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OSTROWSKI AND OSTROWSKI–GRÜSS TYPE INEQUALITIES FOR VECTOR-VALUED FUNCTIONS WITH k POINTS VIA A PARAMETER

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Abstract. In this paper, we provide an extension of the Ostrowski's inequality for vector-valued functions for k points via a parameter. We also provide a sharp Ostrowski-Grüss type inequality for vector valued functions for k points. Our results generalize some of the results for the real-valued case in the literature.

Keywords: Montgomery's identity; Ostrowski's inequality; Grüss inequality; Bochner integral; Banach spaces.
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1. Introduction

Throughout this paper, $(X, \|\cdot\|)$ denotes a Banach space over the real numbers.

Definition 1.1. X is said to be a Radon-Nikodym space if every absolutely continuous X-valued function is almost everywhere differentiable.

It is well-known that every reflexive space is a Radon-Nikodym space. For example every Hilbert space is a Radon-Nikodym space. However, the space $L_1([0,1])$ of all integrable functions defined on the interval $[0,1]$, endowed with the norm

$$\|g\|_1 := \int_0^1 |g(t)|dt$$

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is not a Radon-Nikodym space.

Definition 1.2. Let $a,b \in \mathbb{R}, a<b$ and $f:[a,b] \to X$ be an X-valued function. f is said to be Bochner integrable if and only if the real-valued function $\|f\| : [a,b] \to \mathbb{R}$ defined by $\|f\|(t) := \|f(t)\|$, is Lebesgue integrable on $[a,b]$.

We will denote the Lebesgue integral of a real-valued function g by $\int_a^b g(t)dt$ and denote the Bochner integral of an X-valued function f by $(B)\int_a^b f(t)dt$.

The integration by parts formula holds for the Bochner integral under the following conditions.

Theorem 1.1. Let $a,b \in \mathbb{R}, a<b, g:[a,b] \to \mathbb{R}$ and $f:[a,b] \to X$. Suppose f and g are both differentiable, and the functions gf' and $g'f$ are Bochner integrable on $[a,b]$, then

$$(B)\int_a^b g(t)f'(t)dt = g(b)f(b) - g(a)f(b) - (B)\int_a^b g'(t)f(t)dt$$

Theorem 1.2. If $f:[a,b] \to X$ is Bochner integrable then

$$\left\|(B)\int_a^b f(t)dt\right\| \leq \int_a^b \|f(t)\|dt.$$

For more information on the Bochner integral we refer the reader to [4] and [8].

In 1938, the Ukranian mathematician Alexander Ostrowski [11] obtained an inequality known in the literature as Ostrowski's inequality to provide a bound for the difference between a real-valued function and its integral mean. This result is stated in the following theorem;

Theorem 1.3. Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f':(a,b) \to \mathbb{R}$ is bounded in (a,b). If $M := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)M \tag{1.1}$$

for all $x \in [a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

This inequality has been studied and extended in several different ways by many authors over the past years. For more information about the Ostrowski inequality and its associates, we refer the reader to [1, 2, 5, 6, 7, 9, 10, 12]. In 2002, Barnett et al [2] extended Theorem 1.3 for vector-valued functions in the following theorem.

Theorem 1.4. Let $(X, \|\cdot\|)$ be a Banach space with the Radon-Nikodym property and $f : [a,b] \to X$ an absolutely continuous function on $[a,b]$ with the property that $f' \in L_\infty([a,b],X)$, i.e $f' : [a,b] \to X$ and

$$|||f'|||_{[a,b],\infty} := ess \sup_{t\in[a,b]} \|f'(t)\| < \infty$$

Then we have the inequality

$$\left\| f(s) - \frac{1}{b-a}(B)\int_a^b f(t)dt \right\| \leq \left(\frac{1}{4} + \frac{\left(s - \frac{a+b}{2}\right)^2}{(b-a)^2} \right)(b-a)|||f'|||_{[a,b],\infty} \tag{1.2}$$

for all $s \in [a,b]$.

