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THE ASYMPTOTIC ESTIMATES OF THE NUMBER OF SUBSETS OF $\{1,2,...,m\}$ R-RELATIVELY PRIME TO N

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Abstract. In this paper, Let A be a nonempty subset of $\{1,2,...,n\}$ of positive integers. A is r-relatively prime if greatest r^{th} power common divisor of elements of A is 1. In this case we write $gcd_r(A) = 1$. A is r-relatively prime to n if greatest r^{th} power common divisor of elements of A and n is 1. It is denoted as $gcd_r(A\cup\{n\}) = 1$. For positive integers k,l,m,n we write $[l,m] = \{l,l+1,...,m\}$ and for $l \leq m \leq n$, let $\Phi^{(r)}([l,m],n)$ be the number of subsets of $[l,m]$ which are r-relatively prime to n. The number of sets in $\Phi^{(r)}([l,m],n)$ of cardinality k is $\Phi_k^{(r)}([l,m],n)$. In the present work we consider the case for $l = 1$. That is we obtain the formulae for the functions $\Phi^{(r)}([1,m],n)$ and $\Phi_k^{(r)}([1,m],n)$. We also obtain the exact formulae for the functions $U^{(r)}(m,n)$ which denotes the number of subsets of $[1,n]$, having the elements in both the sets $[1,m]$ and $[m,n]$ which are r-relatively prime to n and the number of sets in $U^{(r)}(m,n)$ of cardinality k is $U_k^{(r)}(m,n)$.

Keywords: r-relatively prime sets; $\Phi^{(r)}([l,m],n)$; $\Phi_k^{(r)}([l,m],n)$.

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1. Introduction

A nonempty set $A \subseteq \{1,2,...,n\}$ of positive integers is said to be relatively prime if gcd of elements of A is 1, and that A is relatively prime to n if gcd of elements of A and n is 1. A is r-relatively prime if greatest r^{th} power common divisor of elements of A is 1. In this case we write $gcd_r(A) = 1$. A is r-relatively prime to n if greatest r^{th} power common divisor of elements of A and n is 1. It is denoted as $gcd_r(A \cup \{n\}) = 1$. For positive integers $l \leq m \leq n$, let $[l,m] = \{l, l+1, ..., m\}$ and $[x]$ denotes the floor of x . For non-negative integers $0 \leq M \leq N$ we have,

$$\sum_{j=k}^{N} \binom{j}{k} = \binom{N+1}{k+1}. \tag{1}$$

2. Preliminaries

In [1] we obtained

$$\Phi^{(r)}([1,n],n) = \sum_{d^r|n} \mu_r(d^r) \left(2^{\frac{n}{d^r}} - 1 \right) \tag{2}$$

$$\Phi_k^{(r)}([1,n],n) = \sum_{d^r|n} \mu_r(d^r) \binom{\frac{n}{d^r}}{k} \tag{3}$$

In [2] we obtained

$$\Phi^{(r)}([m,n],n) = \sum_{d^r|n} \mu_r(d^r) 2^{\frac{n}{d^r}} - \sum_{i=1}^{m-1} \sum_{d^r|gcd_r(i,n)} \mu_r(d^r) 2^{\frac{n-i}{d^r}} \tag{4}$$

$$\Phi_k^{(r)}([m,n],n) = \sum_{d^r|n} \mu_r(d^r) \binom{\frac{n}{d^r}}{k} - \sum_{d^r|n} \mu_r(d^r) \sum_{\substack{i=1 \\ d^r|i}}^{m-1} \binom{\frac{n-i}{d^r}}{k-1} \tag{5}$$

In [3] we proved that

$$\Phi^{(r)}([m,n],n) = \sum_{d^r|n} \mu_r(d^r) \left(2^{\frac{n}{d^r} - \left[\frac{m-1}{d^r}\right]} \right) \tag{6}$$

(7) $$\Phi_k^{(r)}([m,n],n) = \sum_{d^r|n} \mu_r(d^r) \binom{\frac{n}{d^r} - \left[\frac{m-1}{d^r}\right]}{k}$$

Lemma 1. Let

$$\Psi^{(r)}([1,m],n) = \sharp\{A \subseteq \{1,2,...,m\} : m\varepsilon A\ , gcd_r(A \cup \{n\}) = 1\}$$

$$\Psi_k^{(r)}([1,m],n) = \sharp\{A \subseteq \{1,2,...,m\} : \sharp A = k, gcd_r(A \cup \{n\}) = 1\}$$

