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SOME NEW RESULTS ON CERTAIN TYPES OF PROXIMINALITY IN BANACH SPACES

S. ALSURADI AND R. KHALIL*

Department of Mathematics, University of Jordan, Amman, Jordan

Abstract. In this paper, we prove that any convex set in a normed space is $\varepsilon-$proximinal. Consequently, every subspace in a Banach space is $\varepsilon-$proximinal. Some other results of proximinality in tensor product spaces are given.

Keywords: Banach space; tensor product; proximinality; $\varepsilon-$proximinality.

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1. Introduction

Let X be a Banach space and Y be any subset of X. For $x \in X$ we define

$$d(x,Y) = \inf_{y \in Y} \|x-y\|$$

However, such infimum need not to be attained in Y. If for any $x \in X$ there exists some $y_0 \in Y$ such that $\|x-y_0\| = d(x,Y)$, then we say that Y is proximinal in X and y_0 is called a best approximant to x out of Y. Y is called uniquely proximinal if every $x \in X$ has a unique best

*Corresponding author
E-mail addresses: roshdi@ju.edu.jo
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approximant in Y.The problem of whether a set is proximinal or not is a very important problem. It has many applications in approximation theory in function spaces. In fact one of the most classical open conjecture in approximation theory is : If E is a uniquely proximinal set in a Hilbert space X,then E is convex. We refer to [1], [2], [3], and [10] for many results on proximinality. Many other types of proximinality were introduced over the years. The concept of $\varepsilon-$ proximinality was introduced later. Many papers were written on such concept, see [4], [5], [6], [7], [8] and [9] . In this paper we prove that every set in a Banach space is $\varepsilon-$proximinal. Some other results on proximinality in tensor product spaces are presented.

2. $\varepsilon-$Proximinality In Banach Spaces

The notion of $\varepsilon-$ proximinality was introduced in [9], then used in [5], [6], [7], and [8]. In this section we prove that every set in a Banach space is $\varepsilon-$ proximinal. We start with the definition of $\varepsilon-$ proximinality.

Definition 2.1. Let G be a subset of a Banach space X. Let $\varepsilon > 0$ be given and $x \in X$. Then we say that $x_0 \in G$ is an $\varepsilon-$best approximant or $\varepsilon-$best approximation of x in G if

$$\|x - x_0\| \leq \|x - g\| + \varepsilon \ \forall g \in G$$

If this is true for every $x \in X$, then we say G is $\varepsilon-$proximinal in X.

Remark 2.1. Let G be proximinal in X. Then G is $\varepsilon-$proximinal in X for every $\varepsilon > 0$. This is because, if $x \in X$ and x_0 is the best approximant of x in G, then

$$\|x - x_0\| \leq \|x - g\| \leq \|x - g\| + \varepsilon$$

However, the converse need not be true. The set $A = [0, 1)$ is not proximinal in R, but it is $\varepsilon-$proximinal. Indeed: It is clear that A is not proximinal since $\forall x \geq 1$, x has no best approximation in A. Now, to show that A is $\varepsilon-$proximinal, let $x \in \mathbb{R}$. Then

(1) If $x < 0$, then 0 is the best approximation for x in A.
(2) If $x \in A$, then x is the best approximation to itself.

(3) If $x \geq 1$, then for any $\varepsilon > 0$ take $x_0 \in [1-\varepsilon, 1)$. Then x_0 is an ε-best approximation of x in A. This is true since:

$$\begin{aligned} |x - x_0| \leq |x - (1-\varepsilon)| &\leq |x-1| + \varepsilon \\ &\leq |x-g| + \varepsilon \; \forall g \in A \end{aligned}$$

Consequently A is $\varepsilon-$proximinal.

Now we prove the main theorem in this section.

