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A DISCUSSION ON A COMMON FIXED POINT THEOREM ON SEMICOMPATIBLE MAPPINGS

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Abstract: In this paper, we prove a common fixed point theorem which is a generalization of Bijendra Singh and M.S. Chauhan using some weaker conditions namely semi compatible and associated sequence instead of compatibility and completeness of the metric space. Also, we give suitable example to validate our theorem.

Keywords: fixed point; self-maps; semicompatible mappings and associated sequence.

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1. Introduction

A contraction mapping defined on complete metric space is having unique fixed point, this is known as Banach contraction principle and which is first ever result in fixed point theory. This result was further generalized and extended in various ways by many authors. S.Sessa[8] defined weak commutativity and proved common fixed point theorems for weakly commuting maps. Afterwards G.Jungck[1] introduced the concept of compatible mappings which is weaker than weakly commuting mappings. Thereafter Jungck and Rhoades [4] defined weaker class of maps known as weakly compatible maps.

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The concept of semi compatible mappings was introduced by Y.J.Cho, B.K. Sharma and D.R.Sharma [6]. In this paper we prove a common fixed point theorem for four self maps using semicompatible mappings.

2. Definitions and Preliminaries

2.1 Definition [3]. A and S are two self maps of a metric space (X,d) are said to be *compatible mappings* if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

2.2 Example. Let (X,d) be a metric space where $X = [0,2]$ and $d(x,y) = |x-y|$ .We define self maps A and S as

$$A(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{if } x \in (0,1] \\ \frac{x}{2} & \text{if } x \in (1,2] \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ \frac{x+1}{3} & \text{if } x \in (1,2] \end{cases}$$

Let us consider a sequence $\{x_n\}$ by $x_n = 2 - \frac{1}{n}$ for $n \geq 1$ .Then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$.

Thus $\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} SAx_n = 1$. Hence, the pair (A,S) is compatible.

2.3 Definition [6]. A and S are two self maps of a metric space (X,d) are said to be *semicompatible* if $\lim_{n\to\infty} d(ASx_n, St) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$.

2.4 Example. Let $X = [0,2]$ with the usual metric. Define A and S by

$$A(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ \frac{x}{2} & \text{if } 1 \leq x \leq 2 \end{cases} \quad \text{and} \quad S(x) = \begin{cases} 1 & \text{if } x = 1 \\ \frac{x+1}{3} & \text{if otherwise} \end{cases}$$

Consider a sequence $\{x_n\}$ as $x_n = 2 - \frac{1}{n}$, for $n \geq 1$, then

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$. Also $\lim_{n\to\infty} ASx_n = 1$ and $S(1) = 1$ implies $\lim_{n\to\infty} ASx_n = S(1)$.

Hence (A,S) is semi compatible.

Again $\lim_{n\to\infty} SAx_n = \frac{2}{3}$ but $A(1) = 1$ gives $\lim_{n\to\infty} SAx_n \neq A(1)$. Hence (S,A) is not semi compatible.

Further the mappings A and S are not compatible.

2.5 Remark. Semi compatibility of the pair (A,S) does not imply the semi compatibility of the pair (S,A) and semi compatible mappings need not be compatible.

Bijenrda Singh and M.S.Chauhan[5] proved the following theorem.

2.6 Theorem. Let A,B,S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

(2.6.1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2.6.2) one of A,B,S and T is continuous

(2.6.3) $[d(Ax,By)]^2 \leq k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)]$

$$+k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)]$$

where $0 \leq k_1 + 2k_2 < 1, k_1, k_2 \geq 0$

(2.6.4) the pairs (A,S) and (B,T) are compatible on X .

Further if X is a complete metric space then A,B,S and T have a unique common fixed point in X.

Now we generalize the above theorem using semi compatible mappings and an associated sequence.

