), n(i) > i$ such that

(i) $m(i) > n(i)$ and $n(i) \to \infty$ as $i \to \infty$;

(ii) $d(x_{m(i)}, x_{n(i)}, a) > \varepsilon$, but $d(x_{m(i)-1}, x_{n(i)}, a) \leq \varepsilon$.

Lemma 1.2([18-19]) If (X,d) is a 2-metric space and a sequence $\{x_n\}$ in X converges to $x \in X$. Then

$$\lim_{n\to\infty} d(x_n,b,c) = d(x,b,c), \forall b,c \in X.$$

Lemma 1.3([11-12]) Let $\phi \in \Phi$ and $\{r_n\}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = r$. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = \int_0^r \phi(t)dt.$$

Lemma 1.4([11-12]) Let $\phi \in \Phi$ and $\{r_n\}$ be a nonnegative sequence. Then

$$\lim_{n\to\infty} \int_0^{r_n} \phi(t)dt = 0 \Longleftrightarrow \lim_{n\to\infty} r_n = 0.$$

2. Unique Common Fixed Points

Akram et. al[23] introduced the concept of $\mathscr{A}$-contractions (also see [9]) as follows:

Let a nonempty set $\mathscr{A}$ consisting of all functions $\alpha : \mathbb{R}^{+3} \to \mathbb{R}^+$ satisfying:

($\mathscr{A}_1$): α is continuous on $\mathbb{R}^{+3}$ of all triplets of non-negative reals;

($\mathscr{A}_2$): $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or a $\leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$ for all $a,b \in \mathbb{R}^+$.

$\phi \in \Phi$ is said to be sub-additive if ϕ satisfies that for each $a,b \geq 0$,

$$\int_0^{a+b} \phi(t)dt \leq \int_0^a \phi(t)dt + \int_0^b \phi(t)dt.$$

Let $\phi(t) = \frac{1}{1+t}$, then $\phi \in \Phi$, and for $a,b \geq 0$,

$$\int_0^{a+b} \phi(t)dt = \ln^{(1+a+b)} \leq \ln^{(1+a)(1+b)} = \ln^{(1+a)} + \ln^{(1+b)} = \int_0^a \phi(t)dt + \int_0^b \phi(t)dt.$$

Hence $\phi(t)=\frac{1}{1+t}$ is sub-additive.

Now, we give the first common fixed point result for a infinite family of mappings satisfying an implicit $\mathscr{A}$-contractive condition with integral type on 2-metric spaces.

Theorem 2.1 Let (X,d) be a complete 2-metric space and $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X. Suppose that for each $i,j\in\mathbb{N}$ and $a,x,y\in X$,

$$\int_0^{d(f_ix,f_jy,a)}\phi(t)dt\le\alpha\Big(\int_0^{d(x,y,a)}\phi(t)dt,\int_0^{d(x,f_ix,a)}\phi(t)dt,\int_0^{d(y,f_jy,a)}\phi(t)dt\Big),\tag{2.1}$$

where $\phi\in\Phi$. If ϕ is sub-additive, then $\{f_i\}_{i\in\mathbb{N}}$ has a unique common fixed point.

Proof. Take any element $x_0\in X$, and let $x_{n+1}=f_{n+1}x_n$ for all $n=0,1,2,\cdots$, then we obtain a sequence $\{x_n\}_{n=0}^{\infty}$. For any $n\in\mathbb{N}$ and $a\in X$, by (2.1),

$$\begin{aligned}&\int_0^{d(x_n,x_{n+1},a)}\phi(t)dt\\&=\int_0^{d(f_nx_{n-1},f_{n+1}x_n,a)}\phi(t)dt\\&\le\alpha\Big(\int_0^{d(x_{n-1},x_n,a)}\phi(t)dt,\int_0^{d(x_{n-1},f_nx_{n-1},a)}\phi(t)dt,\int_0^{d(x_n,f_{n+1}x_n,a)}\phi(t)dt\Big)\\&=\alpha\Big(\int_0^{d(x_{n-1},x_n,a)}\phi(t)dt,\int_0^{d(x_{n-1},x_n,a)}\phi(t)dt,\int_0^{d(x_n,x_{n+1},a)}\phi(t)dt\Big),\end{aligned}\tag{2.2}$$

so by $(\mathscr{A}_2)$ and mathematical induction,

$$\int_0^{d(x_n,x_{n+1},a)}\phi(t)dt\le k\int_0^{d(x_{n-1},x_n,a)}\phi(t)dt\le k^2\int_0^{d(x_{n-2},x_{n-1},a)}\phi(t)dt\le\cdots\le k^n\int_0^{d(x_0,x_1,a)}\phi(t)dt.$$