Then

$$(i)\ \Psi^{(r)}([1,m],n) = \sum_{d^r|(m,n)_r} \mu_r(d^r) \left(2^{\frac{m}{d^r}-1}\right)$$

$$(ii)\ \Psi_k^{(r)}([1,m],n) = \sum_{d^r|(m,n)_r} \mu_r(d^r) \binom{\frac{m}{d^r} - 1}{k-1}$$

Proof. (i) Let $\alpha(S)$ be the set of subsets of [1,m] containing m and let $\alpha(m,d^r)$ be the set of subsets A of [1,m] such that $m\varepsilon A$ and $gcd_r(A \cup \{n\}) = d^r$. The set $\alpha(S)$ of cardinality 2^{m-1} can be partitioned using the equivalence relation of having the same r^{th} power gcd. We can see that the mapping $A \to \frac{1}{d^r}A$ is a 1-1 correspondence between the subsets of $\alpha(m,d^r)$ and the set of subsets C of $\left[1, \frac{m}{d^r}\right]$ containing $\frac{m}{d^r}$ and $gcd_r\left(C \cup \left\{\frac{n}{d^r}\right\}\right) = 1$. Then

$$\sharp\alpha(m,d^r) = \Psi^{(r)}\left(\left[1, \frac{m}{d^r}\right], n\right).$$

Thus

$$2^{m-1} = \sum_{d^r|(m,n)_r} \sharp\alpha(m,d^r) = \sum_{d^r|(m,n)_r} \Psi^{(r)}\left(\left[1, \frac{m}{d^r}\right], \frac{n}{d^r}\right)$$

By Mobius inversion formula ,we get

$$\Psi^{(r)}\left(\left[1, \frac{m}{d^r}\right], \frac{n}{d^r}\right) = \sum_{d^r|(m,n)_r} \mu_r(d^r) 2^{\frac{m}{d^r}-1}$$

Which proves (i).

(ii) We note that the correspondence $A \to \frac{1}{d^r}A$ preserves the cardinality and using an argument similar in part (i) we have

$$\binom{m-1}{k-1} = \sum_{d^r|(m,n)_r} \Psi_k^{(r)}\left(\left[1, \frac{m}{d^r}\right], \frac{n}{d^r}\right)$$

.

Using the generalized mobius inversion formula we have

$$\Psi_k^{(r)}([1,m],n) = \sum_{d^r|(m,n)_r} \mu_r(d^r) \binom{\frac{m}{d^r}-1}{k-1}$$

which proves (ii). □

3. Main results

Theorem 2. We have

$$(i)\ \Phi^{(r)}([1,m],n) = \sum_{d^r|n} \mu_r(d^r) \left(2^{\left[\frac{m}{d^r}\right]} - 1 \right)$$

$$(ii)\ \Phi_k^{(r)}([1,m],n) = \sum_{d^r|n} \mu_r(d^r) \binom{\left[\frac{m}{d^r}\right]}{k}$$

Proof. (i) From Lemma(1) and equation (2), we get

$$\Phi^{(r)}([1,m],n) = \sharp\{A \subseteq [1,m] : A \neq \emptyset, gcd_r(A \cup \{n\}) = 1\}$$

$$\Phi^{(r)}([1,m+1],n) = \sharp\{A \subseteq [1,m+1] : A \neq \emptyset, gcd_r(A \cup \{n\}) = 1\}$$

$$\Psi^{(r)}([1,m+1],n) = \sharp\{A \subseteq [1,m+1] : m+1 \varepsilon A\ , gcd_r(A \cup \{n\}) = 1\}$$

Therefore

$\Phi^{(r)}([1,m],n) = \Phi^{(r)}([1,m+1],n) - \Psi^{(r)}([1,m+1],n)$

$= \Phi^{(r)}([1,m+2],n) - \left[\Psi^{(r)}([1,m+2],n) + \Psi^{(r)}([1,m+1],n)\right]$

$$\vdots$$

$$= \Phi^{(r)}([1,n],n) - \sum_{i=1}^{n-m} \Psi^{(r)}([1,m+i],n)$$

$$= \sum_{d^r|n} \mu_r(d^r) \left(2^{\frac{n}{d^r}} - 1 \right) - \sum_{i=1}^{n-m} \sum_{d^r|(m+i,n)_r} \mu_r(d^r) \left(2^{\frac{m+i}{d^r}-1} \right)$$

$$= \sum_{d^r|n} \mu_r(d^r) 2^{\frac{n}{d^r}} - \sum_{d^r|n} \mu_r(d^r) - \sum_{d^r|n} \sum_{\substack{i=1 \\ d^r|m+i}}^{n-m} \mu_r(d^r) \left(2^{\frac{m+i}{d^r}-1} \right)$$

