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EXTENSION OF OSTROWKI TYPE INEQUALITY VIA MOMENT GENERATING FUNCTION

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Abstract. In this paper, generalizations of weighted Ostrowski inequality are derived by using moment generating functions in bounded variation, L_∞ and L_p spaces. Applications to composite quadrature formulae are developed in which $\frac{1}{3}$ Simpson's, $\frac{3}{8}$ Simpson's, trapezoidal and midpoint inequalities are derived.

Keywords: moment generating function; Ostrowski inequality; numerical quadrature formulae

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1. Introduction

In 1938 Ostrowski developed an important inequality [11] which states that:

Theorem 1.1. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ be differentiable mapping on I^o the interior of interval I such that $\phi' \in L[u,v]$, where $u,v \in I$ with $u < v$. If $|\phi'(y)| \leq M$, then the following inequality holds

$$\left|\phi(y) - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq (v-u)\left[\frac{1}{4} + \frac{\left(y-\frac{u+v}{2}\right)^2}{(v-u)^2}\right]M \tag{1.1}$$

which holds $\forall\ y \in [u,v]$ and $\frac{1}{4}$ is the best possible constant in a sense that it cannot be replaced by a smaller constant.

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In [6], Dragomir, et. al. proved the generalization of the Ostrowski inequality for L_∞ space using some parameter λ.

Proposition 1.2. Let a function $\phi : [u,v] \to \mathbb{R}$ is continuous on $[u,v]$ and differentiable on (u,v), assume that its derivative is bounded on (u,v) and denote

$$\left\|\phi'\right\|_\infty := \sup_{t\in[u,v]} |\phi'(t)| < \infty.$$

Then

$$\begin{aligned}
&\left|(v-u)\left[\lambda\frac{\phi(u)+\phi(v)}{2}+(1-\lambda)\phi(y)\right]-\int_u^v \phi(t)dt\right| \\
&\leq \left[\frac{(v-u)^2}{4}\left(\lambda^2+(1-\lambda)^2\right)+\left(y-\frac{u+v}{2}\right)^2\right]\left\|\phi'\right\|_\infty
\end{aligned} \tag{1.2}$$

$\forall\ \lambda \in [0,1]$ and the Peano kernel defined as:

$$K(y,t)=\begin{cases} t-\left(u+\lambda\frac{v-u}{2}\right), & t\in[u,y], \\ \\ t-\left(v-\lambda\frac{v-u}{2}\right), & t\in(y,v]. \end{cases} \tag{1.3}$$

where $y \in \left[u+\lambda\frac{v-u}{2}, v-\lambda\frac{v-u}{2}\right]$.

We should also know the definition of bounded variation.

Definition 1.3. If a function $\phi : [a,b] \to \mathbb{R}$ and $[c,d]$ be any closed subinterval of $[a,b]$ and if the set

$$S=\left\{\sum_{i=1}^{n}|\phi(x_i)-\phi(x_{i-1})| : x_i : 1\leq i\leq n \text{is a partition of} [c,d]\right\} \tag{1.4}$$

is bounded then the variation of ϕ on $[c,d]$ is defined to be $\bigvee_c^d(\phi)=\sup S$. If S is unbounded then the variation of ϕ is said to be infinite. A function ϕ is of bounded variation on $[c,d]$ if $\bigvee_c^d(\phi)$.

Also according to [7, p.318]

Definition 1.4. Let $w : (u,v) \to [0,\infty)$ is integrable, i.e., $\int_u^v w(t)dt < \infty$. We denote the first two moments to be *m and M*, where

$$m(u,v)=\int_u^v w(t)dt, \qquad M(u,v)=\int_u^v tw(t)dt$$

In this paper our aim is to give generalization of Preposition 1.2 by using moment generating function and to study three different cases namely ϕ is bounded variation, $\phi' \in L_\infty[u,v]$ and $\phi' \in L_p[u,v]$. We use first two moments of weighted function.

2. If ϕ Is Function of Bounded Variation

Theorem 2.1. Let a function $\phi : [u,v] \to \mathbb{R}$ is bounded variation on $[u,v]$ and $y \in [\mu, \vartheta]$, then the following inequality holds

$$\begin{aligned} &\left| \left[\lambda \frac{\phi(u)+\phi(v)}{2} + (1-\lambda)\phi(y) \right] - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \\ &\leq \frac{1}{2(v-u)} \max \{\lambda(v-u), 2y - [(2-\lambda)u + \lambda v], [\lambda u + (2-\lambda)v] - 2y\} \bigvee_u^v (\phi) \end{aligned} \tag{2.1}$$

here $\bigvee_u^v(\phi)$ is the total variation of ϕ over $[u,v]$.

