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ON MULTIPLICATIVE ZAGREB INDICES OF TWO NEW OPERATIONS OF GRAPHS

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Abstract. Recently, Wang et al. (2017) introduced two new operations of graphs. In this paper the upper bounds of the multiplicative Zagreb indices of the two newly proposed operations of graphs are derived.

Keywords: multiplicative Zagreb indices; graph operations.

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1. Introduction

Throughout the paper we consider only simple finite graphs. $V(G)$ and $E(G)$ are respectively the set of vertices and set of edges of a graph G. The degree of a vertex $u \in V(G)$ is denoted by $d_G(u)$, if their is no confusion we simply write it as $d(u)$. Two vertices u and v are called adjacent if there is an edge connecting them. The connecting edge is usually denoted by uv. Any unexplained graph theoretic symbols and definitions may be found in [16].

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Topological indices are the numerical values which are associated with a graph structure. These graph invariants are utilized for modeling information of molecules in structural chemistry and biology. Over the years many topological indices are proposed and studied based on degree, distance and other parameters of graph. Some of them may be found in [5, 7]. Historically Zagreb indices can be considered as the first degree-based topological indices, which came into picture during the study of total π-electron energy of alternant hydrocarbons by *Gutman* and *Trinajstić* in 1972 [9]. Since these indices were coined, various studies related to different aspects of these indices are reported, for detail see the papers [4, 6, 8, 12, 17] and the references therein.

The first and second Zagreb indices of a graph G are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) = \sum_{uv \in E(G)} (d_G(u) + d_G(v)),$$
$$M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The multiplicative versions of these Zagreb indices are proposed by *Todeschini et al.,* in [14] and latter named as "Multiplicative Zagreb indices" by *Gutman* [10]. These indices can be defined as follows.

$$\prod_1(G) = \prod_{u \in V(G)} d_G^2(u)$$
$$\prod_2(G) = \prod_{uv \in E(G)} d_G(u) d_G(v).$$

Graph operations play a very important role in chemical graph theory, as some chemically interesting graphs can be obtained by different graph operations on some general or particular graphs. Like many other topological indices, studies related to Multiplicative Zagreb indices and coindices of various graph operations [1, 2, 3, 13].

In this paper we consider two new operations of graphs proposed by *Wang et al.* in [15] and derive the upper bounds on the multiplicative Zagreb indices of the two newly proposed operations of graphs. The rest of the paper is organized as follows. In section 2 we reproduce the two graph operations under consideration. In section 3 main results are presented.

2. Two new graph operations and some preliminaries

In this section we first reproduce the definitions of the two newly defined operations of graphs [15] and then some standard results are included.

The Cartesian product of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G)\Box V(H)$=$\{(a,v) : a \in V(G), v \in V(H)\}$, and (a,v) is adjacent to (b,w) whenever $a = b$ and $(v,w) \in E(H)$, or $v = w$ and $(a,b) \in E(G)$. More detail on Cartesian product and some other operations of graphs may be found in [11]. In 2017, Wang et al., proposed the following two operations of graphs and also studied their adjacency spectrum. We reproduce the figure in [15] to make the discussion self expository.

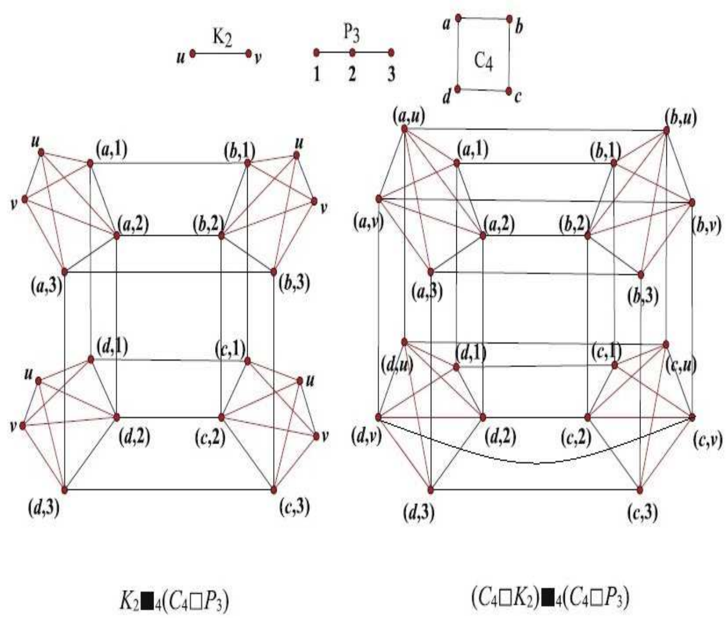

FIGURE 1. Two new operations

Definition 2.1. Let $G_{1i} = G_1$ and $G_{2i} = G_2$ $(1 \leq i \leq k)$ be k copies of graphs G_1 and G_2, respectively, G_j $(j = 3,4)$ is an arbitrary graph.