Hence

$$\lim_{n\to\infty}\int_0^{d(x_n,x_{n+1},a)}\phi(t)dt=0,$$

so by Lemma 1.4,

$$\lim_{n\to\infty}d(x_n,x_{n+1},a)=0,\ \forall\, a\in X.\tag{2.3}$$

Using (2.1) again, we obtain

$$\begin{aligned}
&\int_0^{d(x_n,x_{n+1},x_{n+2})}\phi(t)dt\\
=&\int_0^{d(f_{n+2}x_{n+1},f_{n+1}x_n,x_n)}\phi(t)dt\\
\leq&\alpha\Big(\int_0^{d(x_{n+1},x_n,x_n)}\phi(t)dt,\int_0^{d(x_{n+1},f_{n+2}x_{n+1},x_n)}\phi(t)dt,\int_0^{d(x_n,f_{n+1}x_n,x_n)}\phi(t)dt\Big) \qquad (2.4)\\
=&\alpha\Big(\int_0^{d(x_{n+1},x_n,x_n)}\phi(t)dt,\int_0^{d(x_{n+1},x_{n+2},x_n)}\phi(t)dt,\int_0^{d(x_n,x_{n+1},x_n)}\phi(t)dt\Big)\\
=&\alpha\Big(0,\int_0^{d(x_{n+1},x_{n+2},x_n)}\phi(t)dt,0\Big),
\end{aligned}$$

so by $(\mathscr{A}_2)$,

$$\int_0^{d(x_{n+1},x_{n+2},x_n)}\phi(t)dt=0,$$

hence by property of ϕ,

$$d(x_n,x_{n+1},x_{n+2})=0,\forall\, n\in\mathbb{N}. \qquad (2.5)$$

Fix any $k=0,1,2,\cdots$ and suppose that $d(x_k,x_{n-1},x_n)=0$ for any $n\in\mathbb{N}$ with $(n-1)-k>1$, then by (2.1) and the assumption,

$$\begin{aligned}
&\int_0^{d(x_n,x_{n+1},x_k)}\phi(t)dt\\
=&\int_0^{d(f_nx_{n-1},f_{n+1}x_n,x_k)}\phi(t)dt\\
\leq&\alpha\Big(\int_0^{d(x_{n-1},x_n,x_k)}\phi(t)dt,\int_0^{d(x_{n-1},f_nx_{n-1},x_k)}\phi(t)dt,\int_0^{d(x_n,f_{n+1}x_n,x_k)}\phi(t)dt\Big)\\
=&\alpha\Big(\int_0^{d(x_{n-1},x_n,x_k)}\phi(t)dt,\int_0^{d(x_{n-1},x_n,x_k)}\phi(t)dt,\int_0^{d(x_n,x_{n+1},x_k)}\phi(t)dt\Big)\\
=&\alpha\Big(0,0,\int_0^{d(x_{n+1},x_{n+2},x_k)}\phi(t)dt\Big),
\end{aligned}$$

so by $(\mathscr{A}_2)$, we obtain

$$d(x_{n+1},x_{n+2},x_k)=0.$$

Combining (2.5) and the inductive assumption, we obtain

$$d(x_k,x_n,x_{n+1})=0,\forall\, n,k=0,1,2,\cdots,n\geq k.$$

Therefore, for any $l,m,n\in\mathbb{N}$ with $l<m<n$,

$$\begin{aligned}d(x_l,x_m,x_n)\leq& d(x_l,x_m,x_{n-1})+d(x_l,x_{n-1},x_n)+d(x_m,x_{n-1},x_n)=d(x_l,x_m,x_{n-1})\\ \leq& d(x_l,x_m,x_{n-2})+d(x_l,x_{n-2},x_{n-1})+d(x_m,x_{n-2},x_{n-1})=d(x_l,x_m,x_{n-2})\\ \leq&\cdots\\ \leq& d(x_l,x_m,x_{m+1})\\ =&0,\end{aligned}$$

hence

$$d(x_l,x_m,x_n)=0,\forall\, l,m,n=0,1,2,\cdots. \tag{2.6}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that it is not true, then by Lemma 1,1, there exist $a_0\in X$ and $\varepsilon>0$ such that for any $i\in\mathbb{N}$ there exist $m(i),n(i)\in\mathbb{N}$ with $m(i),n(i)>i$ satisfying

(i) $m(i)>n(i)$ and $n(i)\to\infty$ as $i\to\infty$;

(ii) $d(x_{m(i)},x_{n(i)},a_0)>\varepsilon$, but $d(x_{m(i)-1},x_{n(i)},a_0)\leq\varepsilon, i=1,2,\cdots$.