The purpose of this paper is to extend Theorem 1.4 for k points via a parameter which will be done in Section 2. In Section 3, we provide a sharp Ostrowski-Grüss type inequality for vector-valued functions by using a sharp Grüss inequality obtained by Barnett et al in [3]. Our results extends some of the results in the literature for real-valued functions to the case of vector-valued functions. In addition, we consider some particular cases as examples.

2. Ostrowski inequality for vector-valued functions for k points

To prove our main results, we obtained the following generalized Montgomery identity for vector-valued functions. In what follows, we assume that $(X, \|\cdot\|)$ has the Radon-Nikodym property. We first define the following kernel function which was first provided by Xu and Fang [12];

Definition 2.1. Let

(1) $a, b \in \mathbb{R}$, $\lambda \in [0,1]$, $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a partition of the interval $[a,b]$,

(2) $\alpha_i \in \mathbb{R}$ $(i = 0,1,\cdots,k+1)$ is $k+2$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1,\cdots,k)$ and $\alpha_{k+1} = b$,

we define the kernel function $K(,I_k):[a,b]\to\mathbb{R}$ as follows;

$$K(t,I_k)=\begin{cases} t-\Big(\alpha_1-\lambda\,\frac{\alpha_1-a}{2}\Big),\ t\in[a,\alpha_1),\\ t-\Big(\alpha_1+\lambda\,\frac{\alpha_2-\alpha_1}{2}\Big),\ t\in[\alpha_1,x_1),\\ t-\Big(\alpha_2-\lambda\,\frac{\alpha_2-\alpha_1}{2}\Big),\ t\in[x_1,\alpha_2),\\ \vdots\\ t-\Big(\alpha_{k-1}+\lambda\,\frac{\alpha_k-\alpha_{k-1}}{2}\Big),\ t\in[\alpha_{k-1},x_{k-1}),\\ t-\Big(\alpha_k-\lambda\,\frac{\alpha_k-\alpha_{k-1}}{2}\Big),\ t\in[x_{k-1},\alpha_k),\\ t-\Big(\alpha_k+\lambda\,\frac{\alpha_{k+1}-\alpha_k}{2}\Big),\ t\in[\alpha_k,b], \end{cases} \tag{2.1}$$

for all $t\in[a,b]$.

Lemma 2.1. [Montgomery Identity for vector-valued functions with k points.] Suppose that $f:[a,b]\to X$ is an absolutely continuous function on $[a,b]$. Then for any $\lambda\in[0,1]$, we have the identity

$$\begin{aligned}(B)\int_a^b K(t,I_k)f'(t)dt &= (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\\ &\quad+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt\end{aligned} \tag{2.2}$$

where $K(\cdot,I_k)$ is given in Definition 2.1.

Proof. First, we observe that

$$\begin{aligned}(B)\int_a^b K(t,I_k)f'(t)dt=\sum_{i=0}^{k-1}\Big[&(B)\int_{x_i}^{\alpha_{i+1}}(t-(\alpha_{i+1}-\lambda\,\frac{\alpha_{i+1}-\alpha_i}{2}))f'(t)dt\\ &+(B)\int_{\alpha_{i+1}}^{x_{i+1}}(t-(\alpha_{i+1}+\lambda\,\frac{\alpha_{i+2}-\alpha_{i+1}}{2}))f'(t)dt\Big].\end{aligned}$$