Theorem 2.1. *Let E be any set in a Banach space X. Then for any $\varepsilon > 0$, E is $\varepsilon-$ proximinal in X.*

Proof. Let $x \in X$ be any element.
If $x \in E$, then take $x_0 = x$. So

$$\|x - x_0\| \leq \|x - e\| + \varepsilon \; \forall e \in E$$

Now, let $x \in X - E$, such that $d(x, E) = r$.
Consider $B\left[x, r + \frac{\varepsilon}{2}\right]$. Then $B\left[x, r + \frac{\varepsilon}{2}\right] \cap E \neq \phi$.
Since if not, then $\forall e \in E$ we have $e \notin B\left[x, r + \frac{\varepsilon}{2}\right]$.
That means,

$$\|x - e\| > r + \frac{\varepsilon}{2} \; \forall e \in E.$$

Hence

$$r = \inf_{e \in E} \|x - e\| \geq r + \frac{\varepsilon}{2}.$$

This is a contradiction.
So take any $y \in B\left[x, r + \frac{\varepsilon}{2}\right] \cap E$. Then

$$\begin{aligned} \|x - y\| \leq r + \frac{\varepsilon}{2} &= \inf_{e \in E} \|x - e\| + \frac{\varepsilon}{2} \\ &\leq \|x - e\| + \frac{\varepsilon}{2} \; \forall e \in E \\ &\leq \|x - e\| + \varepsilon \; \forall e \in E \end{aligned}$$

Thus E is $\varepsilon-$ proximinal.

Theorem 2.1 shows that the definition of $\varepsilon-$ proximinality that was introduced and used in [4], [5] , [6], [7], [8] and [9] is really redundant.

3. Proximinality In Injective Tensor Product Spaces

We recall the following definition

Definition 3.1. Let X be a Banach space. Then X is said to have the approximation property if for every compact subset K of X and every $\varepsilon > 0$ there exists a finite rank operator $S : X \to X$ such that

$$\|Sx - x\| \leq \varepsilon \; for \; every \; x \in K$$

For the next result, We need the following two Lemmas.

Lemma 3.1. *[11], Let X and Y be Banach spaces such that X^* has the Radon-Nicodym property and either X^* or Y^* has the approximation property . Then*

$$(X \overset{\vee}{\otimes} Y)^* \cong X^* \overset{\wedge}{\otimes} Y^*$$

Lemma 3.2. *[10], Let X and Y be Banach spaces such that X^* has the approximation property. If every $A \in L(X, Y^*)$ is compact ,then*

$$(X \overset{\wedge}{\otimes} Y)^* \cong X^* \overset{\vee}{\otimes} Y^*$$

Theorem 3.3. *Let X be a reflexive space with the approximation property. If H is a finite dimensional subspace of a Banach space Y, then $X \overset{\vee}{\otimes} H$ is proximinal in $X \overset{\vee}{\otimes} Y$.*

Proof. Since X is reflexive, then so is X^*. Hence, X^* has the Radon-Nicodym property; [11]. Also H^* has the approximation property, since the Identity operator is a finite rank operator on H^* such that

$$\|Ix - x\| \leq \varepsilon \text{ for every } x \in H^* \text{and every } \varepsilon > 0$$

Thus by Lemma 3.1 we have

$$(X \overset{\vee}{\otimes} H)^* \cong X^* \overset{\wedge}{\otimes} H^*$$

Now, since X is reflexive then $X^{**} = X$ has the approximation property. Further, any $A \in L(X^*, H^{**}) = L(X^*, H)$ is compact. This is because for any bounded subset $M \subseteq X^*$ we have $\overline{A(M)}$ is closed and bounded in H and hence is compact. So, by Lemma 3.2 we get

$$(X^* \overset{\wedge}{\otimes} H^*)^* \cong X \overset{\vee}{\otimes} H$$

Consequently, $X \overset{\vee}{\otimes} H$ is reflexive subspace in $X \overset{\vee}{\otimes} Y$. But every reflexive subspace is proximinal [12]. Thus $X \overset{\vee}{\otimes} H$ is proximinal in $X \overset{\vee}{\otimes} Y$.