Proof. We now use the fact from [1], for a continous function $p : [c,d] \to \mathbb{R}$ and a function $f : [c,d] \to \mathbb{R}$ of bounded variation, the following inequality holds

$$\left| \int_c^d p(t)df(t) \right| \leq \sup_{t\in[c,d]} |p(t)| \bigvee_c^d (f).$$

Applying the above inequality for $p(t) = K(y,t)$ and $f(t) = \phi(t)$, we have

$$\left| \int_u^v K(y,t)d\phi(t) \right| \leq \left| \int_u^y K(y,t)d\phi(t) \right| + \left| \int_y^v K(y,t)d\phi(t) \right|$$

Now we us the fact to get

$$\begin{aligned} \sup_{t\in[u,v]} K(y,t) \bigvee_u^v (\phi) &\leq \sup_{t\in[u,y]} K(y,t) \bigvee_u^y (\phi) + \sup_{t\in(y,v]} K(y,t). \bigvee_y^v (\phi) \\ \left| \int_u^v K(y,t)d\phi(t) \right| &\leq \max \left\{ \lambda \frac{v-u}{2}, y - \frac{(2-\lambda)u+\lambda v}{2} \right\} \bigvee_u^y (\phi) \\ &\quad + \max \left\{ \lambda \frac{v-u}{2}, \frac{\lambda u + (2-\lambda)v}{2} - y \right\} \bigvee_y^v (\phi) \\ &\leq \frac{1}{2(v-u)} \max \{\lambda(v-u), 2y - [(2-\lambda)u + \lambda v] \\ &\quad , [\lambda u + (2-\lambda)v] - 2y\} \bigvee_u^v (\phi). \end{aligned}$$

To proof our next theorem we need the following lemma.

Lemma 2.2. Let a function $\phi : [u,v] \to \mathbb{R}$ is absolutely continuous and let weighted function $w : [u,v] \to [0,\infty)$ is integrable and $\int_u^v w(s)ds = m(u,v) < \infty$ then

$$\int_u^v P_w(y,t)\phi'(t)dt = m(\mu,\vartheta)\phi(y) + m(\vartheta,v)\phi(v) - m(\mu,u)\phi(u) - \int_u^v w(t)\phi(t)dt \tag{2.2}$$

where

$$P_w(y,t) = \begin{cases} \int_\mu^t w(s)ds = m(\mu,t), & t \in [u,y], \\ \\ \int_\vartheta^t w(s)ds = m(\vartheta,t), & t \in (y,v]. \end{cases} \tag{2.3}$$

and

$$\mu = u + \lambda\frac{v-u}{2} \qquad , \qquad \vartheta = v - \lambda\frac{v-u}{2}$$

for

$$u \le \mu \le y \le \vartheta \le v$$

Proof. Use Integration-by-parts on kernal (2.3), we get

$$\int_u^y m(\mu,t)d\phi(t) = m(\mu,y)\phi(y) - m(\mu,u)\phi(u) - \int_u^y \phi(t)d(t) \tag{2.4}$$

and

$$\int_y^v m(\vartheta,t)d\phi(t) = m(\vartheta,v)\phi(v) - m(\vartheta,v)\phi(y) - \int_y^v \phi(t)d(t) \tag{2.5}$$

By adding equations (2.4) and (2.5), we get (2.2).

Theorem 2.3. If $\phi : [u,v] \to \mathbb{R}$ is a function of bounded variation on $[u,v]$ and $y \in [u+\lambda\frac{v-u}{2}, v-\lambda\frac{v-u}{2}]$, then the following weighted inequality holds

$$\begin{aligned} &\left|\phi(y)m(\mu,\vartheta) + \phi(u)m(u,\mu) + \phi(v)m(\vartheta,v) - \int_u^v \phi(t)w(t)dt\right| \\ &\le \max\{m(u,\mu), m(\mu,y), m(y,\vartheta), m(\vartheta,v)\}\bigvee_u^v(\phi). \end{aligned} \tag{2.6}$$

