Available online at http://scik.org
Adv. Inequal. Appl. 2019, 2019:2
https://doi.org/10.28919/aia/3785
ISSN: 2050-7461

COUPLED FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS IN FUZZY METRIC SPACE

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Abstract: In this research paper we prove some coupled fixed point theorems for occasionally weakly compatible mappings in fuzzy metric space.

Keywords: occasionally weakly compatible mappings; coupled fixed point, fuzzy metric space.

AMS (2010) Mathematics Classification: 54H25, 47H10.

1. Introduction

Fuzzy set was defined by Zadeh [11]. Fuzzy metric space was introduced by Kramosil and Michalek [7], George and Veermani [3] modified the notion and gave a new notion with the help of continuous t-norms of fuzzy metric spaces. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. The concepts of coupled fixed points and mixed monotone property was recently introduced by Bhaskar and Lakshmikantham [1] they illustrated these

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Received June 29, 2018

results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later these results were extended and generalized by Sedghi et al. [9], Fang [2] and Xin-Qi Hu [10] etc. Fixed point theorems, involving four self-maps, began with the assumption that they are commuted. Sessa [8] weakened the condition of commutativity to that of pairwise weakly commuting. Jungck generalized the notion of weak commutativity to that of pairwise compatible [4] and then pairwise weakly compatible maps [5]. Jungck and Rhoades [6] introduced the concept of occasionally weakly compatible maps.

In this paper we introduce some coupled fixed point theorems for occasionally weakly compatible mappings in fuzzy metric space.

2. Preliminaries

Definition 2.1 A fuzzy set *A* in *X* is a function with domain *X* and values in [0, 1].

Definition 2.2 A binary operation $*: [0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if $*$ is satisfying conditions:

(i) $*$ is an commutative and associative;

(ii) $*$ is continuous;

(iii) $a*1=a$ for all a$\in$ [0,1];

(iv) $a*b\leq c*d$ whenever $a\leq c \, and \, b\leq d$ and $a,b,c,d\in[0,1]$.

Definitions 2.3 A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if *X* is an arbitrary set ,$*$ is a continuous $t-norm$ and *M* is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions, for all $x,y,z\in X, s,t > 0,$

(i) $M(x,y,t)>0;$

(ii) $M(x,y,t)=1 \; if \, and \, only \, if \; x=y;$

(iii) $M(x,y,t) = M(y,x,t)$;

(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;

(v) $M(x,y,\cdot):(0,\infty) \to (0,1]$ is continuous.

Then M is called a *fuzzy metric* on *X*. Then $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.4 Let (X,d) be a metric space. Denote $a * b = ab$ for all $a,b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_d = \frac{t}{t+d(x,y)}.$$

Then $(X, M_d, *)$ is a fuzzy metric space.

Lemma 2.5 Let (X, M,$*$) be a fuzzy metric space. If there exists $q \in (0,1)$ such that M(x, y, qt) $\geq$ M(x ,y ,t) for all x, y $\in$ X and t>0, then x = y.

Definition 2.6 An element $(x,y) \in X \times X$ is called a

(i) Coupled fixed point of the mapping $f: X \times X \to X$ if

$$f(x,y) = x, \quad f(y,x) = y.$$

(ii) Coupled coincidence point of the mapping $f: X \times X \to X$ and $g: X \to X$ if

$$f(x,y) = g(x), \quad f(y,x) = g(y).$$

(iii) Common Coupled coincidence point of the mapping $f: X \times X \to X$ and $g: X \to X$ if

$$x = f(x,y) = g(x), \quad y = f(y,x) = g(y).$$

Definition 2.7 An element $x \in X$ is called a common coupled fixed point of the mappings $f: X \times X \to X$ and $g: X \to X$ if

$$x = f(x, x) = g(x).$$

Definition 2.8 Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four mappings. Then, the pair of maps (B, S) and (A, T) are said to have Common Coupled coincidence point if there exist a, b in X such that $B(a, b) = S(a) = T(a) = A(a, b)$ and $B(b, a) = S(b) = T(b) = A(b, a)$.