Using Lemma 1.1, (2.3), (2.6) and the following fact

$$d(x_{m(i)},x_{n(i)},a_0)\leq d(x_{m(i)},x_{m(i)-1},a_0)+d(x_{m(i)-1},x_{n(i)},a_0)+d(x_{m(i)},x_{n(i)},x_{m(i)-1}),$$

we obtain

$$\lim_{i\to\infty}d(x_{m(i)},x_{n(i)},a_0)=\lim_{i\to\infty}d(x_{m(i)-1},x_{n(i)},a_0)=\varepsilon. \tag{2.7}$$

Since the following two inequalities hold

$$|d(x_{m(i)},x_{n(i)},a_0)-d(x_{m(i)},x_{n(i)-1},a_0)|\leq d(x_{n(i)-1},x_{n(i)},a_0)+d(x_{m(i)},x_{n(i)},x_{n(i)-1})$$

and

$$|d(x_{m(i)-1},x_{n(i)-1},a_0)-d(x_{m(i)},x_{n(i)-1},a_0)|\leq d(x_{m(i)-1},x_{m(i)},a_0)+d(x_{m(i)},x_{m(i)-1},x_{n(i)-1}),$$

so using (2.3), (2.6) and (2.7), we obtain

$$\lim_{i\to\infty}d(x_{m(i)},x_{n(i)-1},a_0)=\lim_{i\to\infty}d(x_{m(i)-1},x_{n(i)-1},a_0)=\varepsilon. \tag{2,8}$$

Using (2.1), we have

$$\begin{aligned}
&\int_0^{d(x_{n(i)},x_{m(i)},a_0)}\phi(t)dt\\
=&\int_0^{d(f_{n(i)}x_{n(i)-1},f_{m(i)}x_{m(i)-1},a_0)}\phi(t)dt\\
\leq&\alpha\Big(\int_0^{d(x_{n(i)-1},x_{m(i)-1},a_0)}\phi(t)dt,\int_0^{d(x_{n(i)-1},f_{n(i)}x_{n(i)-1},a_0)}\phi(t)dt,\int_0^{d(x_{m(i)-1},f_{m(i)}x_{m(i)-1},a_0)}\phi(t)dt\Big)\\
=&\alpha\Big(\int_0^{d(x_{n(i)-1},x_{m(i)-1},a_0)}\phi(t)dt,\int_0^{d(x_{n(i)-1},x_{n(i)},a_0)}\phi(t)dt,\int_0^{d(x_{m(i)-1},x_{m(i)},a_0)}\phi(t)dt\Big).
\end{aligned}$$

Letting $i\to\infty$ and using ($\mathscr{A}_1$), Lemma 1.3, (2.3), (2.7) and (2.8), we obtain

$$\int_0^{\varepsilon}\phi(t)dt\leq\alpha\Big(\int_0^{\varepsilon}\phi(t)dt,0,0\Big),$$

so by ($\mathscr{A}_2$),

$$\int_0^{\varepsilon}\phi(t)dt=0.$$

This is a contradiction, hence $\{x_n\}$ is a Cauchy sequence. Let u be the limit of $\{x_n\}$ by the completeness of X. For any fixed $n\in\mathbb{N}$, take any $i\in\mathbb{N}$ with $i>n$. Since

$$d(f_nu,u,a)\leq d(f_nu,f_{i+1}x_i,a)+d(f_nu,u,x_{i+1})+d(x_{i+1},u,a),\ \forall\ a\in X$$