2.7 Associated sequence [7]: Suppose A, B, S and T are four self maps of a metric space (X,d) satisfying the condition(2.6.1),then for an arbitrary $x_0 \in X$ such that $Ax_0 \in A(X) \subseteq T(X)$ gives $Ax_0 = Tx_1$ for some $x_1 \in X$ .For this point $x_1 \in X$, $Bx_1 \in B(X) \subseteq S(X)$, there exist a point x_2 in X such that $Bx_1 = Sx_2$ .Again $Ax_2 \in A(X) \subseteq T(X)$ gives $Ax_2 = Tx_3$ for some $x_3 \in X$ .Now $Bx_3 \in B(X) \subseteq S(X)$ gives $Bx_3 = Sx_4$ and so on. Proceeding in this similar manner, we can define a sequence $\{y_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n \geq 0$. We shall call this sequence as an "Associated sequence" connected to the four self maps A, B, S and T.

Now we prove a lemma with an example which plays an important role in our main theorem.

2.8 Lemma: Suppose A, B, S and T are self maps of a complete metric space (X, d) into itself satisfying the conditions (2.6.1) and (2.6.3),then the associated sequence $\{y_n\}$ relative to four self maps given in (2.7) is a Cauchy sequence in X.

Proof: Using the conditions (2.6.1), (2.6.3) and from the definition of associated sequence, we have

$$\begin{aligned}\left[d(y_{2n+1},y_{2n})\right]^2 &= \left[d(Ax_{2n},Bx_{2n-1})\right]^2\\ &\le k_1\left[d(Ax_{2n},Sx_{2n})\ d(Bx_{2n-1},Tx_{2n-1})+d(Bx_{2n-1},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1}\ \right]\\ &+k_2\left[d(Ax_{2n},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1})+d(Bx_{2n-1},Tx_{2n-1})\ d(Bx_{2n-1}\ Sx_{2n})\ \right]\\ &= k_1\left[d(y_{2n+1},y_{2n})\ d(y_{2n},y_{2n-1})\ \right]+k_2\left[d(y_{2n+1},y_{2n})\ d(y_{2n+1},y_{2n-1})\right]\end{aligned}$$

this implies

$$d(y_{2n+1},y_{2n})\le k_1\ \ d(y_{2n},y_{2n-1})+k_2\left[d(y_{2n+1},y_{2n})+\ d(y_{2n},y_{2n-1})\ \right]$$

$$d(y_{2n+1},y_{2n})\le \lambda\ \ d(y_{2n},y_{2n-1})$$

$$\text{where } \lambda=\frac{k_1+k_2}{1-k_2}<1.$$

Now for every integer $m>0$, we get

$$\begin{aligned}d(y_n,y_{n+m}) &\le\ d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+............+\ \ d(y_{n+m-1},y_{n+m})\\ &\le \lambda^n d(y_0,y_1)+\lambda^{n+1}d(y_0,y_1)+.............+\lambda^{n+m-1}d(y_0,y_1)\\ &\le\left(\lambda^n+\lambda^{n+1}+.............+\lambda^{n+m-1}\right)d(y_0,y_1)\\ &\le\lambda^n\left(1+\lambda+\lambda^2+.............+\lambda^{m-1}\right)d(y_0,y_1)\end{aligned}$$

Since $\lambda<1$, $\lambda^n\to 0$ as $n\to\infty$, so that $d(y_n,y_{n+m})\to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space, it converges to some point, say $z\in X$.

The converse of the Lemma is not true, that is A,B,S and T are self maps of a metric space (X,d) satisfying the (2.6.1) and (2.6.3), though if for any $x_0\in X$ and for the associated sequence $Ax_0,Bx_1,Ax_2,Bx_3,....,Ax_{2n},Bx_{2n+1},...$ converges, the metric space (X,d) need not be complete.

For this we give an example.

2.9 Example: Let $X=(0,1)$ with $d(x,y)=|x-y|$. Define self maps of A, B, S and T of X by

$$A(x)=B(x)=\begin{cases}\frac{1}{2} & if\ 0<x<\frac{1}{2}\\ x & if\ \frac{1}{2}\le x<1\end{cases}\quad\text{and}\quad S(x)=T(x)=\begin{cases}1-x & if\ 0<x<\frac{1}{2}\\ x & if\ \frac{1}{2}\le x<1\end{cases}$$

Then $A(X) = B(X) = \left[\frac{1}{2}, 1\right)$ and $S(X) = T(X) = \left[\frac{1}{2}, 1\right)$ showing the conditions $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Also it is easy to show that the associated sequence $Ax_0, Bx_1, Ax_2, Bx_3....Ax_{2n}, Bx_{2n+1},....$ converges to the point $\frac{1}{2}$, but X is not a complete metric space.