Definition 2.9 The mappings $f: X \times X \to X$ and $g: X \to X$ of a set X are occasionally weakly compatible (owc) iff there is a point $(x, y) \in X \times X$ which is a coincidence point of f and g at which f and g commute i.e. (f, g) are occasionally weakly compatible maps iff $f(x, y) = g(x)$, $f(y, x) = g(y)$ implies $gf(x, y) = f(gx, gy)$, $gf(y, x) = f(gy, gx)$ for $(x, y) \in X \times X$.

Example 2.10 Let $(X, \mathcal{F}, *)$ be a fuzzy metric space, where $X = [0,1]$ with $a * b = \min\{a, b\}$ and

$$M(x, y, t) = \begin{cases} \dfrac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Let $f: X \times X \to X$ & $g: X \to X$ be defined by

$$f(x, y) = \frac{2x + y}{2}$$

$$g(x) = \begin{cases} x, & \text{if } 0 \leq x < 1; \\ \dfrac{3}{2}, & \text{if } x \geq 1. \end{cases}$$

Here, (0,0) and (1,1) are two coincidence points of f and g. That is

$$f(0,0) = 0 = g(0), f(1,1) = 1 = g(1) \text{ but } gf(0,0) = 0 = f(g0, g0),\ gf(1,1) \neq f(g1, g1).$$

Thus f and g are owc but not weakly compatible.

Example 2.11 Let $(X, \mathcal{F}, *)$ be a fuzzy metric space, where $X = [0,4]$ with $a * b = \min\{a, b\}$ and

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

Let $f: X \times X \to X$ & $g: X \to X$ be defined by

$$f(x,y) = \frac{x+y}{2}$$

$$g(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le x < 1; \\ \frac{x}{2}, & \text{if } x = 1; \\ 3, & \text{if } 1 < x \le 2; \\ x-1, & \text{if } x > 2. \end{cases}$$

Here, (1,0), (0,1), (2,4) and (4,2) are coincidence points of f and g. That is

$$f(1,0) = \frac{1}{2} = g(1), f(0,1) = \frac{1}{2} = g(0) \text{ and } f(2,4) = 3 = g(2), f(4,2) = 3 = g(4) \text{ but}$$

$$gf(1,0) = \frac{1}{2} = f(g1, g0),\ \ gf(0,1) = \frac{1}{2} = f(g0, g1),\ \ gf(2,4) \ne f(g2, g4),\ \ gf(4,2) \ne f(g4, g2).$$

Thus f and g are owc but not weakly compatible.

3. Main results

Theorem: 3.1 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y), B(u,v), qt) \ge min\{M(Sx, Tu, t), M(A(x,y), Sx, t), M(B(u,v), Tu, t)\}$

for all $x, y, u, v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A, S) and (B, T) are owc, then there exists a unique point x in X such that $A(x,x) = T(x) = B(x,x) = S(x) = x$.

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a, b, a', b' in X such that

$A(a,b)=Sa,\ A(b,a)=Sb$ and

$B(a',b')=Ta',\ \ B(b',a')=Tb'$

We claim that $Sa=Ta'$. If not, by inequality (i) we get

$M(A(a,b),B(a',b'),qt)\geq min\{M(Sa,Ta',t),M(A(a,b),Sa,t),M(B(a',b'),Ta',t)\}$

or $M\left(Sa,Ta',qt\right)\geq min\{M(Sa,Ta',t),M(Sa,Sa,t),M(Ta',Ta',t)\}$

$=min\{M(Sa,Ta',t),1,1\}$

$=M(Sa,Ta',t)$

$\Rightarrow Sa=Ta'$

Therefore $A(a,b)=Ta'=Sa=B(a',b')$

Similarly $A(b,a)=Tb'=Sb=B(b',a')$

Thus the pairs (A,S) and (B,T) have common coincidence points.