By applying Theorem 1.1, we have

$$
\begin{aligned}
(B)\int_a^b K(t,I_k)f'(t)dt = \sum_{i=0}^{k-1}\Bigg[&(\alpha_{i+1}-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}))f(\alpha_{i+1})\\
&-(x_i-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}))f(x_i)-(B)\int_{x_i}^{\alpha_{i+1}} f(t)dt\\
&+(x_{i+1}-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}))f(x_{i+1})\\
&-(\alpha_{i+1}-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}))f(\alpha_{i+1})-(B)\int_{\alpha_{i+1}}^{x_{i+1}} f(t)dt\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{i=0}^{k-1}\Bigg[\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(\alpha_{i+1})-(x_i-\alpha_{i+1})f(x_i)-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i)+(x_{i+1}-\alpha_{i+1})f(x_{i+1})\\
&-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(\alpha_{i+1})-(B)\int_{x_i}^{x_{i+1}} f(t)dt\Bigg].
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
(B)\int_a^b K(t,I_k)f'(t)dt &= \sum_{i=0}^{k-1}\left[\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-(B)\int_{x_i}^{x_{i+1}} f(t)dt\right]\\
&+\sum_{i=0}^{k-1}\left[-(x_i-\alpha_{i+1})f(x_i)+(x_{i+1}-\alpha_{i+1})f(x_{i+1})\right]\\
&+\sum_{i=0}^{k-1}\left[-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}f(x_i)-\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}f(x_{i+1})\right]\\
&=\sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-(B)\int_a^b f(t)dt\\
&-x_0f(x_0)+x_kf(x_k)+\sum_{i=0}^{k-1}\alpha_{i+1}(f(x_i)-f(x_{i+1}))\\
&+\sum_{i=0}^{k-1}-\frac{\lambda}{2}\left[(\alpha_{i+1}-\alpha_i)f(x_i)+(\alpha_{i+2}-\alpha_{i+1})f(x_{i+1})\right].
\end{aligned}
$$

That is,

$$
\begin{aligned}
(B)\int_a^b K(t,I_k)f'(t)dt &= \sum_{i=0}^{k-1}\lambda\frac{\alpha_{i+2}-\alpha_i}{2}f(\alpha_{i+1})-(B)\int_a^b f(t)dt\\
&+(\alpha_1-a)f(a)+(b-\alpha_k)f(b)+\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i)\\
&-\frac{\lambda}{2}\left[(\alpha_1-a)f(a)+(b-\alpha_k)f(b)+2\sum_{i=1}^{k-1}(\alpha_{i+1}-\alpha_i)f(x_i)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} \lambda \frac{\alpha_{i+2}-\alpha_i}{2} f(\alpha_{i+1}) - (B)\int_a^b f(t)dt \\
&+ (1-\lambda) \sum_{i=1}^{k-1} (\alpha_{i+1}-\alpha_i) f(x_i) + (1-\frac{\lambda}{2})\Big[(\alpha_1 - a)f(a) + (b-\alpha_k)f(b)\Big].
\end{aligned}
\tag{2.3}
$$

Now, consider the following

$$
\begin{aligned}
&\sum_{i=0}^{k-1}(\alpha_{i+2}-\alpha_i)f(\alpha_{i+1}) = \sum_{i=0}^{k-1}(\alpha_{i+2}-\alpha_{i+1})f(\alpha_{i+1}) + \sum_{i=0}^{k-1}(\alpha_{i+1}-\alpha_i)f(\alpha_{i+1}) \\
&= \sum_{i=1}^{k}(\alpha_{i+1}-\alpha_i)f(\alpha_i) + \sum_{i=0}^{k-1}(\alpha_{i+1}-\alpha_i)f(\alpha_{i+1})
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(\alpha_i) - (\alpha_1-\alpha_0)f(\alpha_0) + \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(\alpha_{i+1}) - (\alpha_{k+1}-\alpha_k)f(\alpha_{k+1}) \\
&= \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)(f(\alpha_i)+f(\alpha_{i+1})) - \Big[(\alpha_1-\alpha_0)f(\alpha_0) + (\alpha_{k+1}-\alpha_k)f(\alpha_{k+1})\Big].
\end{aligned}
$$

So,

$$
\begin{aligned}
\sum_{i=0}^{k-1}\frac{\lambda}{2}(\alpha_{i+2}-\alpha_i)f(\alpha_{i+1}) &= \frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big) \\
&\quad - \frac{\lambda}{2}\Big[(\alpha_1-a)f(a) + (b-\alpha_k)f(b)\Big].
\end{aligned}
\tag{2.4}
$$

Substituting (2.4) in (2.3) gives the identity

$$
\begin{aligned}
(B)\int_a^b K(t,I_k)f'(t)dt &= (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) \\
&\quad + \frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big) - (B)\int_a^b f(t)dt.
\end{aligned}
$$

This completes the proof.