3. Main results

3.1 Theorem: Suppose A, B, S and T are self maps from a metric space (X,d) into itself satisfying the following conditions

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ (3.1.1)

$$[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)] \quad (3.1.2)$$

for all x, y in X where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

S and T are continuous (3.1.3)

the pairs (A,S) and (B,T) are semi compatible mappings. (3.1.4)

Further if

for any $x_0 \in X$ the associated sequence relative to four self maps A,B,S and T such that the sequence $Ax_0, Bx_1, Ax_2, Bx_{3,}...., Ax_{2n}, Bx_{2n+1}...$ converges to $z \in X$ as $n \to \infty$ (3.1.5)

then A,B,S and T have a unique common fixed point in X.

Proof: From the condition (3.1.5), we have

$Ax_{2n} \to z, Tx_{2n+1} \to z, Bx_{2n+1} \to z$ *and* $Sx_{2n} \to z$ as $n \to \infty$. (3.1.6)

Since the pair (A,S) is semi compatible mapping, then $ASx_{2n} \to Sz$ as $n \to \infty$ (3.1.7)

Suppose S is continuous then $SSx_{2n} \to Sz$ and $SAx_{2n} \to Sz$ as $n \to \infty$. (3.1.8)

Put $x = Sx_{2n}, y = x_{2n+1}$ in the condition (3.1.2), we have

$$[d(ASx_{2n}, Bx_{2n+1})]^2 \le k_1[d(ASx_{2n}, SSx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) + d(Bx_{2n+1}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1})] + k_2[d(ASx_{2n}, SSx_{2n})d(ASx_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Tx_{2n+1})d(Bx_{2n+1}, SSx_{2n})]$$

letting $n \to \infty$ and using the conditions (3.1.6),(3.1.7) and (3.1.8), we get

$$[d(Sz,z)]^2 \le k_1[d(Sz,Sz)d(z,z)+d(z,Sz)d(Sz,z)] \\ +k_2[d(Sz,Sz)d(Sz,z)+d(z,Sz)d(z,z)]$$

this gives

$$[d(Sz,z)]^2 \le k_1\left[d(Sz,z)\right]^2$$

and this implies

$(1-k_1)[d(Sz,z)]^2 \le 0$ .Since the distance function can never be negative, we get $d(Sz,z)^2 = 0$, this gives $d(Sz,z)=0$ and this implies $Sz = z$.

Therefore $Sz = z$.

Now put $x = z, y = x_{2n+1}$ in the condition (3.1.2), we have

$$[d(Az,Bx_{2n+1})]^2 \le k_1[d(Az,Sz)d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Sz)d(Az,Tx_{2n+1})] \\ +k_2[d(Az,Sz)d(Az,Tx_{2n+1})+d(Bx_{2n+1},Tx_{2n+1})d(Bx_{2n+1},Sz)]$$

letting $n \to \infty$ and using the conditions (3.1.6) and $Sz = z$,we get

$$[d(Az,z)]^2 \le k_1[d(Az,z)d(z,z)+d(z,z)d(Az,z)] \\ +k_2[d(Az,z)d(Az,z)+d(z,z)d(z,z)]$$

this gives

$[d(Az,z)]^2 \le k_2[d(Az,z)]^2$ and this implies

$[1-k_2][d(Az,z)]^2 \le 0$. Since $0 \le k_1 + 2k_2 < 1$ and distance function can never be negative, we get $d(Az,z)^2 = 0$, this gives $d(Az,z) = 0$ implies $Az = z$.

Hence $Az = Sz = z$ --- (3.1.9), showing that z is a common fixed point of A and S .

