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ON COMPLEMENTATION PROBLEM IN THE LATTICE OF L-TOPOLOGIES

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Abstract. In this paper, we study the lattice structure of the lattice F_X of all L-topologies on a given nonempty set X. It is proved that the lattice F_X is complemented and dually atomic when X is any nonempty set and membership lattice L is a complete atomic boolean lattice. Further we introduce the concept of limit point in the membership lattice and prove that if membership lattice L has a limit point, then for any nonempty set X, the lattice F_X is not complemented.

Keywords: L-topology; lattice; complete atomic boolean lattice; complement; join; meet; lattice homomorphism.

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1. Introduction

Lattice theory and topology are two related branches of mathematics, each influencing the other. Many authors have already undertaken the study of the lattice structure of all topologies on a given set. Birkhoff [1] has described comparision of two topologies and noted that the set

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of all topologies on a fixed set forms a complete lattice with the natural order of set inclusion. Vaidyanathaswamy [11] has showed that this lattice is not distributive in general and also determined atoms in this lattice and proved that it is an atomic lattice. Steiner [9] proved that the lattice of topologies on a set with more than two elements is not even modular. Forlich [2] has determined dual atoms of this lattice and proved that it is also dually atomic and if $|X| = n$, then there are $n(n-1)$ dual atoms in the lattice of topologies on the set X. Steiner [9] has proved that the lattice of topologies on an arbitrary set is complemented. An independent proof that the lattice of topologies is complemented is given by Van Rooij [12]. Analogously the lattice structure of the set of all L-topologies on a given set came into interest. Johnson [5,6] has investigated lattice structure of the set of all L-topologies on a given set X and proved that this lattice is complete, atomic but not modular, not complemented and not dually atomic in general.

In this paper, we study the lattice structure of the lattice F_X of all L-topologies on a given nonempty set X. It has been proved that if membership lattice L is a complete atomic boolean lattice, then for any non-empty set X, the lattice F_X is isomorphic to the lattice $S_{X\times Y}$ of all (classical) topologies on $X \times Y$, where Y is the set of all atoms in L. Further we introduce the concept of limit point in the membership lattice and prove that if membership lattice L has a limit point, then for any nonempty set X, the lattice F_X is not complemented.

2. Preliminaries

Throughout this paper, X stands for a nonempty set, L for a bounded lattice with the least element 0 and the greatest element 1 and F_X stands for the lattice of all L-topologies on X. The constant function in L^X, taking value α is denoted by $\underline{\alpha}$ and x_γ where $\gamma(\neq 0) \in L$, denotes the L- fuzzy point defined by $x_\gamma(y) = \begin{cases} \gamma & if \quad y = x \\ 0 & otherwise \end{cases}$. Any $f \in L^X$ is called as an L-subset of X.

Definition 2.1. An element of L is called an atom if it is a minimal element of $L \setminus \{0\}$.

Definition 2.2. [8] A map from a lattice L to a lattice K is called a lattice homomorphism if it preserves finite meets and joins. The map is called a complete homomorphism if it preserves arbitrary meets and joins.

Definition 2.3. [8] A lattice isomorphism is a lattice homomorphism which is one to one and onto.

Definition 2.4. Let L be a bounded lattice with the least element 0 and the greatest element 1. An element $l(\neq 1) \in L$ is said to be limit point of L if there exists a subset $S \subset L$ such that

(i) $\bigvee S = l$.

(ii) $l \notin S$.

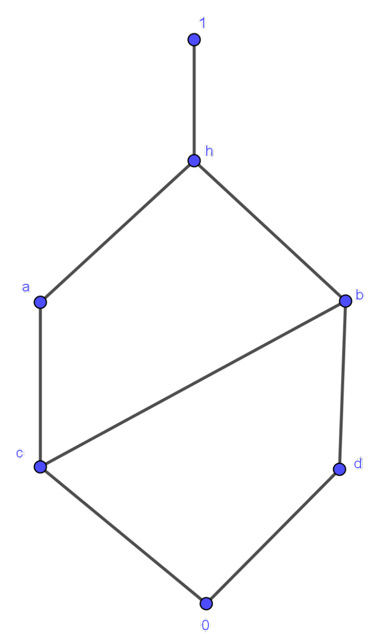
(iii) no element of S can be expressed as arbitrary join or finite meet of members of $L \setminus S$.