Corollary 2.1. If we take $\lambda = 0$ in Lemma 2.1, we have the identity

$$
(B)\int_a^b K(t,I_k)f'(t)dt = \sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i) - (B)\int_a^b f(t)dt \tag{2.5}
$$

where

$$K(t,I_k)=\begin{cases} t-\alpha_1,\ t\in[a,x_1),\\ t-\alpha_2,\ t\in[x_1,x_2),\\ \vdots\\ t-\alpha_{k-1},\ t\in[x_{k-2},x_{k-1}),\\ t-\alpha_k,\ t\in[x_{k-1},b]. \end{cases} \tag{2.6}$$

Corollary 2.2. Let $a,b\in\mathbb{R},a<b$ and let $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$. Then for any $x\in[a,b]$ we have the identity

$$\frac{1}{b-a}(B)\int_a^b K(t,x)f'(t)dt=(1-\lambda)f(x)+\frac{\lambda}{2}\Big(f(a)+f(b)\Big)-\frac{1}{b-a}(B)\int_a^b f(t)dt \tag{2.7}$$

where

$$K(t,x)=\begin{cases} t-\Big(a+\lambda\frac{b-a}{2}\Big),\ t\in[a,x),\\ t-\Big(b-\lambda\frac{b-a}{2}\Big),\ t\in[x,b]. \end{cases} \tag{2.8}$$

Proof. The proof follows directly from Lemma 2.1 by taking $k=2,x_0=\alpha_0=\alpha_1=a,x_1=x$ and $x_2=\alpha_2=\alpha_3=b.$

Theorem 2.1. Suppose f satisfies the conditions of Lemma 2.1, and $f'\in L_\infty([a,b],X)$. Then we have the inequality

$$\begin{aligned} &\|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt\| \\ &\qquad\le M\Bigg\{\sum_{i=0}^{k-1}\Big[\frac{1}{4}\Big(x_{i+1}-x_i\Big)^2+\Big(\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\Big)^2\Big] \\ &\qquad+\frac{\lambda^2}{4}\Big[\Big((b-\alpha_k)^2-(\alpha_1-a)^2\Big)+2\sum_{i=0}^{k-1}\Big(\alpha_{i+1}-\alpha_i\Big)^2\Big] \\ &\qquad+\frac{\lambda}{2}\sum_{i=0}^{k-1}\Big[(\alpha_{i+1}-x_{i+1})(\alpha_{i+2}-\alpha_{i+1})-(\alpha_{i+1}-x_i)(\alpha_{i+1}-\alpha_i)\Big]\Bigg\} \end{aligned} \tag{2.9}$$

where

$$M=ess\sup_{t\in[a,b]}\|f'(t)\|<\infty.$$

Proof. We start with the following computation,

$$
\begin{aligned}
\int_a^b |K(t,I_k)|dt &= \sum_{i=0}^{k-1}\Big[\int_{x_i}^{\alpha_{i+1}}\Big|t-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2})\Big|dt+\int_{\alpha_{i+1}}^{x_{i+1}}\Big|t-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2})\Big|dt\Big]\\
&= \sum_{i=0}^{k-1}\Big[\int_{x_i}^{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}\Big((\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2})-t\Big)dt+\int_{\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2}}^{\alpha_{i+1}}\Big(t-(\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2})\Big)dt\\
&+\int_{\alpha_{i+1}}^{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}\Big((\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2})-t\Big)dt+\int_{\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2}}^{x_{i+1}}\Big(t-(\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2})\Big)dt\Big]\\
&= \frac{1}{2}\sum_{i=0}^{k-1}\Big[\Big((\alpha_{i+1}-\lambda\frac{\alpha_{i+1}-\alpha_i}{2})-x_i\Big)^2+\frac{\lambda^2}{4}\Big(\alpha_{i+1}-\alpha_i\Big)^2\\
&+\Big((\alpha_{i+1}+\lambda\frac{\alpha_{i+2}-\alpha_{i+1}}{2})-x_{i+1}\Big)^2+\frac{\lambda^2}{4}\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\\
&= \frac{1}{2}\sum_{i=0}^{k-1}\Big[\Big((\alpha_{i+1}-x_i\Big)^2-\lambda(\alpha_{i+1}-x_i)(\alpha_{i+1}-\alpha_i)+\frac{\lambda^2}{2}\Big(\alpha_{i+1}-\alpha_i\Big)^2\\
&+\Big((\alpha_{i+1}-x_{i+1}\Big)^2+\lambda(\alpha_{i+1}-x_{i+1})(\alpha_{i+2}-\alpha_{i+1})+\frac{\lambda^2}{2}\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\\
&= \frac{1}{2}\sum_{i=0}^{k-1}\Big[\Big(\alpha_{i+1}-x_i\Big)^2+\Big(\alpha_{i+1}-x_{i+1}\Big)^2+\frac{\lambda^2}{2}\Big(\alpha_{i+1}-\alpha_i\Big)^2+\frac{\lambda^2}{2}\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\\
&-\lambda(\alpha_{i+1}-x_i)(\alpha_{i+1}-\alpha_i)+\lambda(\alpha_{i+1}-x_{i+1})(\alpha_{i+2}-\alpha_{i+1})\Big].
\end{aligned}
$$