so by sub-additive property of ϕ and (2.1),

$$\begin{aligned}
&\int_0^{d(f_nu,u,a)}\phi(t)dt\\
\leq&\int_0^{d(f_nu,f_{i+1}x_i,a)+d(f_nu,u,x_{i+1})+d(x_{i+1},u,a)}\phi(t)dt\\
\leq&\int_0^{d(f_nu,f_{i+1}x_i,a)}\phi(t)dt+\int_0^{d(f_nu,u,x_{i+1})}\phi(t)dt+\int_0^{d(x_{i+1},u,a)}\phi(t)dt\\
\leq&\alpha\Big(\int_0^{d(u,x_i,a)}\phi(t)dt,\int_0^{d(u,f_nu,a)}\phi(t)dt,\int_0^{d(x_i,f_{i+1}x_i,a)}\phi(t)dt\Big)+\int_0^{d(f_nu,u,x_{i+1})}\phi(t)dt+\int_0^{d(x_{i+1},u,a)}\phi(t)dt.
\end{aligned}$$

Letting $i\to\infty$, then by Lemma 1.2, Lemma 1.4, ($\mathscr{A}_1$) and Cauchy property of $\{x_n\}$, we obtain

$$\int_0^{d(f_nu,u,a)}\phi(t)dt\leq\alpha\Big(0,\int_0^{d(f_nu,u,a)}\phi(t)dt,0\Big),$$

so by ($\mathscr{A}_2$),

$$\int_0^{d(f_nu,u,a)}\phi(t)dt=0,$$

which implies that

$$d(f_nu,u,a)=0,\forall a\in X.$$

Hence $f_n u = u$ for all $n \in \mathbb{N}$, that is, $\{f_i\}_{i\in\mathbb{N}}$ has a common fixed point u.

Suppose v is also a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$, then by (2.1), for any $a \in X$,

$$\int_0^{d(u,v,a)} = \int_0^{d(f_1u,f_2v,a)} \le \alpha\Big(\int_0^{d(u,v,a)}, \int_0^{d(u,f_1u,a)}, \int_0^{d(v,f_2v,a)}\Big) = \alpha\Big(\int_0^{d(u,v,a)}, 0, 0\Big).$$

By ($\mathscr{A}_2$) again, $\int_0^{d(u,v,a)} = 0$ for all $a \in X$, that is, $d(u,v,a) = 0$ for all $a \in X$, so $u = v$. This completes that u is the unique common fixed point of $\{f_i\}_{i\in\mathbb{N}}$.

Let $\alpha : \mathbb{R}^{+3} \to \mathbb{R}^+$ is defined by $\alpha(x,y,z) = k_1x + k_2y + k_3z$ for each $x,y,z \in \mathbb{R}^+$, where $k_1,k_2,k_3 \in \mathbb{R}^+$ satisfying $k_1 + k_2 + k_3 < 1$, then α is a $\mathscr{A}$-contractive function.

From Theorem 2.1, we obtain the next results:

Corollary 2.1 Let (X,d) be a complete 2-metric space and f a mapping on X. Suppose that for each $a,x,y \in X$,

$$\int_0^{d(fx,fy,a)} \phi(t)dt \le \alpha\Big(\int_0^{d(x,y,a)} \phi(t)dt, \int_0^{d(x,fx,a)} \phi(t)dt, \int_0^{d(y,fy,a)} \phi(t)dt\Big),$$

where $\phi \in \Phi$. If ϕ is sub-additive, then f has a unique fixed point.

Proof. Let $f_i = f$ for all $i \in \mathbb{N}$, then the conclusion follows from Theorem 2.1.

Corollary 2.2 Let (X,d) be a complete 2-metric space and $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X. Suppose that for each $i,j \in \mathbb{N}$ and $a,x,y \in X$,

$$\int_0^{d(f_ix,f_jy,a)} \phi(t)dt \le k_1\int_0^{d(x,y,a)} \phi(t)dt + k_2\int_0^{d(x,f_ix,a)} \phi(t)dt + k_3\int_0^{d(y,f_jy,a)} \phi(t)dt,$$

where $\phi \in \Phi$ and $k_1,k_2,k_3 \in \mathbb{R}^+$ satisfying $k_1 + k_2 + k_3 < 1$. If ϕ is sub-additive, then $\{f_i\}_{i\in\mathbb{N}}$ has a unique common fixed point.