- The first operation $G_1 \blacksquare_k (G_3 \Box G_2)$ of G_1, G_2 and G_3 is obtained by making the Cartesian product of two graphs G_3 and G_2, thus produces k copies G_{2i} $(1 \leq i \leq k)$ of G_2, then makes k joins $G_{1i} \vee G_{2i}$, $i = 1,2,\ldots,k$.

- The second operation $(G_4\Box G_1)\blacksquare_k(G_3\Box G_2)$ of G_1, G_2, G_3 and G_4 is obtained by making the Cartesian product of two graphs G_3 and G_2, produces k copies G_{2i} $(1\le i\le k)$ of G_2 and making the Cartesian product of two graphs G_4 and G_1, produces k copies G_{1i} $(1\le i\le k)$ of G_1, then makes k joins $G_{1i}\vee G_{2i}$, $i=1,2,\ldots,k$.

Now we propose the following lemma which can easily be proved from the definition 2.1 of the two graph operations.

Lemma 2.1. *Let G_1, G_2, G_3 and G_4 be four graphs with $|V(G_i)|=n_i$, $|E(G_i)|=m_i$ where $i=1,2,3,4$. Then,*

$$d_{(G_1\blacksquare_k(G_3\Box G_2))}(u)=\begin{cases} d_{G_1}(u)+n_2 \quad if \quad u\in V(G_1)\\ d_{G_3}(u_3)+d_{G_2}(u_2)+n_1 \quad if \quad u=(u_3,u_2)\in V(G_3\Box G_2)\end{cases}$$

and

$$d_{((G_4\Box G_1)\blacksquare_k(G_3\Box G_2))}(u)=\begin{cases} d_{G_4}(u_4)+d_{G_1}(u_1)+n_2 \quad if \quad u=(u_4,u_1)\in V(G_4\Box G_1)\\ d_{G_3}(u_3)+d_{G_2}(u_2)+n_1 \quad if \quad u=(u_3,u_2)\in V(G_3\Box G_2).\end{cases}$$

For an example we consider $G_1=K_2$, $G_2=P_2$ and $G_3=G_4=C_4$ and hence obtain the graphs $K_2\blacksquare_4(C_4\Box P_3)$ and $(C_4\times K_2)\blacksquare_4(C_4\Box P_3)$, which are shown in figure 1. It is clear from the definitions of the two operations that $|E(G_1\blacksquare_k(G_3\Box G_2))|=k(m_1+m_2+n_1n_2)+n_2m_3$, $|E((G_4\Box G_1)\blacksquare_k(G_3\Box G_2))|=k(m_1+n_1n_2+m_2)+n_1m_4+m_3n_2$ and $|V(G_1\blacksquare_k(G_3\Box G_2))|=|V((G_4\Box G_1)\blacksquare_k(G_3\Box G_2))|=k(n_1+n_2)$. Also it is to be exclusively mentioned that $|V(G_3)|=|V(G_4)|=k$ but $G_3\neq G_4$ in general.

Here we highlight a standard lemma and two known results without proof. These results will be expedited in the coming section.

Lemma 2.2. (AM-GM Inequality) Let $x_1,x_2,\ldots,x_n$ be nonnegative numbers. Then

$$\frac{x_1+x_2+\ldots+x_n}{n}\ge\sqrt[n]{x_1x_2\ldots x_n},$$

where equality holds iff all the x_i's all equal.