(iv) if $d \in S$ can not be expressed as arbitrary join or finite meet of members of $S \setminus \{d\}$, then $d \neq \left(\bigwedge_{i=1}^{n} \alpha_i\right) \wedge \left(\bigwedge_{j=1}^{m} \beta_j\right)$, where $\alpha_i \in L \setminus S$ for $1 \leq i \leq n$ and $\beta_j \in S \setminus \{d\}$ for $1 \leq j \leq m$ and $d \neq \bigvee_{\beta \in T} \beta$, where $T \subseteq M = \Big\{\eta = \left(\bigwedge_{i=1}^{n} \alpha_i\right) \wedge \left(\bigwedge_{j=1}^{m} \beta_j\right)$, where $\alpha_i \in L \setminus S$ for $1 \leq i \leq n$ and $\beta_j \in S \setminus \{d\}$ for $1 \leq j \leq m\Big\}$.

Then we say that S is a limit set of l.

Remark 2.5. Clearly, 0 can not be a limit point of L.

Example 2.6. Consider the lattice determined by the Hasse diagram given below :



Then $S = \{a, b, c\}$ is a limit set of h and h is a limit point of L.

Example 2.7. Consider the lattice $(L,\leq)$, where $L=[0,1]$ and '$\leq$' is the usual relation of 'less than or equal to' on numbers. For any $l(\neq 0,1)\in L, S=\{x\in L : l/2<x<l\}$ is a limit set of l.

Thus, every element $l(\neq 0,1)\in L$ is a limit point of L.

Example 2.8. A finite chain has no limit point.

Example 2.9. Let $A=\{a,b,c\}$ and $L=P(A)$. Then the lattice $\big(L,\subseteq\big)$ has no limit point.

3. Complementation-I

Theorem 3.1. Let X be a non-empty set and L be a bounded latttice with the least element 0 and the largest element 1. If L has atleast one limit point, then F_X is not complemented.

Proof. Let $x\in X$, $l\in L$ be a limit point of L and S be a limit set of l. Consider the set $\mathfrak{F}$ = $\{\underline{0},\underline{1}\}\cup\{f\in L^X : f(x)=0 \text{ or } f(x)\in L\setminus S\}$. Then $\mathfrak{F}$ is an L-topology since $L\setminus S$ is closed with respect to finite meet and arbitrary join.

We claim that $\mathfrak{F}$ has no complement. If possible, let $\mathfrak{F}^\star$ be a complement of $\mathfrak{F}$.

Since S is limit set of l and $L^X=\mathfrak{F}\vee\mathfrak{F}^\star$, therefore for each $\alpha\in S$ there exists L-subset $g_\alpha\in\mathfrak{F}^\star$ such that $g_\alpha(x)=\alpha$. Then $\bigvee_{\alpha\in S} g_\alpha\in\mathfrak{F}^\star$ and $\bigvee_{\alpha\in S} g_\alpha(x)=l$.

Since $\mathfrak{F}$ contains all L-subsets $h\in L^X$ such that $h(x)=l$, we have $\bigvee_{\alpha\in S} g_\alpha\in\mathfrak{F}\Rightarrow\bigvee_{\alpha\in S} g_\alpha\in\mathfrak{F}\wedge\mathfrak{F}^\star=\{\underline{0},\underline{1}\}$, a contradiction since $l\neq 0,1\Rightarrow\mathfrak{F}$ has no complement.

Hence F_X is not complemented.

Remark 3.2. If there exists a subset $S\subset L$ such that

(i) $\bigvee S=1$.

(ii) $1\notin S$.