Let $A(a,b)=Ta'=Sa=B(a',b')=x$

and $A(b,a)=Tb'=Sb=B(b',a')=y$

Since (A,S) and (B,T) are owc

So $Sx=SA(a,b)=A(Sa,Sb)=A(x,y)$

and $Sy=SA(b,a)=A(Sb,Sa)=A(y,x)$

Also $Tx=TB(a',b')=B(Ta',Tb')=B(x,y)$

and $Ty=TB(b',a')=B(Tb',Ta')=B(y,x)$

Next we show that $x=y$, for this

putting $x=a$, $y=b$, $u=b'$, $v=a'$ in (ii),

$M(x,y,qt)=M(A(a,b),B(b',a'),qt)\geq min\{M(Sa,Tb',t),M(A(a,b),Sa,t),M(B(b',a'),Tb',t)\}$

$$= min\{M(Sa, Tb', t), M(Sa, Sa, t), M(Tb', Tb', t)\}$$

$$= M(Sa, Tb', t)$$

$$= M(x, y, t)$$

$\Rightarrow x = y$

Now we prove that $Sx = Tx$

$$M(Sx, Tx, qt) = M(Sx, Ty, qt) = M(A(x, y), B(y, x), qt)$$
$$\geq min\{M(Sx, Ty, t), M(A(x, y), Sx, t), M(B(y, x), Ty, t)\}$$

$$= min\{M(Sx, Tx, t), M(Sx, Sx, t), M(Ty, Ty, t)\}$$

$$= min\{M(Sx, Tx, t), 1, 1\}$$

$$= M(Sx, Tx, t)$$

$\Rightarrow Sx = Tx$

Also by condition (ii) we have,

$x = B(x, x)$

Thus $A(x, x) = T(x) = B(x, x) = S(x) = x.$

Example 3.1.1 Let $X = [0,1]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0,1]$, define

$$M(x, y, t) = \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, *)$ be a fuzzy metric space, with $a * b = \min\{a, b\}$. Let $S, T: X \to X$ and $A, B: X \times X \to X$ defined by

$$A(x,y) = \frac{2x+y}{2}$$

$$S(X) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ \frac{3}{2}, & \text{if } x \ge 1. \end{cases}$$

$$B(x,y) = y$$

$$T(X) = \begin{cases} x, & \text{if } 0 \le x < 1; \\ 5, & \text{if } x \ge 1. \end{cases}$$

Clearly all the conditions of the above theorem are satisfied. Also

$SA(0,0) = A(S0, S0)$ and $BT(0,0) = B(T0, T0)$

So, (A, S) and (B, T) are owc maps and (0,0) is the common coupled fixed point of A, B, S and T.

Theorem: 3.2 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y), B(u,v), qt) \geq$

$$min\{M(Sx, Tu, t), M(A(x,y), Sx, t), M(A(x,y), Tu, t), M(Sx, B(u,v), t), M(B(u,v), Tu, t)\}$$

for all $x, y, u, v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A, S) and (B, T) are owc, then there exists a unique point x in X such that $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.3 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y), B(u,v), qt) \geq M(Sx, Tu, t)$ for all $x, y, u, v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that

$A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.4 Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$ for all $t \in [0,1]$. Let $A,B: X \times X \to X$ and $S,T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y),B(u,v),qt) \geq \frac{1}{3}\{M(Sx,Tu,t) + M(A(x,y),Tu,t), M(Sx,B(u,v),t)\}$

for all $x,y,u,v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that

$A(x,x) = T(x) = B(x,x) = S(x) = x.$

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a, b, a', b' in X such that

$A(a,b) = Sa$, $A(b,a) = Sb$ and

$B(a',b') = Ta'$, $B(b',a') = Tb'$

We claim that $Sa = Ta'$. If not, by inequality (i) we get

$$M(A(a,b),B(a',b'),qt) \geq \frac{1}{3}\{M(Sa,Ta',t) + M(A(a,b),Ta',t) + M(Sa,B(a',b'),t)\}$$

or $M(Sa,Ta',qt) \geq \frac{1}{3}\{M(Sa,Ta',t) + M(Sa,Ta',t) + M(Sa,Ta',t)\}$

$$= M(Sa,Ta',t)$$

$\Rightarrow Sa = Ta'$

Therefore $A(a,b) = Ta' = Sa = B(a',b')$

Similarly $A(b,a) = Tb' = Sb = B(b',a')$

Thus the pairs (A, S) and (B, T) have common coincidence points.