4. Proximinality In Projective Tensor Product Spaces

Let X and Y be two Banach spaces, and let $X \overset{\wedge}{\otimes} Y$ denote the completed projective tensor product of X and Y. Then

$$X \overset{\wedge}{\otimes} Y = \{\sum_{i=1}^{\infty} x_i \otimes y_i : \sum_{i=1}^{\infty} \|x_i\|\|y_i\| < \infty, \text{where } x_i \in X \text{ and } y_i \in Y\ \forall i \in \mathbb{N}\}; \text{ see [10]}$$

Theorem 4.1. *Let E and F be two subsets of X and Y respectively. We let $[E]$ and $[F]$ denote the span of the sets E and F respectively. Assume that $[E]$ is separable dual space in X and $[F]$ is finite dimensional in Y . Then $[E] \overset{\wedge}{\otimes} [F]$ is proximinal in $X \overset{\wedge}{\otimes} Y$.*

Proof. Let $h \in X \overset{\wedge}{\otimes} Y$ such that

$$d(h, [E] \overset{\wedge}{\otimes} [F]) = \inf_{w \in [E] \overset{\wedge}{\otimes} [F]} \|h - w\| = r$$

By the definition of the infimum; there exists a sequence $(w_m) \in [E] \overset{\wedge}{\otimes} [F]$ such that $\lim_{m \to \infty} \|h - w_m\| = r$. Since $[F]$ is finite dimensional, then any element $z \in [E] \overset{\wedge}{\otimes} [F]$ can be written

$$z = \sum_{i=1}^{n} x_i \otimes e_i \text{ where } x_i \in [E] \text{ and } \{e_1, e_2, , , , e_n\} \text{ is a basis for } [F]; \text{ [10]}$$

Thus $w_m = \sum_{i=1}^{n} x_i^m \otimes e_i$, where

$$\begin{aligned} w_1 &= x_1^1 \otimes e_1 + x_2^1 \otimes e_2 + ... + x_n^1 \otimes e_n \\ w_2 &= x_1^2 \otimes e_1 + x_2^2 \otimes e_2 + ... + x_n^2 \otimes e_n \\ &\vdots \end{aligned}$$

Further we have $\|w_m\| \leq 2\|h\|$ is a bounded sequence; [12].

Now, Consider the sequences (x_1^m), (x_2^m), ,,(x_n^m).

Then each one of them has a w^*-convergent subsequence; being a bounded sequence in a separable dual space space $[E]$, (Helly's selection theorem).

We can extract w^*-convergent subsequences with uniform index, say $(x_1^{m_j})$, $(x_2^{m_j})$, ,,$(x_n^{m_j})$.

Now, take the sequence

$$u_{m_j} = x_1^{m_j} \otimes e_1 + x_2^{m_j} \otimes e_2 + .. + x_n^{m_j} \otimes e_n$$

Then (u_{m_j}) is a subsequence of (w_m) which is w^*-convergent, say to u.

Thus we have for any f in the unit ball of the predual space of $[E] \overset{\wedge}{\otimes} [F] = (G \overset{\vee}{\otimes} H)^*$ where G and H are the predual spaces of $[E]$ and $[F]$ respectively; Lemma 3.1.

$$\begin{aligned} |< h-u, f >| &= \lim_{j\to\infty} |< |h-u_{m_j}, f >| \\ &\leq \lim_{j\to\infty} \|h-u_{m_j}\| \\ &= \inf_{w\in[E]\overset{\wedge}{\otimes}[F]} \|h-w\| \end{aligned}$$

Hence

$$\|h-u\| \leq d(h, [E] \overset{\wedge}{\otimes} [F])$$

This implies that $[E] \overset{\wedge}{\otimes} [F]$ is proximinal and u is a best approximation to h.

Corollary 4.2. *Let Y be a finite dimensional subspace of a Banach space X, Then $\ell^p \overset{\wedge}{\otimes} Y$ is proximinal in $\ell^p \overset{\wedge}{\otimes} X$ for $1 < p < \infty$*

Corollary 4.3. *Let $[E]$ be reflexive in X and $[F]$ be a finite dimensional subspace in Y. Then $[E] \overset{\wedge}{\otimes} [F]$ is proximinal in $[X] \overset{\wedge}{\otimes} [Y]$.*

Proof. It follows by proceeding as the proof in Theorem 4.1 and using the fact that every bounded sequence in reflexive space has a $w-$ convergent subsequence; [11].

Conflict of Interests

The authors declare that there is no conflict of interests.

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