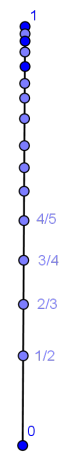
(iii) no element of S can be expressed as arbitrary join or finite meet of members of $L\setminus S$.

(iv) if $d\in S$ can not be expressed as arbitrary join or finite meet of members of $S\setminus\{d\}$, then $d\neq\big(\bigwedge_{i=1}^{n}\alpha_i\big)\wedge\big(\bigwedge_{j=1}^{m}\beta_j\big)$, where $\alpha_i\in L\setminus S$ for $1\leq i\leq n$ and $\beta_j\in S\setminus\{d\}$ for $1\leq j\leq m$ and $d\neq\bigvee_{\beta\in T}\beta$, where $T\subseteq M=\Big\{\eta=\big(\bigwedge_{i=1}^{n}\alpha_i\big)\wedge\big(\bigwedge_{j=1}^{m}\beta_j\big)$, where $\alpha_i\in L\setminus S$ for $1\leq i\leq n$ and $\beta_j\in S\setminus\{d\}$ for $1\leq j\leq m\Big\}$,

then the lattice F_X may or may not be complemented.

Example 3.3. Consider the lattice $(L,\leq)$, where $L=[0,1]$ and '$\leq$' is the usual relation of 'less than or equal to' on numbers. Then $S=\{x : .9<x<1\}\subset L$ satisfies all the conditions given in the remark 3.2. The lattice F_X is not complemented since every element $l(\neq 0,1)\in L$ is a limit point of L and then the result follows by theorem 3.1.

Example 3.4. Let $X=\{x,y\}$ and $L=\{1-1/n : n\in N\}\cup\{1\}$ be the lattice determined by the Hasse diagram given below:



Then $S=\{1-1/n : n\geq 10000\}$ satisfies all the conditions given in the remark 3.2. But the lattice F_x is complemented as shown below :

Let $\mathfrak{F}$ be an arbitrary L-topology in F_X such that $\mathfrak{F}\neq L^X,\{\underline{0},\underline{1}\}$.

Consider the element $x\in X$ and define the sets :

$\mathscr{A}_x=\{\lambda(\neq 0,1) : f(x)=\lambda$ for some $f\in\mathfrak{F}\}$

and $\mathscr{B}_x=\{\eta(\neq 0,1) : g(x)\neq\eta,\forall g\in\mathfrak{F}\}$.

There are two cases :

Case 1: $x_1\in\mathfrak{F}$.

In this case, define $\mathfrak{F}_x=\{g\in L^X : g(y)=1$ and $g(x)=\lambda$ where $\lambda\in\mathscr{B}_x\}$.

Case 2: $x_1\notin\mathfrak{F}$.

In this case, define $\mathfrak{F}_x=\{x_\lambda : \lambda\in\mathscr{B}_x\}\cup\{x_1\}$.

In the similar way, define $\mathfrak{F}_y$.

Let $\mathfrak{A}=\{\underline{0},\underline{1}\}\cup\mathfrak{F}_x\cup\mathfrak{F}_y$ and $\mathfrak{F}^\star$ be the L-topology generated by the set $\mathfrak{A}$.

Since no element $a(\neq 1) \in L$ can be written as finite meet or arbitrary join of members of $L \setminus \{a\}$, it follows that $\mathfrak{F} \wedge \mathfrak{F}^\star = \{\underline{0}, \underline{1}\}$.

Clearly, either $x_1 \in \mathfrak{F}$ or $x_1 \in \mathfrak{F}^1$.

For any $\lambda(\neq 0) \in L$, either $\lambda \in \mathscr{A}_x$ or $\lambda \in \mathscr{B}_x$ and hence the following four cases arise :

Case 1 : $x_1 \in \mathfrak{F}$ and $\lambda \in \mathscr{A}_x$.

Then $x_\lambda \in \mathfrak{F}$.