Let $A(a, b) = Ta' = Sa = B(a', b') = x$

and $A(b, a) = Tb' = Sb = B(b', a') = y$

Since (A, S) and (B, T) are owc

So $Sx = SA(a, b) = A(Sa, Sb) = A(x, y)$

and $Sy = SA(b, a) = A(Sb, Sa) = A(y, x)$

Also $Tx = TB(a', b') = B(Ta', Tb') = B(x, y)$

and $Ty = TB(b', a') = B(Tb', Ta') = B(y, x)$

Next we show that $x = y$, for this

putting $x = a$, $y = b$, $u = b'$, $v = a'$ in (ii),

$$M(x, y, qt) = M(A(a, b), B(b', a'), qt) \geq \frac{1}{3}\{M(Sa, Tb', t) + M(A(a, b), Tb', t) + M(Sa, B(b', a'), t)\}$$

$$= \frac{1}{3}\{M(Sa, Tb', t) + M(Sa, Tb', t) + M(Sa, Tb', t)\}$$

$$= M(Sa, Tb', t)$$

$$= M(x, y, t)$$

$\Longrightarrow x = y$

Now we prove that $Sx = Tx$

$$M(Sx, Tx, qt) = M(Sx, Ty, qt) = M(A(x, y), B(y, x), qt)$$

$$\geq \frac{1}{3}\{M(Sx, Ty, t) + M(A(x, y), Ty, t) + M(Sx, B(y, x), t)\}$$

$$= \frac{1}{3}\{M(Sx, Ty, t) + M(Sx, Ty, t) + M(Sx, Ty, t)\}$$

$$= M(Sx, Tx, t)$$

$\Longrightarrow Sx = Tx$

Also by condition (ii) we have,

$x = B(x,x)$

Thus $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.5 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $$M(A(x,y), B(u,v), qt) \geq aM(A(x,y), Tu, t) + bM(Sx, B(u,v), t) + c\left[\frac{3M(Sx,Tu,t)}{M(A(x,y),Sx,t)+M(B(u,v),Tu,t)+1}\right]$$

where $(a + b + c) \geq 1$ for all $x, y, u, v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A, S) and (B, T) are owc, then there exists a unique point x in X such that

$A(x,x) = T(x) = B(x,x) = S(x) = x.$

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a, b, a', b' in X such that

$A(a,b) = Sa, \ A(b,a) = Sb$ and

$B(a',b') = Ta', \ \ B(b',a') = Tb'$

We claim that $Sa = Ta'$. If not, by inequality (i) we get

$$M(A(a,b), B(a',b'), qt) \geq aM(A(a,b), Ta', t) + bM(Sa, B(a',b'), t) + c\left[\frac{3M(Sa,Ta',t)}{M(A(a,b),Sa,t)+M(B(a',b'),Ta',t)+1}\right]$$

or $M(Sa, Ta', qt) \geq aM(Sa, Ta', t) + bM(Sa, Ta', t) + c\left[\frac{3M(Sa,Ta',t)}{M(Sa,Sa,t)+M(Ta',Ta',t)+1}\right]$

$$= aM(Sa, Ta', t) + bM(Sa, Ta', t) + c\left[\frac{3M(Sa, Ta', t)}{3}\right]$$

$$= (a + b + c)M(Sa, Ta', t) \ , \qquad \text{where } (a + b + c) \geq 1$$

$\Rightarrow Sa = Ta'$

Therefore $A(a, b) = Ta' = Sa = B(a', b')$

Similarly $A(b, a) = Tb' = Sb = B(b', a')$

Thus the pairs (A, S) and (B, T) have common coincidence points.