Theorem 2.1. [12] *Let $G_1,G_2,\ldots,G_n$ be graphs with $V_i=V(G_i)$ and $E_i=E(G_i)$, $1\le i\le n$, and $V=V(\Box_{i=1}^n G_i)$. Then $M_1(\Box_{i=1}^n G_i)=|V|\sum_{i=1}^n\frac{M_1(G_i)}{|V_i|}+4|V|\sum_{i\neq j,j=1}^n\frac{|E_i||E_j|}{|V_i||V_j|}$.*

Theorem 2.2. [12] *Let $G_1, G_2, ..., G_n$ be graphs with $V_i = V(G_i)$ and $E_i = E(G_i), 1 \leq i \leq n$, and $V = V(\Box_{i=1}^n G_i)$ and $E = E(\Box_{i=1}^n G_i)$. Then $M_2(\Box_{i=1}^n G_i) = |V| \sum_{i=1}^n (\frac{M_2(G_i)}{|V_i|} + 3M_1(G_i)(\frac{|E|}{|V_i|} - \frac{|V||E_i|}{|V_i|^2})) + 4|V| \sum_{i,j,k=1 i\neq j, i\neq k, j\neq k}^n \frac{|E_i||E_j||E_k|}{|V_i||V_j||V_k|}$.*

3. Main Results

Theorem 2.3. *Let G_1, G_2 and G_3 be graphs with $|V_i| = |V(G_i)| = n_i$, $|E_i| = |E(G_i)| = m_i$, $1 \leq i \leq 3$ and $n_3 = k$. Then*

$$\prod_1 (G_1 \blacksquare_k (G_3 \Box G_2)) \leq \left(\frac{M_1(G_1) + 4n_2 m_1 + n_2^2 n_1}{n_1} \right)^{n_1 k} \times \left(\frac{kM_1(G_2) + n_2 M_1(G_3) + 8m_2 m_3 + n_1^2 n_2 k + 4n_1 k m_2 + 4n_1 n_2 m_3}{n_2 k} \right)^{n_2 k},$$

and

$$\prod_2 (G_1 \blacksquare_k (G_3 \Box G_2)) \leq \left(\frac{n_2 M_1(G_1) + M_2(G_1) + n_2^2 m_1}{m_1} \right)^{m_1 k} \times \left(\frac{\gamma_1}{n_1 n_2 k} \right)^{n_1 n_2 k} \times \left(\frac{(3m_3 + n_1 k) M_1(G_2) + (3m_2 + n_1 n_2) M_1(G_3) + kM_2(G_2) + n_2 M_2(G_3) + \gamma_2}{\beta} \right)^{\beta},$$

where $\gamma_1 = 8n_1 m_2 m_3 + n_1^2 n_2 m_3 + n_1^2 k m_2$, $\gamma_2 = (n_2 m_3 + k m_2)(4m_1 + 2n_1 n_2) + k(n_1^2 n_2^2 + 2m_1 n_1 n_2)$ and $\beta = km_2 + n_2 m_3$. Moreover equalities hold iff all G_1, G_2 and G_3 are regular graphs.

Proof. Since the set of vertices of $G_1 \blacksquare_k (G_3 \Box G_2)$ can be divided into two categories viz., $u \in V(G_1)$ and $u \in V(G_3 \Box G_2)$, so we have

$$\prod_1 (G_1 \blacksquare_k (G_3 \Box G_2)) = \prod_{u \in V(G_1 \blacksquare_k (G_3 \Box G_2))} d^2_{G_1 \blacksquare_k (G_3 \Box G_2)}(u)$$

$$= \left(\prod_{u \in V(G_1)} d^2_{G_1 \blacksquare_k (G_3 \Box G_2)}(u) \right)^k \times \prod_{u=(u_3,u_2) \in V(G_3 \Box G_2)} d^2_{G_1 \blacksquare_k (G_3 \Box G_2)}(u). \tag{1}$$

Using lemma 2.1, expression (1) can be written as

$$\prod_1(G_1\blacksquare_k(G_3\square G_2)) = \left(\prod_{u\in V(G_1)}(d_{G_1}(u)+n_2)^2\right)^k \times$$
$$\prod_{u=(u_3,u_2)\in V(G_3\square G_2)}(d_{G_3}(u_3)+d_{G_2}(u_1)+n_1))^2.$$