Case 2 : $x_1 \in \mathfrak{F}$ and $\lambda \in \mathscr{B}_x$.

$\lambda \in \mathscr{B}_x \Rightarrow$ there exists some $g \in \mathfrak{F}^\star$ such that $g(x) = \lambda$ and $g(y) = 1$. Then $g \wedge x_1 = x_\lambda \in \mathfrak{F} \vee \mathfrak{F}^\star$.

Case 3 : $x_1 \in \mathfrak{F}^\star$ and $\lambda \in \mathscr{A}_x$.

$\lambda \in \mathscr{A}_x \Rightarrow$ there exists some $g \in \mathfrak{F}$ such that $g(x) = \lambda$. Then $g \wedge x_1 = x_\lambda \in \mathfrak{F} \vee \mathfrak{F}^\star$.

Case 4 : $x_1 \in \mathfrak{F}^\star$ and $\lambda \in \mathscr{B}_x$.

Then $x_\lambda \in \mathfrak{F}^\star$.

$\Rightarrow x_\lambda \in \mathfrak{F} \vee \mathfrak{F}^\star, \forall \lambda(\neq 0) \in L$.

Similarly, we can show that $y_\lambda \in \mathfrak{F} \vee \mathfrak{F}^\star, \forall \lambda(\neq 0) \in L$.

$\Rightarrow \mathfrak{F} \vee \mathfrak{F}^\star = L^X \Rightarrow \mathfrak{F}^\star$ is complement of $\mathfrak{F}$.

Remark 3.5. If the membership lattice L has no limit point, then the lattice F_X may or may not be complemented.

Example 3.6. Let X be a non-empty set and $L = \{0, 1\}$. Then L has no limit point and F_X is complemented.

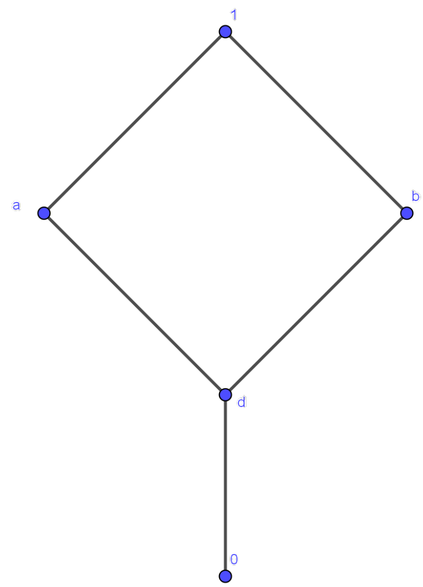
Example 3.7. Let $X = \{x, y, z\}$ and L be the membership lattice determined by the Hasse diagram given below:

Clearly, L has no limit point.

Consider the L-topology $\mathfrak{F} = \{\underline{0}, \underline{1}\} \cup \{f \in L^X : f(x) = d\}$. If $\mathfrak{F}$ has a complement say $\mathfrak{F}^\star$, then $L^X = \mathfrak{F} \vee \mathfrak{F}^\star$ and $f(x) = d$ for any $f(\neq \underline{0}, \underline{1}) \in \mathfrak{F}$, it follows that there must exist some L-subsets say $g_1, g_2 \in \mathfrak{F}^\star$ such that $g_1(x) = a$ and $g_2(x) = b$.

Then $g = g_1 \wedge g_2 \in \mathfrak{F}^\star$ and $g(x) = d$. Since $\mathfrak{F}$ contains all L-subsets $h \in L^X$ such that $h(x) = d$, we have $g \in \mathfrak{F} \Rightarrow g \in \mathfrak{F} \wedge \mathfrak{F}^\star = \{\underline{0}, \underline{1}\}$, a contradiction $\Rightarrow \mathfrak{F}$ is not complemented.

Hence F_X is not complemented.



Theorem 3.8. Let X be a non-empty set and L be a bounded lattice with the least element 0 and the largest element 1. If there exists some element $d(\neq 0,1) \in L$ and a finite subset $S \subset L$ such that

(i) $\bigwedge S = d$.