Proof. By using the same fact we used in the previous theorem

$$\left|\int_c^d p(t)df(t)\right| \quad \le \quad \sup_{t\in[c,d]}|p(t)|\bigvee_c^d(f).$$

Applying the above inequality for $p(t) = P_w(y,t)$ and $f(t) = \phi(t)$, we have

$$\begin{aligned}
& \left| \int_u^v P_w(y,t) d\phi(t) \right| \\
\leq\ & \left| \int_u^y P_w(y,t) d\phi(t) \right| + \left| \int_y^v P_w(y,t) d\phi(t) \right| \\
& \text{By using the same fact} \\
\leq\ & \sup_{t\in[u,y]} P_w(y,t) \bigvee_u^y (\phi) + \sup_{t\in(y,v]} P_w(y,t) \bigvee_y^v (\phi) \\
\leq\ & \max\{m(u,\mu), m(\mu,y)\} \bigvee_u^y (\phi) + \max\{m(y,\vartheta), m(\vartheta,v)\} \bigvee_y^v (\phi) \\
\leq\ & \max\{m(u,\mu), m(\mu,y), m(y,\vartheta), m(\vartheta,v)\} \left[\bigvee_u^y (\phi) + \bigvee_y^v (\phi) \right] \\
\leq\ & \max\{m(u,\mu), m(\mu,y), m(y,\vartheta), m(\vartheta,v)\} \bigvee_u^v (\phi)
\end{aligned}$$

which implies

$$\left| \phi(y)m(\mu,\vartheta) + \phi(u)m(u,\mu) + \phi(v)(\vartheta,v) - \int_u^v \phi(t)w(t)dt \right| \\ \leq \max\{m(u,\mu), m(\mu,y), m(y,\vartheta), m(\vartheta,v)\} \bigvee_u^v (\phi).$$

Corollary 2.4. By replacing $w(s) = \frac{1}{v-u}$, and the values of μ and ϑ in (2.6), we will get the inequality (2.1).

Corollary 2.5. By replacing $\lambda = 0$ in (2.1), we will get the following inequality

$$\left| \phi(y)m(u,v) - \int_u^v w(t)\phi(t)dt \right| \leq \max\{m(u,y), m(y,v)\} \bigvee_u^v (\phi). \tag{2.7}$$

Remark 2.6. By replacing $w(s) = \frac{1}{v-u}$ in (2.7), we will get the following inequality

$$\left| \phi(y) - \frac{1}{v-u} \int_u^v \phi(t)dt \right| \leq \frac{1}{v-u} \left[\max\{(y-u), (v-y)\} \bigvee_u^v (\phi) \right]. \tag{2.8}$$

Corollary 2.7. By replacing $y = \frac{u+v}{2}$ in (2.7), we will get the following inequality

$$\begin{aligned} &\left| \left[\lambda \frac{\phi(u)+\phi(v)}{2} + (1-\lambda)\phi\left(\frac{u+v}{2}\right) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \\ &\leq \frac{1}{2}\left[\max\{\lambda, (1-\lambda)\} \bigvee_u^v(\phi) \right] \\ &= \frac{1}{2}\left[\left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right] \bigvee_u^v(\phi) \right] \end{aligned} \tag{2.9}$$

the inequality (2.9) is the result of Corollary 1 of [1].

Corollary 2.8. In (2.9)

1. by replacing $\lambda = 0$, we will get midpoint inequality

$$\left| \phi\left(\frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{2}\bigvee_u^v(\phi) \tag{2.10}$$

2. by replacing $\lambda = \frac{1}{4}$ we will get $\frac{3}{8}$ Simpson's inequaity

$$\left| \frac{3}{8}\left[\frac{\phi(u)+\phi(v)}{3} + 2\phi\left(\frac{u+v}{2}\right) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{3}{8}\bigvee_u^v(\phi) \tag{2.11}$$

3. by replacing $\lambda = \frac{1}{3}$, we will get $\frac{1}{3}$ Simpson's inequaity

$$\left| \frac{1}{6}\left[\phi(u) + 4\phi\left(\frac{u+v}{2}\right) + \phi(v) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{3}\bigvee_u^v(\phi) \tag{2.12}$$

4. by replacing $\lambda = \frac{1}{2}$, we will get perturbad trapezoidal inequality

$$\left| \frac{1}{4}\left[\phi(u) + 2\phi\left(\frac{u+v}{2}\right) + \phi(v) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{4}\bigvee_u^v(\phi) \tag{2.13}$$

5. by replacing $\lambda = 1$, we will get trapezoidal inequality

$$\left| \frac{\phi(u)+\phi(v)}{2} - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{2}\bigvee_u^v(\phi). \tag{2.14}$$

The inequalities (2.10), (2.12), (2.13) and (2.14) are the results of Corollary 2 of [1].