In 2011, M Saha and Anamika Ganguly[24] obtained the next fixed point theorem:

Let (X,d) be a complete 2-metric space and let $T : X \to X$ a self-mapping such that for all $x,y,a \in X$,

$$\psi(d(Tx,Ty,a)) \le \psi(d(x,y,a)) - \varphi(d(x,y,a)),$$

where $\psi,\varphi : [0,\infty) \to [0,\infty)$ are continuous and strictly increasing functions with $\psi(t) = \varphi(t) = 0 \Longleftrightarrow t = 0$. Then T has a unique fixed point.

The above result is a generalization of the corresponding conclusion in [25] from a metric space to a 2-metric space.

Now, we give the integral version of M Saha and Anamika Ganguly' theorem for infinite mappings as follows

Theorem 2.2 Let (X,d) be a complete 2-metric space, $\{f_i\}_{i\in\mathbb{N}}$ a family of self mappings on X satisfying $f_i(X)\subset f_{i+1}(X), \forall\, n\in\mathbb{N}$. If for each $i,j,k\in\mathbb{N}$ with $i\neq j, i\neq k, j\neq k$ and $x,y,z,a\in X$,

$$\psi\Big(\int_0^{d(f_ix,f_jy,a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(f_jy,f_kz,a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(f_jy,f_kz,a)}\phi(t)dt\Big), \tag{2.9}$$

where $\phi\in\Phi$, ψ is a continuous and strictly increasing function with $\psi(t)=0\Leftrightarrow t=0$ and $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$. Then $\{f_i\}_{i\in\mathbb{N}}$ have a unique common fixed point.

Proof. Take any $x_0\in X$. By the condition $f_i(X)\subset f_{i+1}(X)$ for all $n=1,2,\cdots$, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$f_nx_{n-1}=f_{n+1}x_n=y_n,\ \forall\, n=1,2,3,\cdots.$$

Take $i=n+2, j=n+1, k=n, x=x_{n+1}, y=x_n, z=x_{n-1}$, then by (2.9), for any $a\in X$,

$$\psi\Big(\int_0^{d(f_{n+2}x_{n+1},f_{n+1}x_n,a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(f_{n+1}x_n,f_nx_{n-1},a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(f_{n+1}x_n,f_nx_{n-1},a)}\phi(t)dt\Big),$$

that is,

$$\psi\Big(\int_0^{d(y_{n+1},y_n,a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(y_n,y_{n-1},a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(y_n,y_{n-1},a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(y_n,y_{n-1},a)}\phi(t)dt\Big), \tag{2.10}$$

hence by the strictly increasing increasing property of ψ,

$$\int_0^{d(y_{n+1},y_n,a)}\phi(t)dt\leq\int_0^{d(y_n,y_{n-1},a)}\phi(t)dt,\ \forall\, a\in X, n=2,3,\cdots. \tag{2.11}$$

Now, we show that

$$d(y_{n+1},y_n,a)\leq d(y_n,y_{n-1},a),\ \forall\, a\in X, n=2,3,\cdots. \tag{2.12}$$

Suppose that (2.12) does not hold, that is, there exist $k\in\mathbb{N}$ and $b\in X$ such that

$$d(y_{k+1},y_k,b)>d(y_k,y_{k-1},b).$$

If $d(y_k,y_{k-1},b)=0$, then $d(y_{k+1},y_k,b)=0$ by (2.11) and the property of ϕ, hence

$$d(y_{k+1},y_k,b)=d(y_k,y_{k-1},b),$$

this is a contradiction, therefore $d(y_{k+1},y_k,b) > d(y_k,y_{k-1},b) > 0$. By the properties of ϕ and φ,

$$\varphi\Big(\int_0^{d(y_k,y_{k-1},b)}\phi(t)dt\Big) > 0.$$

So using (2.10), the property of ψ and the above result, we have

$$\begin{aligned}0 <&\psi\Big(\int_0^{d(y_{k+1},y_k,b)}\phi(t)dt\Big)\\ \leq&\psi\Big(\int_0^{d(y_k,y_{k-1},b)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(y_k,y_{k-1},b)}\phi(t)dt\Big)\\ <&\psi\Big(\int_0^{d(y_k,y_{k-1},b)}\phi(t)dt\Big).\end{aligned}$$