(ii) $d \notin S$.

(iii) no element of S can be expressed as arbitrary join or finite meet of members of $L \setminus S$.

(iv) if $l \in S$ can not be expressed as arbitrary join or finite meet of members of $S \setminus \{l\}$, then $l \neq \left(\bigwedge_{i=1}^{n} \alpha_i\right) \wedge \left(\bigwedge_{j=1}^{m} \beta_j\right)$, where $\alpha_i \in L \setminus S$ for $1 \leq i \leq n$ and $\beta_j \in S \setminus \{l\}$ for $1 \leq j \leq m$ and $l \neq \bigvee_{\beta \in T} \beta$, where $T \subseteq M = \Big\{\eta = \left(\bigwedge_{i=1}^{n} \alpha_i\right) \wedge \left(\bigwedge_{j=1}^{m} \beta_j\right)$, where $\alpha_i \in L \setminus S$ for $1 \leq i \leq n$ and $\beta_j \in S \setminus \{l\}$ for $1 \leq j \leq m\Big\}$,

then F_X is not complemented.

Proof. Let $x \in X$ and $\mathfrak{F} = \{\underline{0}, \underline{1}\} \cup \{f \in L^X : f(x) = 0$ or $f(x) \in L \setminus S\}$. Then $\mathfrak{F}$ is an L-topology since $L \setminus S$ is closed with respect to finite meet and arbitrary join.

We claim that $\mathfrak{F}$ has no complement. If possible, let $\mathfrak{F}^\star$ be a complement of $\mathfrak{F}$.

Since $L^X = \mathfrak{F} \vee \mathfrak{F}^\star$ and by the definition of the set S, for each $\alpha \in L$ there exists L-subset $g_\alpha \in \mathfrak{F}^\star$ such that $g_\alpha(x) = \alpha$. Then $\bigwedge_{\alpha \in S} g_\alpha \in \mathfrak{F}^\star$ and $\bigwedge_{\alpha \in S} g_\alpha(x) = d$.

Since $\mathfrak{F}$ contains all L-subsets $h \in L^X$ such that $h(x) = d$, we have $\bigwedge_{\alpha \in S} g_\alpha \in \mathfrak{F} \Rightarrow \bigwedge_{\alpha \in S} g_\alpha \in \mathfrak{F} \wedge \mathfrak{F}^\star = \{\underline{0}, \underline{1}\}$, a contradiction since $d \neq 0, 1 \Rightarrow \mathfrak{F}$ has no complement.

Hence F_X is not complemented.

4. Complementation-II

Throughout this section, L stands for a complete atomic boolean lattice and $P(A)$ for power set of the set A. Since every complete atomic boolean lattice is isomorphic to some power set of some set, namely the set of all of its atoms. Assume that L is isomorphic to the power set algebra $(P(Y), \subseteq)$, where Y is the set of all atoms in L.

Theorem 4.1. For a L-subset $f \in L^X$, define a subset of $X \times Y$ by $f^\star = \{(x,y) : f(x) \geq y\}$. Then the map $\Phi : L^X \to P(X \times Y)$ defined by $\Phi(f) = f^\star, \forall f \in L^X$, is a lattice isomorphism.

Proof. $\underline{\Phi \text{ is a lattice homomorphism}}$: Let $f, g \in L^X$.

Then $\Phi(f \wedge g) = (f \wedge g)^\star$

= $\{(x,y) : (f \wedge g)(x) \geq y\}$

= $\{(x,y) : f(x) \geq y$ and $g(x) \geq y\}$

= $\{(x,y) : f(x) \geq y\} \cap \{(x,y) : g(x) \geq y\}$

= $f^\star \cap g^\star$

= $\Phi(f) \cap \Phi(g)$.