3. For the Case $\phi' \in L_\infty[u,v]$

Theorem 3.1. Let a function $\phi : [u,v] \to \mathbb{R}$ is absolutely continuous and ϕ' is bounded on $[u,v]$ i.e.,

$$\left\|\phi'\right\|_\infty = \sup_{t\in[u,v]} |\phi'(t)| < \infty.$$

Then the following inequality holds

$$\begin{aligned} &\left|\phi(y)m(\mu,\vartheta) + \phi(u)m(u,\mu) + \phi(v)m(\vartheta,v) - \int_u^v \phi(t)w(t)dt\right| \\ &\leq [um(\mu,u) + y(m(\mu,y) + m(\vartheta,y)) + vm(\vartheta,v) \\ &+ M(u,\mu) + M(y,\mu) + M(y,\vartheta) + M(v,\vartheta)] \left\|\phi'\right\|_\infty . \end{aligned} \tag{3.1}$$

Proof. By using the identity (2.2), kernal (2.3) and the fact

$$\int_c^d \left(\int_a^t w(s)ds\right) dt = \int_c^d m(a,t)dt = tm(a,t)|_c^d - M(c,d)$$

we have

$$\begin{aligned} &\left|\phi(y)m(\mu,\vartheta) + \phi(u)m(u,\mu) + \phi(v)m(\vartheta,v) - \int_u^v \phi(t)w(t)dt\right| \\ = &\left|\int_u^v P_w(y,t)\phi'(t)dt\right| \\ \leq &\int_u^v |P_w(y,t)|dt \left\|\phi'\right\|_\infty \\ \leq &\left[-\int_u^\mu m(\mu,t)dt + \int_\mu^y m(\mu,t)dt - \int_y^\vartheta m(\vartheta,t)dt + \int_\vartheta^v m(\vartheta,t)dt\right] \left\|\phi'\right\|_\infty \\ \leq &[um(\mu,u) + y(m(\mu,y) + m(\vartheta,y)) + vm(\vartheta,v) + M(u,\mu) + M(y,\mu) \\ &+ M(y,\vartheta) + M(v,\vartheta)] \left\|\phi'\right\|_\infty . \end{aligned}$$

Corollary 3.2. By replacing $w(s) = \frac{1}{v-u}$, μ and ϑ in (3.1), we will get the inequality of Proposition 1.2.

Corollary 3.3. By replacing $\lambda = 0$, we will get the following inequality

$$\left|\phi(y)m(u,v) - \int_u^v w(t)\phi(t)dt\right| \leq [ym(u,y) + ym(v,y) + M(y,u) + M(y,v)dt] \left\|\phi'\right\|_\infty . \tag{3.2}$$

Remark 3.4. By replacing $w(s) = \frac{1}{v-u}$ in (3.2), we will get the following inequality

$$\left|\phi(y) - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{1}{v-u}\left[\left(y-\frac{u+v}{2}\right)^2 + \frac{(v-u)^2}{4}\right]\left\|\phi'\right\|_\infty. \tag{3.3}$$

Corollary 3.5. By replacing $y = \frac{u+v}{2}$ in Corollary 3.2, we will get following inequality

$$\left|\left[\lambda\frac{\phi(u)+\phi(v)}{2} + (1-\lambda)\phi\left(\frac{u+v}{2}\right)\right] - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{(v-u)}{4}[\lambda^2+(\lambda-1)^2]\left\|\phi'\right\|_\infty \tag{3.4}$$

This inequality is a result of Corollary 4 in [1].