This is also a contradiction and hence (2.12) holds. Therefore, for any fixed $a\in X$, $\{d(y_n,y_{n-1},a)\}$ is a non-increasing and non-negative real sequence, hence there exists $r(a)\geq 0$ such that

$$\lim_{n\to\infty} d(y_n,y_{n-1},a) = r(a)$$

Let $n\to\infty$, then from (2.10) and by Lemma 1.4, we obtain

$$\psi\Big(\int_0^{r(a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{r(a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{r(a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{r(a)}\phi(t)dt\Big),$$

hence

$$\varphi\Big(\int_0^{r(a)}\phi(t)dt\Big)=0,$$

which implies that $r(a)=0$ by the property of φ. Therefore, we have the following fact:

$$\lim_{n\to\infty} d(y_n,y_{n-1},a) = 0, \forall a\in X. \tag{2.13}$$

Take $a=y_{n-1}$ in (2.10), then we obtain

$$\psi\Big(\int_0^{d(y_{n+1},y_n,y_{n-1})}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(y_n,y_{n-1},y_{n-1})}\phi(t)dt\Big)=\psi(0)=0, \forall n=1,2,\cdots,$$

hence by the property of ψ,

$$d(y_{n+2},y_{n+1},y_n)=0, \forall n=1,2,\cdots. \tag{2.14}$$

Fix any $\alpha\in\mathbb{N}$, then $d(y_\alpha,y_{\alpha+1},y_{\alpha+2})=0$ by (2.14). Suppose that $d(y_\alpha,y_n,y_{n+1})=0$, where $n>\alpha+1$. Take $i=n+3, j=n+2, k=n+1, x=x_{n+2}, y=x_{n+1}, z=x_n, a=y_\alpha$, then by (2.9) and the above assumption,

$$\begin{aligned}
&\psi\Big(\int_0^{d(y_{n+2},y_{n+1},y_\alpha)}\phi(t)dt\Big)\\
=&\psi\Big(\int_0^{d(f_{n+3}x_{n+2},f_{n+2}x_{n+1},y_\alpha)}\phi(t)dt\Big)\\
\le&\psi\Big(\int_0^{d(f_{n+2}x_{n+1},f_{n+1}x_n,y_\alpha)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(f_{n+2}x_{n+1},f_{n+1}x_n,y_\alpha)}\phi(t)dt\Big)\\
=&\psi\Big(\int_0^{d(y_{n+1},y_n,y_\alpha)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(y_{n+1},y_n,y_\alpha)}\phi(t)dt\Big)\\
=&\psi(0)-\varphi(0)=0.
\end{aligned}$$

Hence by the property of ψ,

$$d(y_{n+2},y_{n+1},y_\alpha)=0.$$

So using the mathematical induction, we obtain

$$d(y_\alpha,y_n,y_{n+1})=0,\ \forall n\ge\alpha\ge 1. \tag{2.15}$$

Therefor, for all $m,n,k\in\mathbb{N}$ with $k>n>m$, using (2.15), we obtain

$$\begin{aligned}
&d(y_m,y_n,y_k)\\
\le&d(y_m,y_n,y_{k-1})+d(y_m,y_{k-1},y_k)+d(y_n,y_{k-1},y_k)=d(y_m,y_n,y_{k-1})\\
\le&\cdots\le d(y_m,y_n,y_{n+1})=0.
\end{aligned}$$

So we have the following fact

$$d(y_m,y_n,y_k)=0,\forall m,n,k\in\mathbb{N}. \tag{2.16}$$

Suppose that $\{y_n\}$ is not a Cauchy sequence, then there exist $a_0\in X$ and $\varepsilon>0$ such that for any $i\in\mathbb{N}$ there exist $m(i),n(i)\in\mathbb{N}$ with $m(i),n(i)>i$ satisfying

(i) $m(i)>n(i)+1$ and $n(i)\to\infty$ as $i\to\infty$;

(ii) $d(y_{m(i)},y_{n(i)},a_0)>\varepsilon$, but $d(y_{m(i)-1},y_{n(i)},a_0)\le\varepsilon, i=1,2,\cdots$.