Now $(x,y) \in f^\star \cup g^\star = \Phi(f) \cup \Phi(g)$

$\iff (x,y) \in f^\star$ or $(x,y) \in g^\star$

$\iff f(x) \geq y$ or $g(x) \geq y$

$\iff (f \vee g)(x) \geq y$

$\iff (x,y) \in (f \vee g)^\star = \Phi(f \vee g)$.

$\underline{\Phi \text{ is one-one}}$: Let $f, g \in L^X$ such that $f \neq g \Rightarrow$ there exists some $x \in X$ such that $f(x) \neq g(x)$. Sine L is atomic, there exists an atom say $y \in Y$ such that $f(x) \geq y$ but $g(x) \ngeq y \Rightarrow (x,y) \in f^\star$ but $(x,y) \notin g^\star \Rightarrow f^\star \neq g^\star \Rightarrow \Phi(f) \neq \Phi(g)$.

$\underline{\Phi \text{ is onto}}$: Let A be any subset of $X \times Y$. Consider the L-subset $f \in L^X$ defined by $f(x) = \bigvee_{y \in Y} y$ such that $(x,y) \in A$. Then $\Phi(f) = A$.

Theorem 4.2. The map $\Phi : L^X \to P(X \times Y)$ defined by $\Phi(f) = f^\star, \forall f \in L^X$, is a complete lattice homomorphism.

Proof. $\underline{\Phi \text{ preserves arbitrary meet}}$: Let $f_i \in L^X, \forall i \in \Delta$.

Then $\Phi(\bigwedge_{i\in\Delta} f_i) = (\bigwedge_{i\in\Delta} f_i)^\star$

$= \{(x,y) : \bigwedge_{i\in\Delta} f_i(x) \geq y\}$

$= \bigcap_{i\in\Delta}\{(x,y) : f_i(x) \geq y\}$

$= \bigcap_{i\in\Delta} f_i^\star$

$= \bigcap_{i\in\Delta} \Phi(f_i)$.

$\underline{\Phi \text{ preserves arbitrary join}}$: Let $(x,y) \in \bigcup_{i\in\Delta} f_i^\star$

$\iff (x,y) \in f_i^\star$ for some $i \in \Delta$

$\iff f_i(x) \geq y$ for some $i \in \Delta$

$\iff \bigvee_{i\in\Delta} f_i(x) \geq y$

$\iff (x,y) \in (\bigvee_{i\in\Delta} f_i)^\star$

$\Rightarrow (\bigvee_{i\in\Delta} f_i)^\star = \bigcup_{i\in\Delta} f_i^\star$

$\Rightarrow \Phi(\vee_{i\in\Delta} f_i) = \bigcup_{i\in\Delta} \Phi(f_i)$.

Remark 4.3. Let $\mathfrak{F}$ be an L-topology on X. Define $\mathfrak{F}^\star = \{f^\star : f \in \mathfrak{F}\}$.

Now $\mathfrak{F}$ is an L-topology on X

$\Rightarrow \underline{0}, \underline{1}, f \wedge g, \bigvee_{i\in\Delta} h_i \in \mathfrak{F}$, where $f, g, h_i \in \mathfrak{F}, \forall i \in \Delta$

$\Rightarrow \phi, X \times Y, (f \wedge g)^\star, (\bigvee_{i\in\Delta} h_i)^\star \in \mathfrak{F}^\star$

$\Rightarrow \phi, X \times Y, f^\star \cap g^\star, \bigcup_{i\in\Delta} h_i^\star \in \mathfrak{F}^\star$, where $f^\star, g^\star, h_i^\star \in \mathfrak{F}^\star, \forall i \in \Delta$

$\Rightarrow \mathfrak{F}^\star$ is a (classical) topology on $X \times Y$.

Similarly, it can be shown that if $\mathfrak{F}^\star$ is a (classical) topology on $X \times Y$, then $\mathfrak{F}$ is an L-topology on X.

Thus $\mathfrak{F}$ is an L-topology on $X \iff \mathfrak{F}^\star$ is a (classical) topology on $X \times Y$.