Corollary 3.6. In (3.4)

1. by replacing $\lambda = 0$, we get mid-point inequality

$$\left|\phi\left(\frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{1}{4}(v-u)\left\|\phi'\right\|_\infty \tag{3.5}$$

2. by replacing $\lambda = \frac{1}{4}$, we get $\frac{3}{8}$ Simpson's inequality

$$\left|\frac{3}{8}\left[\frac{\phi(u)+\phi(v)}{3} + 2\phi\left(\frac{u+v}{2}\right)\right] - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{5}{32}(v-u)\left\|\phi'\right\|_\infty \tag{3.6}$$

3. by replacing $\lambda = \frac{1}{3}$, we get $\frac{1}{3}$ Simpson's inequality

$$\left|\frac{1}{6}\left[\phi(u) + 4\phi\left(\frac{u+v}{2}\right) + \phi(v)\right] - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{5}{36}(v-u)\left\|\phi'\right\|_\infty \tag{3.7}$$

4. by replacing $\lambda = \frac{1}{2}$, we get perturbed trapezoidal inequality

$$\left|\frac{1}{4}\left[\phi(u) + 2\phi\left(\frac{u+v}{2}\right) + \phi(v)\right] - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{1}{8}(v-u)\left\|\phi'\right\|_\infty \tag{3.8}$$

5. by replacing $\lambda = 1$, we get trapezoidal inequality

$$\left|\frac{1}{2}[\phi(u)+\phi(v)] - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{1}{4}(v-u)\left\|\phi'\right\|_\infty. \tag{3.9}$$

The inequalities (3.5), (3.7), (3.8) and (3.9) are the results of Corollary 5 of [1].

4. For the Case $\phi' \in L_p[u,v]$

Theorem 4.1. Let $\phi : I \subset \mathbb{R} \to \mathbb{R}$ is absolutely continuous mapping on I^o, the interior of the interval I, where $u, v \in I$ with $u < v$. If $\phi' \in L_p[u,v]$, $p > 1$. Then the following inequality holds

$$\begin{aligned}
&\left|\left[\lambda\frac{v-u}{2}+(1-\lambda)\phi(y)\right]-\frac{1}{v-u}\int_u^v \phi(t)dt\right| \\
&\leq \frac{1}{v-u}\frac{1}{(q+1)^{\frac{1}{q}}}\left[2\left(\lambda\frac{v-u}{2}\right)^{q+1}+\left(y-\frac{(2-\lambda)u+\lambda v}{2}\right)^{q+1}\right. \\
&\left.+\left(\frac{\lambda u+(2-\lambda)v}{2}-y\right)^{q+1}\right]^{\frac{1}{q}}\|\phi'\|_p
\end{aligned} \tag{4.1}$$

$\forall\, \lambda \in [0,1]$ and $\frac{1}{p}+\frac{1}{q}=1$, $p>1$ and

$$u+\lambda\frac{v-u}{2} \leq y \leq v-\lambda\frac{v-u}{2}.$$

Proof. We have

$$\begin{aligned}
&\left|\left[\lambda\frac{v-u}{2}+(1-\lambda)\phi(y)\right]-\frac{1}{v-u}\int_u^v \phi(t)dt\right| \\
=\ &\left|\frac{1}{v-u}\int_u^v K(y,t)\phi'(t)dt\right| \\
&\text{Applying Hölder's Inequality} \\
\leq\ &\frac{1}{v-u}\left(\int_u^v |K(y,t)|^q dt\right)^{\frac{1}{q}}\left(\int_u^v |\phi'(t)|^p dt\right)^{\frac{1}{p}} \\
\leq\ &\frac{1}{v-u}\left[\int_u^y \left|t-\left(u+\lambda\frac{v-u}{2}\right)\right|^q dt+\int_y^v \left|t-\left(v-\lambda\frac{v-u}{2}\right)\right|^q dt\right]^{\frac{1}{q}}\|\phi'\|_p \\
\leq\ &\frac{1}{v-u}\frac{1}{(q+1)^{\frac{1}{q}}}\left[2\left(\lambda\frac{v-u}{2}\right)^{q+1}+\left(y-\frac{(2-\lambda)u+\lambda v}{2}\right)^{q+1}\right. \\
&\left.+\left(\frac{\lambda u+(2-\lambda)v}{2}-y\right)^{q+1}\right]^{\frac{1}{q}}\|\phi'\|_p .
\end{aligned}$$