As $d_{G_3\square G_2}(u_3,u_2) = d_{G_3}(u_3)+d_{G_3}(u_2)$ the last expression is

$$= \left(\prod_{u\in V(G_1)}(d_{G_1}(u)+n_2)^2\right)^k \times \prod_{u\in V(G_3\square G_2)}(d_{G_3\square G_2}(u)+n_1)^2 \tag{2}$$

and by lemma 2.2 we can write expression (2) as

$$\prod_1(G_1\blacksquare_k(G_3\square G_2)) \leq \left(\frac{\sum_{u\in V(G_1)}(d^2_{G_1}(u)+2n_2d_{G_1}(u)+n_2^2)}{n_1}\right)^{n_1k}$$
$$\times\left(\frac{\sum_{u\in V(G_3\square G_2)}(d^2_{G_3\square G_2}(u)+2n_1d_{G_3\square G_2}(u)+n_1^2}{n_2k}\right)^{n_2k}. \tag{3}$$

Now

$$\sum_{u\in V(G_1)}(d^2_{G_1}(u)+2n_2d_{G_1}(u)+n_2^2) = M_1(G_1)+4n_2m_1+n_2^2n_1, \tag{4}$$

and

$$\sum_{u\in V(G_3\square G_2)}(d^2_{G_3\square G_2}(u)+2n_1d_{G_3\square G_2}(u)+n_1^2) = M_1(G_3\square G_2)+n_1^2n_2k+4n_1|E(G_3\square G_2)|. \tag{5}$$

Using theorem 2.1 and the fact that $|E(G_3\square G_2)| = km_2+n_2m_3$, we write the expression (5) as

$$\sum_{u\in V(G_3\square G_2)}(d^2_{G_3\square G_2}(u)+2n_1d_{G_3\square G_2}(u)+n_1^2) = n_2M_1(G_3)+kM_1(G_2)+8m_2m_3+n_1^2n_2k$$
$$+4n_1km_2+4n_1n_2m_3. \tag{6}$$

From (3), (4) and (6), we have the first inequality.

Moreover equality holds iff

$$d^2_{G_1}(u_i)+2n_2d_{G_1}(u_i)+n_2^2 = d^2_{G_1}(u_j)+2n_2d_{G_1}(u_j)+n_2^2 \ \ (u_i,u_j\in V(G_1))$$
$$\implies (d_{G_1}(u_i)-d_{G_1}(u_j))(d_{G_1}(u_i)+d_{G_1}(u_j)+2n_2) = 0 \tag{7}$$

and

$$d^2_{G_3\square G_2}(u)+2n_1d_{G_3\square G_2}(u)+n_1^2=d^2_{G_3\square G_2}(u')+2n_1d_{G_3\square G_2}(u')+n_1^2 \;\; (u,u'\in V(G_3\square G_2))$$

$$\Longrightarrow (d_{G_3\square G_2}(u)-d_{G_3\square G_2}(u'))(d_{G_3\square G_2}(u)+d_{G_3\square G_2}(u')+2n_1)=0. \tag{8}$$

So from (7) and (8), we can conclude that equality holds iff both G_1 and $G_3\square G_2$ are regular.

Since the set of edges of $G_1\blacksquare_k(G_3\square G_2)$ can again be divided into three categories viz., $uv\in E(G_1)$, $uv\in E(G_3\square G_2)$, and $uv\in E(G_1\blacksquare_k(G_3\square G_2))$ s.t. $u\in V(G_1)$ and $v\in V(G_3\square G_2)$, so we have

$$\begin{aligned}\prod_2(G_1\blacksquare_k(G_3\square G_2)) &= \left(\prod_{uv\in E(G_1)}(d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2)\right)^k\\ &\times \prod_{uv\in E(G_3\square G_2)}(d_{G_3\square G_2}(u)+n_1)(d_{G_3\square G_2}(v)+n_1)\\ &\times \prod_{uv\in E(G_1\blacksquare_k(G_3\square G_2),\, u\in V(G_1) \text{ and } v\in V(G_3\square G_2)}(d_{G_1}(u)+n_2)(d_{G_3\square G_2}(v)+n_1) \end{aligned} \tag{9}$$

Again using lemma 2.1, expression (9) can be written as

$$\begin{aligned}\prod_2(G_1\blacksquare_k(G_3\square G_2)) &\leq \left(\frac{\sum_{uv\in E(G_1)}(d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2)}{m_1}\right)^{m_1k}\\ &\times\left(\frac{\sum_{uv\in E(G_3\square G_2)}(d_{G_3\square G_2}(u)+n_1)(d_{G_3\square G_2}(v)+n_1)}{km_2+n_2m_3}\right)^{km_2+n_2m_3}\\ &\times\left(\frac{\sum_{uv\in E(G_1\blacksquare_k(G_3\square G_2)}(d_{G_1}(u)+n_2)(d_{G_3\square G_2}(v)+n_1)}{n_1n_2k}\right)^{n_1n_2k}\end{aligned} \tag{10}$$