Also since the pair (B,T) is semi compatible mapping, then $BTx_{2n} \to Tz$ as $n \to \infty$ (3.1.10)

Suppose T is continuous then $TTx_{2n} \to Tz$ and $TBx_{2n} \to Tz$ as $n \to \infty$. (3.1.11)

Put $x = x_{2n}$ and $y = Tx_{2n}$ in the condition (3.1.2), we have

$$[d(Ax_{2n},BTx_{2n})]^2 \le k_1[d(Ax_{2n},Sx_{2n})d(BTx_{2n},TTx_{2n})+d(BTx_{2n},Sx_{2n})d(Ax_{2n},TTx_{2n})] \\ +k_2[d(Ax_{2n},Sx_{2n})d(Ax_{2n},TTx_{2n})+d(BTx_{2n},Sx_{2n})d(BTx_{2n},TTx_{2n})]$$

letting $n \to \infty$ and using the conditions(3.1.10) and (3.1.11), we get

$$[d(z,Tz)]^2 \le k_1[d(z,z)d(Tz,Tz)+d(Tz,z)d(z,Tz)] \\ +k_2[d(z,z)d(z,Tz)+d(Tz,z)d(Tz,Tz)]$$

this gives

$[d(z,Tz)]^2 \le k_1[d(Tz,z)]^2$ and this implies

$(1-k_1)[d(z,Tz)]^2 \le 0$. Since $0 \le k_1 + 2k_2 < 1$ and distance function can never be negative, we get $d(z,Tz)^2 = 0$, this gives $d(z,Tz) = 0$ and this implies $Tz = z$.

Now put $x = z, y = z$ in the condition (3.1.2), we have

$$[d(Az,Bz)]^2 \le k_1[d(Az,Sz)d(Bz,Tz) + d(Bz,Sz)d(Az,Tz)]$$
$$+ k_2[d(Az,Sz)d(Az,Tz) + d(Bz,Tz)d(Bz,Sz)]$$

Using the conditions $Az = Sz = z$ and $Tz = z$, we have

$$[d(z,Bz)]^2 \le k_1[d(z,z)d(Bz,Bz) + d(Bz,z)d(z,Bz)]$$
$$+ k_2[d(z,z)d(z,Bz) + d(Bz,Bz)d(Bz,z)]$$

this gives

$[d(z,Bz)]^2 \le k_1[d(Bz,z)]^2$ and this implies

$(1-k_1)[d(z,Bz)]^2 \le 0$. Since $0 \le k_1 + 2k_2 < 1$ and distance function can never be negative, we get $d(z,Bz)^2 = 0$, this gives $d(Bz,z) = 0$ implies $Bz = z$.

Hence $Bz = Tz = z$ … (3.1.12), showing that z is a common fixed point of B and T.

From the conditions (3.1.9) and (3.1.12), we have $Az = Sz = Bz = Tz = z$.

Since $Bz = Tz = Az = Sz = z$, we get z is a common fixed point of A, B, S and T.

3.2 Conclusion: From the example (2.6), we prove that the pairs (A, S) and (B, T) are semicompatible and S and T are continuous.

For this, take a sequence $x_n = \frac{1}{2} - \frac{1}{n}$, for $n \ge 1$, then

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} = t$ (Say), then $\lim_{n\to\infty} ASx_n = S(t) = \frac{1}{2}$ and $\lim_{n\to\infty} BTx_n = T(t) = \frac{1}{2}$ so that the pairs (A,S) and (B,T) are semicompatible mappings. Further the condition (3.1.2) holds for the values of $k_1, k_2 \ge 0$, satisfying the condition $0 \le k_1 + 2k_2 < 1$. It can be seen that X is not a complete metric space and it can be easily verified that the associated sequence $Ax_0, Bx_1, Ax_2, Bx_3 \ldots Ax_{2n}, Bx_{2n+1}, \ldots$ converges to the point $\frac{1}{2}$ which is a common fixed point of A, B, S and T. We observe that $\frac{1}{2}$ is the unique common fixed point of A, B, S and T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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