Theorem 4.2. Let a function $\phi : [u,v] \to \mathbb{R}$ is absolutely continuous and ϕ' is bounded on $[u,v]$. If ϕ' belongs to $L_p[u,v], p > 1$, then the following inequality holds

$$\left|\phi(y)m(\mu,\vartheta) + \phi(u)m(a,\mu) + \phi(v)m(\vartheta,v) - \int_u^v \phi(t)w(t)dt\right|$$
$$\leq \frac{1}{(q+1)^{\frac{1}{q}}}\left[\int_u^\mu [m(t,\mu)]^q dt + \int_\mu^y [m(\mu,t)]^q dt + \int_y^\vartheta [m(t,\vartheta)]^q dt + \int_\vartheta^v [m(\vartheta,t)]^q dt\right]^{\frac{1}{q}} \left\|\phi'\right\|_p. \tag{4.2}$$

Proof. By using the identity (2.2) and kernel (2.3) and applying Hölder's Inequality

$$\begin{aligned}
&\left|\int_u^v P_w\phi'(t)dt\right| \\
\leq\ &\left[\int_u^v |P_w(y,t)|^q dt\right]^{\frac{1}{q}} \left\|\phi'\right\|_p \\
\leq\ &\left[\int_u^y |m(\mu,t)|^q dt + \int_y^v |m(\vartheta,t)|^q dt\right]^{\frac{1}{q}} \left\|\phi'\right\|_p \\
\leq\ &\left[\int_u^\mu [m(t,\mu)]^q dt + \int_\mu^y [m(\mu,t)]^q dt + \int_y^\vartheta [m(t,\vartheta)]^q dt\right. \\
&\left. + \int_\vartheta^v [m(\vartheta,t)]^q dt\right]^{\frac{1}{q}} \left\|\phi'\right\|_p
\end{aligned}$$

which completes the proof.

Corollary 4.3 By replacing $w(s) = \frac{1}{v-u}$ and the values of μ and ϑ in (4.1), we will get the following inequality

$$\left|\frac{\lambda}{2}[\phi(u) + \phi(v)] + (1-\lambda)\phi(y) - \frac{1}{v-u}\int_u^v \phi(t)dt\right|$$
$$\leq \frac{1}{(q+1)^{\frac{1}{q}}(v-u)^{\frac{q+1}{q}}}\left[\left(y - \frac{(2-\lambda)u + \lambda v}{2}\right)^{q+1} + 2\left(\lambda\frac{v-u}{2}\right)^{q+1} + \left(\frac{\lambda u + (2-\lambda)v}{2} - y\right)^{q+1}\right] \left\|\phi'\right\|_p. \tag{4.3}$$

Corollary 4.4 By replacing $\lambda = 0$ in (4.1), we will get the following inequality

$$\left|\phi(y)m(u,v) - \int_u^v w(t)\phi(t)dt\right| \leq \left[\int_u^y (m(u,t))^q + \int_y^v (m(t,v))^q\right]^{\frac{1}{q}} \left\|\phi'\right\|_p. \tag{4.4}$$

Remark 4.5 By replacing $w(s) = \frac{1}{v-u}$ in (4.4), we will get the following inequality

$$\left|\phi(y) - \frac{1}{v-u}\int_u^v \phi(t)dt\right| \leq \frac{1}{v-u}\frac{1}{(q+1)^{\frac{1}{q}}}\left[(y-u)^{q+1} + (v-y)^{q+1}\right]^{\frac{1}{q}} \left\|\phi'\right\|_p \tag{4.5}$$

Corollary 4.6 By replacing $y = \frac{u+v}{2}$ in (4.3), we will get the following inequality

$$\left| \left[\lambda \frac{\phi(u)+\phi(v)}{2} + (1-\lambda)\phi\left(\frac{u+v}{2}\right) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{2}\left(\frac{(1-\lambda)^{q+1}+\lambda^{q+1}}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\|\phi'\right\|_p \tag{4.6}$$

The inequality (4.6) is the result of Corollary 7 in [1].

Corollary 4.7 In (4.6)

1. by replacing $\lambda = 0$, we get mid-point inequality

$$\left| \phi\left(\frac{u+v}{2}\right) - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{2}\left(\frac{1}{q+1}\right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\|\phi'\right\|_p \tag{4.7}$$

2. by replacing $\lambda = \frac{1}{4}$, we get $\frac{3}{8}$ Simpson's inequality

$$\left| \frac{3}{8}\left[\frac{\phi(u)+\phi(v)}{3} + 2\phi\left(\frac{u+v}{2}\right) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{8}\left(\frac{3^{q+1}+1}{4(q+1)} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\|\phi'\right\|_p \tag{4.8}$$