Using (2.13) and (2.16) and the method in proof of Theorem 2.1, we also obtain

$$\lim_{n\to\infty}d(y_{m(i)},y_{n(i)},a_0)=\lim_{n\to\infty}d(y_{m(i)-1},y_{n(i)},a_0)=\lim_{i\to\infty}d(y_{m(i)},y_{n(i)-1},a_0)=\lim_{i\to\infty}d(y_{m(i)-1},y_{n(i)-1},a_0)=\varepsilon. \tag{2.17}$$

Take $i=m(i)+1, j=n(i)+1, k=m(i), x=x_{m(i)}, y=x_{n(i)}, z=x_{m(i)-1}$, then by (2.9),

$$\begin{aligned}&\psi\Big(\int_0^{d(f_{m(i)+1}x_{m(i)},f_{n(i)+1}x_{n(i)},a_0)}\phi(t)dt\Big)\\ \leq&\psi\Big(\int_0^{d(f_{n(i)+1}x_{n(i)},f_{m(i)}x_{m(i)-1},a_0)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(f_{n(i)+1}x_{n(i)},f_{m(i)}x_{m(i)-1},a_0)}\phi(t)dt\Big),\end{aligned}$$

i.e.,

$$\psi\Big(\int_0^{d(y_{m(i)},y_{n(i)},a_0)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(y_{n(i)},y_{m(i)-1},a_0)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(y_{n(i)},y_{m(i)-1},a_0)}\phi(t)dt\Big).$$

Let $i\to\infty$, then by (2.17), Lemma 1.4 and the above formula,

$$\psi\Big(\int_0^{\varepsilon}\phi(t)dt\Big)\leq\psi\Big(\int_0^{\varepsilon}\phi(t)dt\Big)-\varphi\Big(\int_0^{\varepsilon}\phi(t)dt\Big),$$

hence

$$\varphi\Big(\int_0^{\varepsilon}\phi(t)dt\Big)=0,$$

so $\int_0^{\varepsilon}\phi(t)dt=0$. This is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence.

Since X is complete, there exists $u\in X$ such that $y_n\to u$ as $n\to\infty$. For any fixed $n\in\mathbb{N}$, take $l\in\mathbb{N}$ satisfying $l>n+1$. Let $i=n, j=l+1, k=l, x=u, y=x_l, z=x_{l-1}$, then by (2.9), for any $a\in X$,

$$\psi\Big(\int_0^{d(f_nu,f_{l+1}x_l,a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(f_{l+1}x_l,f_lx_{l-1},a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(f_{l+1}x_l,f_lx_{l-1},a)}\phi(t)dt\Big),$$

i.e.,

$$\psi\Big(\int_0^{d(f_nu,y_l,a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(y_l,y_{l-1},a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(y_l,y_{l-1},a)}\phi(t)dt\Big).$$

Let $l\to\infty$, then by (2.13), the above formula reduces to

$$\psi\Big(\int_0^{d(f_nu,u,a)}\phi(t)dt\Big)\leq\psi(0))-\varphi(0)=0,\forall\ a\in X.$$

Hence $d(f_nu,u,a)=0$ for all $a\in X$ by the property of ψ, threrfore $f_nu=u$ for all $n=1,2,\cdots$. This shows that u is a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$.

Suppose that v is also a common fixed point of $\{f_i\}_{i\in\mathbb{N}}$. Take $i=1, j=2, k=3, x=u, y=z=v$, then by (2.9), for each $a\in X$, we have

$$\psi\Big(\int_0^{d(f_1u,f_2v,a)}\phi(t)dt\Big)\leq\psi\Big(\int_0^{d(f_2v,f_3v,a)}\phi(t)dt\Big)-\varphi\Big(\int_0^{d(f_2v,f_3v,a)}\phi(t)dt\Big),$$

hence

$$\psi\Big(\int_0^{d(u,v,a)}\phi(t)dt\Big) < \psi(0)-\varphi(0)=0,$$

i.e.,

$$\psi\Big(\int_0^{d(u,v,a)}\phi(t)dt\Big) = 0,\ \forall\ a\in X,$$

therefore $u=v$ by the property of ψ. So u is the unique common fixed point of $\{f_i\}_{i\in\mathbb{N}}$.

Remark 2.1 In Theorem 2.2, we do not need the monotone condition of φ.

Corollary 2.3 Let (X,d) be a complete 2-metric space, f a self mapping on X. If for each $x,y,z,a\in X$,

$$\psi\Big(\int_0^{d(fx,fy,a))}\phi(t)dt\Big) \le \psi\Big(\int_0^{d(fy,fz,a)}\phi(t)dt\Big) - \varphi\Big(\int_0^{d(fy,fz,a)}\phi(t)dt\Big),$$

where $\phi\in\Phi$, ψ is is a continuous and strictly increasing function with $\psi(t)=0 \Leftrightarrow t=0$ and $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$. Then f has a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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