Now

$$\sum_{uv\in E(G_1)}(d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2)=k\left(M_2(G_1)+n_2M_1(G_1)+n_2^2m_1\right), \tag{11}$$

also using theorem 2.1 and theorem 2.2, we have

$$\begin{aligned}&\textstyle\sum_{uv\in E(G_3\square G_2)}(d_{G_3\square G_2}(u)+n_1)(d_{G_3\square G_2}(v)+n_1)\\ &=kM_2(G_2)+n_2M_2(G_3)+(3m_3+n_1k)M_1(G_2)+(3m_2+n_1n_2)M_1(G_3)\\ &+8n_1m_2m_3+n_1^2n_2m_3+n_1^2km_2,\end{aligned} \tag{12}$$

and

$$\sum_{uv\in E(G_1\blacksquare_k(G_3\square G_2))}(d_{G_1}(u)+n_2)(d_{G_3\square G_2}(v)+n_1) = (n_2m_3+km_2)(4m_1+2n_1n_2) + k(n_1^2n_2^2+2m_1n_1n_2). \tag{13}$$

Using (11), (12), (13) in (10), we have the second inequality.

From lemma 2.2 for connected graphs G_1, G_2 and G_3 equality holds iff

$$(d_{G_1}(u)+n_2)(d_{G_1}(v)+n_2) = (d_{G_1}(u)+n_2)(d_{G_1}(v')+n_2) \text{ (for any } uv, uv' \in E(G_1)),$$

for any $uv, uv' \in E(G_3\square G_2)$,

$$d_{G_3\square G_2}(u)+n_1)(d_{G_3\square G_2}(v)+n_1 = d_{G_3\square G_2}(u)+n_1)(d_{G_3\square G_2}(v')+n_1,$$

and for any $uv, uv' \in E(G_1\blacksquare_k(G_3\square G_2)$, $u\in V(G_1)$ and $v, v' \in V(G_3\square G_2)$,

$$d_{G_1}(u)+n_2)(d_{G_3\square G_2}(v)+n_1 = d_{G_1}(u)+n_2)(d_{G_3\square G_2}(v')+n_1.$$

Hence equality holds in the second inequality iff all the graphs G_1, G_2 and G_3 are connected regular graphs. $\square$

Theorem 2.4. *Let G_1, G_2, G_3 and G_4 be graphs with $|V_i| = |V(G_i)| = n_i$, $|E_i| = |E(G_i)| = m_i$, $1\le i\le 3$ and $n_3 = k = n_4$. Then*

$$\prod_1((G_4\square G_1)\blacksquare_k(G_3\square G_2)) \le \left(\frac{kM_1(G_1)+n_1M_1(G_4)+8m_1m_4+4n_2(km_1+n_1m_4)+n_2^2kn_1}{n_1k}\right)^{n_1k} \times \left(\frac{kM_1(G_2)+n_2M_1(G_3)+8m_2m_3+4n_1(km_2+n_2m_3)+n_1^2kn_2}{n_2k}\right)^{n_2k},$$

and

$$\prod_2((G_4\square G_1)\blacksquare_k(G_3\square G_2))$$
$$\le \left(\frac{kM_2(G_1)+n_1M_2(G_4)+3m_4M_1(G_1)+3m_1M_1(G_4)+n_2(kM_1(G_1)+n_1M_1(G_4)+\delta_{14})}{m_1k+m_4n_1}\right)^{m_1k+m_4n_1}$$
$$\left(\frac{kM_2(G_2)+n_2M_2(G_3)+3m_3M_1(G_2)+3m_2M_1(G_3)+n_1(kM_1(G_2)+n_2M_1(G_3)+\delta_{23})}{m_2k+m_3n_2}\right)^{m_2k+m_3n_2}$$
$$\times\left(\frac{\gamma_2+n_1n_2\sum_{i=1}^k d_{G_4}(u_4^i)d_{G_3}(v_3^i)}{n_1n_2k}\right)^{n_1n_2k},$$

where $u_4^i \in V(G_4)$, $v_3^i \in V(G_3)$ and for fixed i, (u_4^i, u_1) is adjacent to (v_3^i, v_2) for any $u_1 \in V(G_1)$, $v_2 \in V(G_2)$ and $\delta_{pq} = 8m_pm_q + n_pn_2m_q + kn_qm_p$, $\gamma_2 = 4n_1m_2m_4 + 2n_1^2n_2m_4 + 4n_2m_1m_3 + 4m_1m_2k + 2kn_1m_1n_2 + 2n_1n_2^2m_3 + 2kn_1n_2m_2 + kn_1^2n_2^2$. Moreover equalities hold iff all G_1, G_2, G_3 and G_4 are regular graphs.