Let $A(a, b) = Ta' = Sa = B(a', b') = x$

and $A(b, a) = Tb' = Sb = B(b', a') = y$

Since (A, S) and (B, T) are owc

So $Sx = SA(a, b) = A(Sa, Sb) = A(x, y)$

and $Sy = SA(b, a) = A(Sb, Sa) = A(y, x)$

Also $Tx = TB(a', b') = B(Ta', Tb') = B(x, y)$

and $Ty = TB(b', a') = B(Tb', Ta') = B(y, x)$

Next we show that $x = y$, for this

putting $x = a$, $y = b$, $u = b'$, $v = a'$ in (ii),

$$M(x, y, qt) = M(A(a, b), B(b', a'), qt)$$
$$\geq aM(A(a, b), Tb', t) + bM(Sa, B(b', a'), t) + c\left[\frac{3M(Sa, Tb', t)}{M(A(a, b), Sa, t) + M(B(b', a'), Tb', t) + 1}\right]$$

$$= aM(Sa, Tb', t) + bM(Sa, Tb', t) + c\left[\frac{3M(Sa, Tb', t)}{M(Sa, Sa, t) + M(Tb', Tb', t) + 1}\right]$$

$$= aM(Sa,Tb',t) + bM(Sa,Tb',t) + c\left[\frac{3M(Sa,Tb',t)}{3}\right]$$

$$= (a+b+c)M(Sa,Tb',t)$$

$$= (a+b+c)M(x,y,t), \qquad \text{where } (a+b+c) \geq 1$$

$\Longrightarrow x = y$

Now we prove that $Sx = Tx$

$$M(Sx,Tx,qt) = M(Sx,Ty,qt) = M(A(x,y),B(y,x),qt)$$
$$\geq aM(A(x,y),Ty,t) + bM(Sx,B(y,x),t)$$
$$+ c\left[\frac{3M(Sx,Ty,t)}{M(A(x,y),Sx,t) + M(B(y,x),Ty,t) + 1}\right]$$

$$= aM(Sx,Ty,t) + bM(Sx,Ty,t) + c\left[\frac{3M(Sx,Ty,t)}{M(Sx,Sx,t) + M(Ty,Ty,t) + 1}\right]$$

$$= aM(Sx,Ty,t) + bM(Sx,Ty,t) + c\left[\frac{3M(Sx,Ty,t)}{3}\right]$$

$$= (a+b+c)M(Sx,Tx,t), \qquad \text{where } (a+b+c) \geq 1$$

$\Longrightarrow Sx = Tx$

Also by condition (ii) we have,

$x = B(x,x)$

Thus $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.6 Let $(X,M,*)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A,B: X \times X \to X$ and $S,T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y),B(u,v),qt) \geq \alpha M(Sx,Tu,t) + \beta M(Tu,A(x,y),t) + \gamma M(B(u,v),Sx,t)$

where $(\alpha+\beta+\gamma) \geq 1$, for all $x,y,u,v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that

$A(x,x) = T(x) = B(x,x) = S(x) = x.$

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a, b, a', b' in X such that

$A(a,b) = Sa,\ A(b,a) = Sb$ and

$B(a',b') = Ta',\ \ B(b',a') = Tb'$

We claim that $Sa = Ta'$. If not, by inequality (i) we get

$M(A(a,b), B(a',b'), qt) \geq \alpha M(Sa, Ta', t) + \beta M(Ta', A(a,b), t) + \gamma M(B(a',b'), Sa, t)$

or $M\big(Sa, Ta', qt\big) \geq \alpha M(Sa, Ta', t) + \beta M(Ta', Sa, t) + \gamma M(Ta', Sa, t)$

$= (\alpha + \beta + \gamma) M(Sa, Ta', t),$ where $(\alpha + \beta + \gamma) \geq 1$

$\Rightarrow Sa = Ta'$

Therefore $A(a,b) = Ta' = Sa = B(a',b')$

Similarly $A(b,a) = Tb' = Sb = B(b',a')$

Thus the pairs (A,S) and (B,T) have common coincidence points.