That is,

$$
\begin{aligned}
\int_a^b |K(t,I_k)|dt = &\frac{1}{2}\sum_{i=0}^{k-1}\Big[\Big(\alpha_{i+1}-x_i\Big)^2+\Big(\alpha_{i+1}-x_{i+1}\Big)^2\Big]\\
&+\frac{\lambda^2}{4}\sum_{i=0}^{k-1}\Big[\Big(\alpha_{i+1}-\alpha_i\Big)^2+\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\\
&+\frac{\lambda}{2}\sum_{i=0}^{k-1}\Big[(\alpha_{i+1}-x_{i+1})(\alpha_{i+2}-\alpha_{i+1})-(\alpha_{i+1}-x_i)(\alpha_{i+1}-\alpha_i)\Big].
\end{aligned}
$$

By applying the parallelogram law for numbers on the terms in the first sum and rearranging the terms in the second sum, we have

$$
\begin{aligned}
\int_a^b |K(t,I_k)|dt = &\sum_{i=0}^{k-1}\Big[\frac{1}{4}\Big(x_{i+1}-x_i\Big)^2+\Big(\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\Big)^2\Big]\\
&+\frac{\lambda^2}{4}\Big[\Big((b-\alpha_k)^2-(\alpha_1-a)^2\Big)+2\sum_{i=0}^{k-1}\Big(\alpha_{i+1}-\alpha_i\Big)^2\Big] \qquad (2.10)\\
&+\frac{\lambda}{2}\sum_{i=0}^{k-1}\Big[(\alpha_{i+1}-x_{i+1})(\alpha_{i+2}-\alpha_{i+1})-(\alpha_{i+1}-x_i)(\alpha_{i+1}-\alpha_i)\Big].
\end{aligned}
$$

Now from Lemma 2.1 we have

$$\begin{aligned}(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt \\ =(B)\int_a^b K(t,I_k)f'(t)dt.\end{aligned} \tag{2.11}$$

By taking the norm on both sides of (2.11) and applying Theorem 1.2, we obtain

$$\begin{aligned}\|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt\| \\ \leq M\int_a^b |K(t,I_k)|dt\end{aligned} \tag{2.12}$$

where

$$M = ess\sup_{t\in[a,b]}\|f'(t)\| < \infty$$

The desired inequality follows by substituting (2.10) in (2.12).

Corollary 2.3. If we take $\lambda = 0$ in Theorem 2.1, then we have the inequality

$$\begin{aligned}\left\|\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)-(B)\int_a^b f(t)dt\right\| \\ \leq M\sum_{i=0}^{k-1}\left[\frac{1}{4}\Big(x_{i+1}-x_i\Big)^2+\Big(\alpha_{i+1}-\frac{x_{i+1}+x_i}{2}\Big)^2\right]\end{aligned} \tag{2.13}$$

where

$$M = ess\sup_{t\in[a,b]}\|f'(t)\| < \infty.$$

Remark 2.1. If $k = 1$ in Corollary 2.3, then we obtain Theorem 1.4.