Proof. There are three two types of vertices present in $(G_4\square G_1)\blacksquare_k(G_3\square G_2)$, they are $u \in G_4\square G_1$ and $u \in G_3\square G_2$.

$$\begin{aligned}\prod_1((G_4\square G_1)\blacksquare_k(G_3\square G_2)) &= \prod_{u\in V((G_4\square G_1)\blacksquare_k(G_3\square G_2))} d^2_{(G_4\square G_1)\blacksquare_k(G_3\square G_2)}(u)\\ &= \prod_{u=(u_3,u_2)\in V(G_3\square G_2)} d^2_{(G_4\square G_1)\blacksquare_k(G_3\square G_2)}(u)\\ &\times \prod_{u=(u_4,u_1)\in V(G_3\square G_2)} d^2_{(G_4\square G_1)\blacksquare_k(G_3\square G_2)}(u), \end{aligned} \tag{14}$$

using lemma 2.1 and 2.2, we can write (14) as

$$\begin{aligned}\prod_1((G_4\square G_1)\blacksquare_k(G_3\square G_2)) \le& \left(\frac{\sum_{u\in V(G_4\square G_1)}(d_{(G_4\square G_1)}(u)+n_2)^2}{n_1k}\right)^{n_1k}\\ &\times\left(\frac{\sum_{u\in V(G_3\square G_2)}(d_{(G_3\square G_2)}(u)+n_1)^2}{n_2k}\right)^{n_2k}. \end{aligned} \tag{15}$$

Now

$$\begin{aligned}\sum_{u\in V(G_4\square G_1)}(d_{(G_4\square G_1)}(u)+n_2)^2 &= kM_1(G_1)+n_1M_1(G_4)+8m_1m_4+4n_2(km_1+n_1m_4)\\ &+n_2^2kn_1,\end{aligned}$$

and

$$\begin{aligned}\sum_{u\in V(G_3\square G_2)}(d_{(G_3\square G_2)}(u)+n_1)^2 &= kM_1(G_2)+n_2M_1(G_3)+8m_2m_3+4n_1(km_2+n_2m_3)\\ &+n_1^2kn_2.\end{aligned}$$

Hence we have the first inequality of the theorem.

For equality we must have

$$(d_{(G_4\square G_1)}(u)+n_2)^2 = (d_{(G_4\square G_1)}(v)+n_2)^2 \text{ (for any } uv\in E(G_4\square G_1)), \tag{16}$$

and

$$(d_{(G_3\square G_2)}(u)+n_1)^2=(d_{(G_3\square G_2)}(v)+n_1)^2 \text{ (for any } uv\in E(G_3\square G_2)). \tag{17}$$

From (16) and (17) it is easy to show that the equality holds iff all the graphs G_1, G_2, G_3 and G_4 are regular.

Now the set of edges of $(G_4\square G_1)\blacksquare_k(G_3\square G_2)$ can again classified into three categories viz., $uv\in E(G_4\square G_1)$, $uv\in E(G_3\square G_2)$, and $uv\in E((G_4\square G_1)\blacksquare_k(G_3\square G_2))$ where $u\in V(G_4\square G_1)$, $v\in V(G_3\square G_2)$. Hence we have

$$\begin{aligned}\prod_1((G_4\square G_1)\blacksquare_k(G_3\square G_2)) = &\prod_{uv\in E(G_4\square G_1)}(d_{(G_4\square G_1)}(u)+n_2)(d_{(G_4\square G_1)}(v)+n_2)\\ &\times\prod_{uv\in E(G_3\square G_2)}(d_{(G_3\square G_2)}(u)+n_1)(d_{(G_3\square G_2)}(v)+n_1)\\ &\times\prod_{uv\in E((G_4\square G_1)\blacksquare_k(G_3\square G_2)}(d_{(G_4\square G_1)}(u)+n_2)(d_{(G_3\square G_2)}(v)+n_1)\end{aligned} \tag{18}$$