Let $A(a,b) = Ta' = Sa = B(a',b') = x$

and $A(b,a) = Tb' = Sb = B(b',a') = y$

Since (A,S) and (B,T) are owc

So $Sx = SA(a,b) = A(Sa,Sb) = A(x,y)$

and $Sy = SA(b,a) = A(Sb,Sa) = A(y,x)$

Also $Tx = TB(a',b') = B(Ta',Tb') = B(x,y)$

and $Ty = TB(b',a') = B(Tb',Ta') = B(y,x)$

Next we show that $x = y$, for this

putting $x = a$, $y = b$, $u = b'$, $v = a'$ in (ii),

$$M(x,y,qt) = M(A(a,b),B(b',a'),qt) \geq \alpha M(Sa,Tb',t) + \beta M(Tb',A(a,b),t) + \gamma M(B(b',a'),Sa,t)$$

$$= \alpha M(Sa,Tb',t) + \beta M(Tb',Sa,t) + \gamma M(Tb',Sa,t)$$

$$= (\alpha+\beta+\gamma) M(Sa,Tb',t)$$

$$= (\alpha+\beta+\gamma) M(x,y,t), \quad \text{where } (\alpha+\beta+\gamma) \geq 1$$

$$\Rightarrow x = y$$

Now we prove that $Sx = Tx$

$$M(Sx,Tx,qt) = M(Sx,Ty,qt) = M(A(x,y),B(y,x),qt) \geq$$

$$\alpha M(Sx,Ty,t) + \beta M(Ty,A(x,y),t) + \gamma M(B(y,x),Sx,t)$$

$$= \alpha M(Sx,Ty,t) + \beta M(Ty,Sx,t) + \gamma M(Ty,Sx,t)$$

$$= (\alpha+\beta+\gamma) M(Sx,Tx,t), \quad \text{where } (\alpha+\beta+\gamma) \geq 1$$

$$\Rightarrow Sx = Tx$$

Also by condition (ii) we have,

$$x = B(x,x)$$

Thus $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.7 Let $(X,M,*)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A,B: X \times X \to X$ and $S,T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y),B(u,v),qt) \geq \alpha \min[M(Sx,Ty,t), M(Sx,A(x,y),t)] + \beta \min[M(B(u,v),Tv,t), M(A(x,y),Tu,t)] + \gamma M(B(u,v),Sx,t)$

where $(\alpha+\beta+\gamma) \geq 1$, for all $x,y,u,v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that

$A(x,x) = T(x) = B(x,x) = S(x) = x$

Theorem: 3.8 Let $(X,M,*)$ be a fuzzy metric space with $t*t=t$ for all $t \in [0,1]$. Let $A,B: X \times X \to X$ and $S,T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y),B(u,v),qt) \geq$

$$\psi\{M(Sx,Tu,t), M(Sx,A(x,y),t), M(B(u,v),Tu,t), M(A(x,y),Tu,t), M(Sx,B(u,v),t)\}$$

where $\psi: [0,1]^5 \to [0,1]$ and $\psi(t,1,1,t,t) \geq t,$ for all $x,y,u,v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that

$A(x,x) = T(x) = B(x,x) = S(x) = x.$

Proof: Since the pairs (A,S) and (B,T) are owc so there are points a,b,a',b' in X such that