Corollary 2.4. Let $a,b\in\mathbb{R}, a<b$ and let $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ such that $f'\in L_\infty([a,b],X)$. Then for any $x\in[a,b]$ we have the inequality

$$\begin{aligned}\left\|(1-\lambda)f(x)+\frac{\lambda}{2}\Big(f(a)+f(b)\Big)-\frac{1}{b-a}(B)\int_a^b f(t)dt\right\| \\ \leq M\left\{\frac{1}{4}(b-a)((1-\lambda)^2+\lambda^2)+\Big(x-\frac{a+b}{2}\Big)^2\right\}\end{aligned} \tag{2.14}$$

where

$$M = ess\sup_{t\in[a,b]}\|f'(t)\| < \infty.$$

Proof. The proof follows directly from Theorem 2.1 by taking $k = 2, x_0 = \alpha_0 = \alpha_1 = a, x_1 = x$ and $x_2 = \alpha_2 = \alpha_3 = b$.

3. Ostrowski-Grüss type inequality for vector-valued functions for k points

The following lemma is a particular case of Theorem 2 in [3].

Lemma 3.1. Let $a,b \in \mathbb{R}$ with $a<b$. If $g:[a,b]\to X$ is Bochner integrable on $[a,b]$ and there exist $v\in X$ and $r>0$ such that $g(x)\in \overline{B}(v,r):=\{y\in X: \|y-v\|\le r\}$ for a.e $x\in[a,b]$ and $\alpha:[a,b]\to\mathbb{R}$ a Lebesgue integrable function with αg Bochner integrable on $[a,b]$, then we have the sharp inequality

$$\begin{aligned}&\left\|(B)\int_a^b \alpha(t)g(t)dt-\frac{1}{b-a}\int_a^b\alpha(t)dt\cdot(B)\int_a^b g(t)dt\right\|\\&\qquad\le r\int_a^b\left|\alpha(t)-\frac{1}{b-a}\int_a^b\alpha(s)ds\right|dt.\end{aligned}\tag{3.1}$$

Theorem 3.1. Let $a,b\in\mathbb{R}$ with $a<b$, If $f:[a,b]\to X$ be absolutely continuous on $[a,b]$ such that f' is Bochner integrable on $[a,b]$. If there exist $v\in X$ and $r>0$ such that $f'(x)\in\overline{B}(v,r):=\{y\in X:\|y-v\|\le r\}$ for a.e $x\in[a,b]$, then we have the sharp inequality

$$\begin{aligned}&\Bigg\|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt\\&\quad-\frac{1}{8(b-a)}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2\\&\quad+\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2-\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\Big(f(b)-f(a)\Big)\Bigg\|\\&\quad\le r\int_a^b\Bigg|K(t,I_k)-\frac{1}{8(b-a)}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2\\&\quad+\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2-\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\Bigg|dt.\end{aligned}\tag{3.2}$$

Proof. By applying Lemma 3.1 to the functions $\alpha(t):=K(t,I_k)$ and $g(t):=f'(t)$ we have

$$\begin{aligned}&\left\|(B)\int_a^b K(t,I_k)f'(t)dt-\frac{1}{b-a}\int_a^b K(t,I_k)dt\cdot(B)\int_a^b f'(t)dt\right\|\\&\qquad\le r\int_a^b\left|K(t,I_k)-\frac{1}{b-a}\int_a^b K(s,I_k)ds\right|dt.\end{aligned}\tag{3.3}$$

But we observe that

$$(B)\int_a^b f'(t)dt=f(b)-f(a)\tag{3.4}$$

and

$$\begin{aligned}\int_a^b K(t,I_k)dt=\frac{1}{8}\sum_{i=0}^{k-1}&\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2\\&+\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2-\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big].\end{aligned}\tag{3.5}$$

From Lemma 2.1, we have

$$
\begin{aligned}
(B)\int_a^b K(t,I_k)f'(t)d &= (1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)t \\
&+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt
\end{aligned}
\tag{3.6}
$$

The desired inequality is obtained from (3.3) by using (3.4)–(3.6).