3. by replacing $\lambda = \frac{1}{3}$, we get $\frac{1}{3}$ Simpson's inequality

$$\left| \frac{1}{6}\left[\phi(u) + 4\phi\left(\frac{u+v}{2}\right) + \phi(v) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{6}\left(\frac{2^{q+1}+1}{3(q+1)} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\|\phi'\right\|_p \tag{4.9}$$

4. by replacing $\lambda = \frac{1}{2}$, we get perturbed trapezoidal inequality

$$\left| \frac{1}{4}\left[\phi(u) + 2\phi\left(\frac{u+v}{2}\right) + \phi(v) \right] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{4}\left(\frac{1}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\|\phi'\right\|_p \tag{4.10}$$

5. by replacing $\lambda = 1$, we get trapezoidal inequality

$$\left| \frac{1}{2}[\phi(u)+\phi(v)] - \frac{1}{v-u}\int_u^v \phi(t)dt \right| \leq \frac{1}{2}\left(\frac{1}{q+1} \right)^{\frac{1}{q}} (v-u)^{\frac{1}{q}} \left\|\phi'\right\|_p . \tag{4.11}$$

The inequalities (4.7), (4.9), (4.10) and (4.11) are the results of Corollary 8 in [1].

5. Application to Quadrature Formula

If $I_n : u = y_0 < y_1 < y_2 < ... < y_n = v$ be a partition of the interval $[u,v]$ and let $h_i = y_{i+1} - y_i$ for $i \in \{0,1,2,...,n-1\}$

Consider a general Quadrature Formula

$$Q_n(I_n,\phi) := \sum_{i=0}^{n-1} [\phi(\xi_i)m(\mu_i,\vartheta_i) + \phi(y_i)m(y_i,\mu_i)ds + \phi(y_{i+1})m(\vartheta_i,y_{i+1})] \tag{5.1}$$

$\forall\ \lambda \in [0,1]$ and

$$\mu_i = y_i + \lambda \frac{h_i}{2} \leq \xi_i \leq y_{i+1} - \lambda \frac{h_i}{2} = \vartheta_i$$

and

$$R_n(I_n, \phi) = \int_u^v \phi(t)w(t)dt - Q_n(I_n, \phi)$$

which yields following theorems.

Theorem 5.1. Let ϕ be as defined in Theorem 3.1 and we have

$$\int_u^v \phi(t)w(t)dt = R_n(I_n, \phi) + Q_n(I_n, \phi)$$

where $Q_n(I_n, \phi)$ is given in (5.1), then the remainder satisfies the estimation

$$|R_n(I_n, \phi)| \leq \sum_{i=0}^{n} \max\{m(y_i, \mu_i), m(\mu_i, \xi_i), m(\xi_i, \vartheta_i), m(\vartheta_i, y_{i+1})\} \bigvee_{y_i}^{y_{i+1}} (\phi). \tag{5.2}$$

Proof. Applying inequality (2.1) on the interval $[y_i, y_{i+1}]$, we get

$$R_i(I_i, \phi) = \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t)w(t)dt - \sum_{i=0}^{n} [\phi(\xi_i)m(\mu_i, \vartheta_i) + \phi(y_i)m(y_i, \mu_i)ds + \phi(y_{i+1})m(\vartheta_i, y_{i+1})].$$

Sum the equalities presented above over i from 0 to n, we get

$$\begin{aligned} R_n(I_n, \phi) &= \sum_{i=0}^{n} \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t)w(t)dt \\ &\quad - \sum_{i=0}^{n} [\phi(\xi_i)m(\mu_i, \vartheta_i) + \phi(x_i)m(y_i, \mu_i)ds + \phi(y_{i+1})m(\vartheta_i, y_{i+1})] \end{aligned}$$

which implies

$$\begin{aligned} |R_n(I_n, \phi)| &= \left| \sum_{i=0}^{n} \frac{1}{h_i} \int_{y_i}^{y_{i+1}} \phi(t)w(t)dt \right. \\ &\quad \left. - \sum_{i=0}^{n} [\phi(\xi_i)m(\mu_i, \vartheta_i) + \phi(y_i)m(y_i, \mu_i)ds + \phi(y_{i+1})m(\vartheta_i, y_{i+1})] \right| \\ &\leq \sum_{i=0}^{n} \max\{m(y_i, \mu_i), m(\mu_i, \xi_i), m(\xi_i, \vartheta_i), m(\vartheta_i, y_{i+1})\} \bigvee_{y_i}^{y_{i+1}} (\phi). \end{aligned}$$