$$= \sum_{d^r|n} \mu_r(d^r)\left(2^{\frac{n}{d^r}}-1\right) - \sum_{d^r|n} \mu_r(d^r)\left(\sum_{j=\left[\frac{m}{d^r}\right]+1}^{\frac{n}{d^r}} 2^{j-1}\right)$$

$$= \sum_{d^r|n} \mu_r(d^r)2^{\frac{n}{d^r}} - \sum_{d^r|n} \mu_r(d^r) - \sum_{d^r|n} \mu_r(d^r) \sum_{\substack{i=1 \\ d^r|m+i}}^{n-m} \mu_r(d^r)\left(2^{\frac{m+i}{d^r}-1}\right)$$

$$= \sum_{d^r|n} \mu_r(d^r)\left(2^{\frac{n}{d^r}}-1\right) - \sum_{d^r|n} \mu_r(d^r)\left[2^{\left[\frac{m}{d^r}\right]} + 2^{\left[\frac{m}{d^r}\right]+1} + \ldots + 2^{\frac{n}{d^r}-1}\right]$$

$$= \sum_{d^r|n} \mu_r(d^r)\left(2^{\frac{n}{d^r}}-1\right) - \sum_{d^r|n} \mu_r(d^r)2^{\left[\frac{m}{d^r}\right]}\left[1+2+\ldots+2^{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]-1}\right]$$

$$= \sum_{d^r|n} \mu_r(d^r)\left(2^{\frac{n}{d^r}}-1\right) - \sum_{d^r|n} \mu_r(d^r)2^{\left[\frac{m}{d^r}\right]}\left[2^{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]}-1\right]$$

$$= \sum_{d^r|n} \mu_r(d^r)\left(2^{\frac{n}{d^r}}-1\right) - \sum_{d^r|n} \mu_r(d^r)2^{\frac{n}{d^r}} + \sum_{d^r|n} \mu_r(d^r)2^{\left[\frac{m}{d^r}\right]}$$

$= \sum_{d^r|n} \mu_r(d^r)\left[2^{\left[\frac{m}{d^r}\right]}-1\right]$.Which proves (i)

(ii) From Lemma(1) and equation (3), we get

$\Phi_k^{(r)}([1,m],n) = \Phi_k^{(r)}([1,m+1],n) - \Psi_k^{(r)}([1,m+1],n)$
$= \Phi_k^{(r)}([1,m+2],n) - \left[\Psi_k^{(r)}([1,m+2],n) + \Psi_k^{(r)}([1,m+1],n)\right]$

$$\vdots$$

$$= \Phi_k^{(r)}([1,n],n) - \sum_{i=1}^{n-m} \Psi_k^{(r)}([1,m+i],n)$$

$$= \sum_{d^r|n} \mu_r(d^r) \binom{\frac{n}{d^r}}{k} - \sum_{i=1}^{n-m} \sum_{d^r|(m,n)_r} \mu_r(d^r) \binom{\frac{m+i}{d^r} - 1}{k}$$

$$= \sum_{d^r|n} \mu_r(d^r) \binom{\frac{n}{d^r}}{k} - \sum_{d^r|n} \mu_r(d^r) \sum_{\substack{i=1 \\ d^r|m+i}}^{n-m} \mu_r(d^r) \binom{\frac{m+i}{d^r} - 1}{k-1}$$

$$= \sum_{d^r|n} \mu_r(d^r) \left[\binom{\frac{n}{d^r}}{k} - \sum_{j=\left[\frac{m}{d^r}\right]+1}^{\frac{n}{d^r}} \binom{j-1}{k-1} \right]$$

$$= \sum_{d^r|n} \mu_r(d^r) \left[\binom{\frac{n}{d^r}}{k} - \sum_{j=1}^{\frac{n}{d^r}} \binom{j-1}{k-1} + \sum_{j=1}^{\left[\frac{m}{d^r}\right]} \binom{j-1}{k-1} \right]$$

$$= \sum_{d^r|n} \mu_r(d^r) \left[\binom{\frac{n}{d^r}}{k} - \binom{\frac{n}{d^r}}{k} + \binom{\left[\frac{m}{d^r}\right]}{k} \right]$$

$$= \sum_{d^r|n} \mu_r(d^r) \binom{\left[\frac{m}{d^r}\right]}{k}$$

(by equation (1)). Which proves (ii). □

Corollary 3. When $U^{(r)}(m,n)$ denote number of nonempty subsets of $[1,n]$ having elements in both the sets $[1,m]$ and $[m,n]$ which are r-relatively prime to n and $U_k^{(r)}(m,n)$ be the number of sets in $U^{(r)}(m,n)$ of cardinality k , then

$$(i)\ U^{(r)}(m,n) = \sum_{d^r|n} \mu_r(d^r) \left(2^{\frac{n}{d^r}} - 2^{\left[\frac{m-1}{d^r}\right]} - 2^{\frac{n}{d^r} - \left[\frac{m}{d^r}\right]} \right)$$

$$(ii)\ U_k^{(r)}(m,n) = \sum_{d^r|n} \mu_r(d^r) \left(\binom{\frac{n}{d^r}}{k} - \binom{\left[\frac{m-1}{d^r}\right]}{k} - \binom{\frac{n}{d^r} - \left[\frac{m}{d^r}\right]}{k} \right)$$