Again using lemma 2.2, we have

$$\begin{aligned}\prod_1((G_4\square G_1)\blacksquare_k(G_3\square G_2)) \leq &\left(\frac{\sum_{uv\in E(G_4\square G_1)}(d_{(G_4\square G_1)}(u)+n_2)(d_{(G_4\square G_1)}(v)+n_2)}{n_1m_4+km_1}\right)^{n_1m_4+km_1}\\ &\times\left(\frac{\sum_{uv\in E(G_3\square G_2)}(d_{(G_3\square G_2)}(u)+n_1)(d_{(G_3\square G_2)}(v)+n_1)}{n_2m_3+m_2k}\right)^{n_2m_3+m_2k}\\ &\times\left(\frac{\sum_{uv\in E((G_4\square G_1)\blacksquare_k(G_3\square G_2)}(d_{(G_4\square G_1)}(u)+n_2)(d_{(G_3\square G_2)}(v)+n_1)}{kn_1n_2}\right)^{kn_1n_2}.\end{aligned} \tag{19}$$

Hence we have the second inequality of the theorem from the following expressions.

$$\begin{aligned}\sum_{uv\in E(G_4\square G_1)}(d_{(G_4\square G_1)}(u)+n_2)(d_{(G_4\square G_1)}(v)+n_2) =&kM_2(G_1)+n_1M_2(G_4)+3m_4M_1(G_1)\\ &+3m_1M_1(G_4)+n_2(kM_1(G_1)+n_1M_1(G_4)\\ &+8m_1m_4+n_1n_2m_4+km_1),\end{aligned}$$

$$\sum_{uv\in E(G_3\Box G_2)} (d_{(G_3\Box G_2)}(u)+n_1)(d_{(G_3\Box G_2)}(v)+n_1) = kM_2(G_2)+n_2M_2(G_3)+3m_3M_1(G_2) + 3m_2M_1(G_3)+n_2(kM_1(G_2)+n_1M_1(G_3) + 8m_2m_3+n_1n_2m_3+km_2),$$

and finally let $u\in V(G_4\Box G_1)$ and $v\in V(G_3\Box G_2)$ s.t. $u=(u_4^i,u_1)$, $v=(v_3^i,v_2)$, where $V(G_3)=\{u_3^1,u_3^2,\ldots,u_3^k\}$ and $V(G_4)=\{v_4^1,v_4^2,\ldots,v_4^k\}$. Also for fixed i, u is adjacent to v for all $u_1\in V(G_1)$ and $v_2\in V(G_2)$. Then we have

$$\sum_{uv\in E((G_4\Box G_1)\blacksquare_k(G_3\Box G_2))} d(u)d(v) = \sum_{i=1}^{k}\sum_{u_2\in V(G_2)}\sum_{v_1\in V(G_1)} (d_{G_3}(u_3^i)+d_{G_2}(u_2)+n_1)(d_{G_4}(u_4^i)+ d_{G_1}(v_1)+n_2)$$
$$=4n_1m_2m_4+2n_1^2n_2m_4+4n_2m_1m_3+4m_1m_2k+2kn_1m_1n_2 +2n_1n_2^2m_3+2kn_1n_2m_2+kn_1^2n_2^2+n_1n_2\sum_{i=1}^{k} d_{G_4}(u_4^i)d_{G_3}(v_3^i).$$

For equality of the second result in the theorem, we must have

for any $uv\ uv'\in E(G_4\Box G_1)$

(20) $$(d_{(G_4\Box G_1)}(u)+n_2)(d_{(G_4\Box G_1)}(v)+n_2) = (d_{(G_4\Box G_1)}(u)+n_2)(d_{(G_4\Box G_1)}(v')+n_2),$$

and for any $uv\ uv'\in E(G_3\Box G_2)$

(21) $$(d_{(G_3\Box G_2)}(u)+n_1)(d_{(G_3\Box G_2)}(v)+n_1) = (d_{(G_3\Box G_2)}(u)+n_1)(d_{(G_3\Box G_2)}(v')+n_1)$$

From expression (20) and (21), we can easily see that equality holds iff all the graphs G_1, G_2, G_3 and G_4 are regular. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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