Theorem 5.2. Let ϕ be as defined in Theorem 4.1 and we have

$$\int_u^v \phi(t)w(t)dt = R_n(I_n, \phi) + Q_n(I_n, \phi)$$

where $Q_n(I_n,\phi)$ is given in (5.1), then the remainder satisfies the estimation

$$
\begin{aligned}
|R_n(I_n,\phi)| &\le [y_i m(\mu_i,y_i)+\xi_i\left(m(\mu_i,\xi_i)+m(\vartheta_i,\xi_i)\right)\\
&+y_{i+1}m(\vartheta_i,y_{i+1})+M(y_i,\mu_i)+M(\xi_i,\mu_i)\ +M(\xi_i,\vartheta_i)+M(y_{i+1},\vartheta_i)]\left\|\phi'\right\|_\infty.
\end{aligned}
\tag{5.3}
$$

Proof. By using the similar technique use in Theorem 5.1, we get

$$
\begin{aligned}
|R_n(I_n,\phi)| &= \Bigg|\sum_{i=0}^{n}\frac{1}{h_i}\int_{y_i}^{y_{i+1}}\phi(t)w(t)dt\\
&\quad -\sum_{i=0}^{n}[\phi(\xi_i)m(\mu_i,\vartheta_i)+\phi(y_i)m(y_i,\mu_i)ds+\phi(y_{i+1})m(\vartheta_i,y_{i+1})]\Bigg|\\
&\le [y_i m(\mu_i,y_i)+\xi_i\left(m(\mu_i,\xi_i)+m(\vartheta_i,\xi_i)\right)+y_{i+1}m(\vartheta_i,y_{i+1})\\
&\quad +M(y_i,\mu_i)+M(\xi_i,\mu_i)+M(\xi_i,\vartheta_i)+M(y_{i+1},\vartheta_i)]\left\|\phi'\right\|_\infty.
\end{aligned}
$$

Theorem 5.3. Let ϕ be as defined in Theorem 5.1 and we have

$$\int_u^v\phi(t)w(t)dt=R_n(I_n,\phi)+Q_n(I_n,\phi)$$

where $Q_n(I_n,\phi)$ is given in (5.1), then the remainder satisfies the estimation

$$
\begin{aligned}
|R_n(I_n,\phi)| &\le \frac{1}{(q+1)^{\frac{1}{q}}}\left[\int_{y_i}^{\mu_i}(m(y_i,\mu_i))^q\,dt+\int_{\mu_i}^{\xi_i}(m(\mu_i,\xi_i))^q\right.\\
&\left.+\int_{\xi_i}^{\vartheta_i}(m(\xi_i,\vartheta_i))^q\,dt+\int_{\vartheta_i}^{y_{i+1}}(m(\vartheta_i,y_{i+1}))^q\,dt\right]^{\frac{1}{q}}\left\|\phi'\right\|_p.
\end{aligned}
\tag{5.4}
$$

Proof By using the similar technique use in Theorem 5.1, we get

$$
\begin{aligned}
|R_n(I_n,\phi)| &= \Bigg|\sum_{i=0}^{n}\frac{1}{h_i}\int_{y_i}^{y_{i+1}}\phi(t)w(t)dt\\
&\quad -\sum_{i=0}^{n}[\phi(\xi_i)m(\mu_i,\vartheta_i)+\phi(y_i)m(y_i,\mu_i)ds+\phi(y_{i+1})m(\vartheta_i,y_{i+1})]\Bigg|\\
&\le \frac{1}{(q+1)^{\frac{1}{q}}}\left[\int_{y_i}^{\mu_i}(m(y_i,\mu_i))^q\,dt+\int_{\mu_i}^{\xi_i}(m(\mu_i,\xi_i))^q\right.\\
&\quad\left.+\int_{\xi_i}^{\vartheta_i}(m(\xi_i,\vartheta_i))^q\,dt+\int_{\vartheta_i}^{y_{i+1}}(m(\vartheta_i,y_{i+1}))^q\,dt\right]^{\frac{1}{q}}\left\|\phi'\right\|_p.
\end{aligned}
$$

6. Conclusion

We have derived three different versions of Ostrowski type inequality, namely for bounded variation, L_p and L_∞ space involving weights in terms of moment generating functions and by using them we also discussed their few applications in numerical integration.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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