$A(a,b) = Sa,\ \ A(b,a) = Sb$ and

$B(a',b') = Ta',\ \ B(b',a') = Tb'$

We claim that $Sa = Ta'$. If not, by inequality (i) we get

$M(A(a,b),B(a',b'),qt) \geq$
$\psi\{M(Sa,Ta',t), M(Sa,A(a,b),t), M(B(a',b'),Ta',t), M(A(a,b),Ta',t), M(Sa,B(a',b'),t)\}$

or $M(Sa,Ta',qt) \geq \psi\{M(Sa,Ta',t), M(Sa,Sa,t), M(Ta',Ta',t), MSa,Ta',t), M(Sa,Ta',t)\}$

$$\geq M(Sa,Ta',t)$$

$\Rightarrow Sa = Ta'$

Therefore $A(a,b) = Ta' = Sa = B(a',b')$

Similarly $A(b,a) = Tb' = Sb = B(b',a')$

Thus the pairs (A,S) and (B,T) have common coincidence points.

Let $A(a,b) = Ta' = Sa = B(a',b') = x$

and $A(b,a) = Tb' = Sb = B(b',a') = y$

Since (A,S) and (B,T) are owc

So $Sx = SA(a,b) = A(Sa,Sb) = A(x,y)$

and $Sy = SA(b,a) = A(Sb,Sa) = A(y,x)$

Also $Tx = TB(a',b') = B(Ta',Tb') = B(x,y)$

and $Ty = TB(b',a') = B(Tb',Ta') = B(y,x)$

Next we show that $x = y$, for this

putting $x = a$, $y = b$, $u = b'$, $v = a'$ in (ii),

$M(x,y,qt) = M(A(a,b),B(b',a'),qt) \geq$

$\psi\left\{M\left(Sa,Tb',t\right), M(Sa,A(a,b),t), M\left(B\left(b',a'\right),Tb',t\right), M(A(a,b),Tb',t), M\left(Sa,B\left(b',a'\right),t\right)\right\}$

$= \psi\{M(Sa,Tb',t), M(Sa,Sa,t), M(Tb',Tb',t), M(Sa,Tb',t), M(Sa,Tb',t)\}$

$\geq M\left(Sa,Tb',t\right)$

$= M(x,y,t)$

$\Rightarrow x = y$

Now we prove that $Sx = Tx$

$M(Sx,Tx,qt) = M(Sx,Ty,qt) = M(A(x,y),B(y,x),qt)$
$\geq \psi\{M(Sx,Ty,t), M(Sx,A(x,y),t), M(B(y,x),Ty,t), M(A(x,y),Ty,t), M(Sx,B(y,x),t)\}$

$= \psi\{M(Sx,Ty,t), M(Sx,Sx,t), M(Ty,Ty,t), M(Sx,Ty,t), M(Sx,Ty,t)\}$

$\geq M(Sx,Tx,t)$

$\Rightarrow Sx = Tx$

Also by condition (ii) we have,

$x = B(x,x)$

Thus $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.9 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y), B(u,v), qt) \geq \psi\{M(Sx, Tu, t), M(A(x,y), Tu, t), M(Sx, B(u,v), t)\}$

where $\psi: [0,1]^3 \to [0,1]$ and $\psi(t,t,t) \geq t,$ for all $x, y, u, v \in X$

(ii) $y = B(x,y)$

Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Theorem: 3.10 Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$. Let $A, B: X \times X \to X$ and $S, T: X \to X$ be four self-mappings satisfying the following conditions:

(i) $M(A(x,y), B(u,v), qt) \geq$

$\psi max\{M(Sx, Tu, t), M(Sx, A(x,y), t), M(B(u,v), Tu, t), M(A(x,y), Tu, t), M(Sx, B(u,v), t)\}$

where $\psi: [0,1] \to [0,1]$ and $\psi(t) \geq t,$ for all $x, y, u, v \in X$

(ii) $y = B(x,y)$

(iii) Moreover if the pairs (A,S) and (B,T) are owc, then there exists a unique point x in X such that $A(x,x) = T(x) = B(x,x) = S(x) = x.$

Conflict of Interests

The authors declare that there is no conflict of interests.

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