Theorem 4.4. The map $\Psi : F_X \to S_{X\times Y}$, defined by $\Psi(\mathfrak{F}) = \mathfrak{F}^\star, \forall \mathfrak{F} \in F_X$ is a lattice isomorphism.

Proof. $\underline{\Psi \text{ is a lattice homomorphism}}$: Let $\mathfrak{F}_1, \mathfrak{F}_2 \in F_X$.

Then $\Psi(\mathfrak{F}_1 \wedge \mathfrak{F}_2) = (\mathfrak{F}_1 \wedge \mathfrak{F}_2)^\star$

$= (\mathfrak{F}_1 \cap \mathfrak{F}_2)^\star$

$= \{f^\star : f \in \mathfrak{F}_1 \cap \mathfrak{F}_2\}$

$= \{f^\star : f \in \mathfrak{F}_1\} \cap \{f^\star : f \in \mathfrak{F}_2\}$

$= \mathfrak{F}_1^\star \cap \mathfrak{F}_2^\star$

$= \Psi(\mathfrak{F}_1) \wedge \Psi(\mathfrak{F}_2)$.

and $\Psi(\mathfrak{F}_1) \vee \Psi(\mathfrak{F}_2) = \mathfrak{F}_1^\star \vee \mathfrak{F}_2^\star$

$= \{f^\star : f^\star \in \mathfrak{F}_1^\star$ or $f^\star \in \mathfrak{F}_2^\star$ or $f^\star = \bigcup_{i\in\Delta} g_i^\star$, where $g_i^\star = \bigcap_{j=1}^n h_j^\star, h_j^\star \in \mathfrak{F}_1^\star \cup \mathfrak{F}_2^\star\}$

$= \{f^\star : f \in \mathfrak{F}_1$ or $f \in \mathfrak{F}_2$ or $f = \bigvee_{i\in\Delta} g_i$, where $g_i = \bigwedge_{j=1}^n h_i, h_i \in \mathfrak{F}_1 \cup \mathfrak{F}_2\}$

$= (\mathfrak{F}_1 \vee \mathfrak{F}_2)^\star$

$= \Psi(\mathfrak{F}_1 \vee \mathfrak{F}_2)$.

Ψ is one-one : Let $\mathfrak{F}_1, \mathfrak{F}_2 \in F_X$ such that $\mathfrak{F}_1 \neq \mathfrak{F}_2 \Rightarrow$ there exists some L-subset $f \in L^X$ such that $f \in \mathfrak{F}_1$ but $f \notin \mathfrak{F}_2 \Rightarrow f^\star \in \mathfrak{F}_1^\star$ but $f^\star \notin \mathfrak{F}_2^\star \Rightarrow \mathfrak{F}_1^\star \neq \mathfrak{F}_2^\star \Rightarrow \Psi(\mathfrak{F}_1) \neq \Psi(\mathfrak{F}_2)$.

Ψ is onto : Since the map Φ (defined in theorem 4.1) is onto, for any topology $\mathfrak{S} \in S_{X\times Y}$, consider $\mathfrak{S}^\spadesuit = \{f \in L^X : \Phi(f) \in \mathfrak{S}\}$. Then $\mathfrak{S}^\spadesuit$ is an L-topology and $\Psi(\mathfrak{S}^\spadesuit) = \mathfrak{S}$.

Remark 4.5. It has been proved that the lattice of all (classical) topologies is complete, atomic, dually atomic, complemented but neither modular nor distributive in general. Therefore, it follows that if membership lattice L is a complete atomic boolean lattice, then for any non-empty set X, the lattice of all L-topologies F_X is complete, atomic, dually atomic, complemented but neither modular nor distributive in general.

5. Conclusion

In this paper, we have determined the lattice structure of the lattice F_X on a non-empty set X when membership lattice L is a complete atomic boolean lattice. Complementation problem and lattice structure of the lattice F_X on a non-empty set X when membership lattice L is other than a complete atomic boolean lattice, will be discussed in future papers.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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