Corollary 3.1. Let $a,b\in\mathbb{R}, a<b, f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ and f' is Bochner integrable. If there exist vectors $v,V\in X$ such that

$$
\left\|f'(x)-\frac{1}{2}(v+V)\right\|\le\frac{1}{2}\|V-v\|\ a.e\ x\in[a,b]
$$

then we have the inequality

$$
\begin{aligned}
&\Bigg\|(1-\lambda)\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)+\frac{\lambda}{2}\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\Big(f(\alpha_i)+f(\alpha_{i+1})\Big)-(B)\int_a^b f(t)dt \\
&-\frac{1}{8(b-a)}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2 \\
&+\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2-\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\Big(f(b)-f(a)\Big)\Bigg\| \\
&\le\frac{1}{2}\|V-v\|\int_a^b\Bigg|K(t,I_k)-\frac{1}{8(b-a)}\sum_{i=0}^{k-1}\Big[\lambda^2\Big(\alpha_{i+1}-\alpha_i\Big)^2-\Big(2x_i-\lambda\alpha_i+(\lambda-2)\alpha_{i+1}\Big)^2 \\
&+\Big(2x_{i+1}-\lambda\alpha_{i+2}+(\lambda-2)\alpha_{i+1}\Big)^2-\lambda^2\Big(\alpha_{i+2}-\alpha_{i+1}\Big)^2\Big]\Bigg|dt.
\end{aligned}
\tag{3.7}
$$

Corollary 3.2. If we take $\lambda=0$ in Theorem 3.1, then we have the inequality

$$
\begin{aligned}
&\Bigg\|\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)-(B)\int_a^b f(t)dt \\
&\quad-\frac{1}{2(b-a)}\Big[b^2-a^2-\sum_{i=0}^{k-1}\alpha_{i+1}\Big(x_{i+1}-x_i\Big)^2\Big]\Big(f(b)-f(a)\Big)\Bigg\| \\
&\quad\le r\int_a^b\Bigg|K(t,I_k)-\frac{1}{2(b-a)}\Big[b^2-a^2-\sum_{i=0}^{k-1}\alpha_{i+1}\Big(x_{i+1}-x_i\Big)^2\Big]\Bigg|dt,
\end{aligned}
\tag{3.8}
$$

where $K(\cdot,I_k)$ is given by (2.6).

Corollary 3.3. Let $a,b\in\mathbb{R}, a<b$ and let $f:[a,b]\to X$ be an absolutely continuous function on $[a,b]$ such that f' is Bochner integrable on $[a,b]$. If there exist $v\in X$ and $r>0$ such that $f'(x)\in\overline{B}(v,r):=\{y\in$

$X : \|y-v\| \leq r\}$ for a.e $x \in [a,b]$, then for any $x \in [a,b]$, we have the sharp inequality

$$\begin{aligned}
&\Bigg\|(1-\lambda)f(x)+\frac{\lambda}{2}\Big(f(a)+f(b)\Big)-\frac{1}{b-a}(B)\int_a^b f(t)dt \\
&-\frac{1}{8(b-a)}\left[\Big(2x-\lambda a+(\lambda-2)b\Big)^2-\Big(2x-\lambda b+(\lambda-2)a\Big)^2\right]\Big(f(b)-f(a)\Big)\Bigg\| \qquad (3.8)\\
&\leq r\int_a^b \left|K(t,x)-\frac{1}{8(b-a)}\left[\Big(2x-\lambda a+(\lambda-2)b\Big)^2-\Big(2x-\lambda b+(\lambda-2)a\Big)^2\right]\right|dt.
\end{aligned}$$

where $K(\cdot,x)$ is given by (2.8).

Proof. The proof follows directly from Theorem 3.1 by taking $k=2, x_0=\alpha_0=\alpha_1=a, x_1=x$ and $x_2=\alpha_2=\alpha_3=b$.

4. Conclusion

Two main results have been established. The first result extends a result of Barnet et al in [2] for k points via a parameter $\lambda \in [0,1]$. In the second part, a sharp Ostrowski–Güss type inequality for vector valued functions has been provided. Some particular cases have been considered as examples. These results generalizes the results in the literature for vector-valued functions.

Conflict of Interests

The author declares that there is no conflict of interests.

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