Proof. we have

$$U^{(r)}(m,n)=\Phi^{(r)}([1,n],n)-\Phi^{(r)}([1,m-1],n)-\Phi^{(r)}([m+1,n],n)$$

$$=\sum_{d^r|n}\mu_r(d^r)\left(2^{\frac{n}{d^r}}-1\right)-\sum_{d^r|n}\mu_r(d^r)\left(2^{\left[\frac{m-1}{d^r}\right]}-1\right)-\sum_{d^r|n}\mu_r(d^r)\left(2^{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]}\right)$$

(By equations (2) and (6))

$$=\sum_{d^r|n}\mu_r(d^r)\left(2^{\frac{n}{d^r}}-1-2^{\left[\frac{m-1}{d^r}\right]}+1-2^{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]}\right)$$

$$=\sum_{d^r|n}\mu_r(d^r)\left(2^{\frac{n}{d^r}}+2^{\left[\frac{m-1}{d^r}\right]}-2^{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]}\right)$$

which proves (i).

(ii) We have

$$U_k^{(r)}(m,n)=\Phi_k^{(r)}([1,n],n)-\Phi_k^{(r)}([1,m-1],n)-\Phi_k^{(r)}([m+1,n],n)$$

$$=\sum_{d^r|n}\mu_r(d^r)\binom{\frac{n}{d^r}}{k}-\sum_{d^r|n}\mu_r(d^r)\binom{\left[\frac{m-1}{d^r}\right]}{k}-\sum_{d^r|n}\mu_r(d^r)\binom{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]}{k}$$

(By equations (3) and (7))

$$=\sum_{d^r|n}\mu_r(d^r)\left(\binom{\frac{n}{d^r}}{k}-\binom{\left[\frac{m-1}{d^r}\right]}{k}-\binom{\frac{n}{d^r}-\left[\frac{m}{d^r}\right]}{k}\right)$$

which proves (ii). □

Asymptotic Estimates for $\Phi^{(r)}([1,m],n)$ and $\Phi_k^{(r)}([1,m],n)$:

Theorem 4. Let the smallest r-prime power divisor of n in $[1,m]$ is p^r. Then

$$(i)\ 0\le 2^m-2^{\left[\frac{m}{p^r}\right]}-\Phi^{(r)}([1,m],n)\le m\left(2^{\left[\frac{m}{p^r}\right]}-1\right)$$

$$(ii)\ 0 \leq \binom{m}{k} - \binom{\left[\frac{m}{p^r}\right]}{k} - \Phi_k^{(r)}([1,m],n) \leq m \binom{\left[\frac{m}{p^r}\right]}{k}$$

Proof. (i) The upper bound for the function $\Phi^{(r)}([1,m],n)$ is

obtained by deleting the subsets of $[1,m]$ consisting of multiples of p^r. So we have

$$\Phi^{(r)}([1,m],n) \leq 2^m - 2^{\left[\frac{m}{p^r}\right]}.$$

As for the lower bound we have

$$\Phi^{(r)}([1,m],n) = \sum_{d^r|n} \mu_r(d^r) \left(2^{\left[\frac{m}{d^r}\right]} - 1\right)$$

$$= \mu_r(1)\,(2^m - 1) + \mu_r(p^r)\left(2^{\left[\frac{m}{p^r}\right]} - 1\right) + \sum_{d^r|n,d>p} \mu_r(d^r)\left(2^{\left[\frac{m}{d^r}\right]} - 1\right)$$

$$= (2^m - 1) - \left(2^{\left[\frac{m}{p^r}\right]} - 1\right) + \sum_{d^r|n,d>p} \mu_r(d^r)\left(2^{\left[\frac{m}{d^r}\right]} - 1\right)$$

$$= 2^m - 2^{\left[\frac{m}{d^r}\right]} + \sum_{d^r|n,d>p} \mu_r(d^r)\left(2^{\left[\frac{m}{d^r}\right]} - 1\right)$$

$$\therefore \Phi^{(r)}([1,m],n) - 2^m + 2^{\left[\frac{m}{d^r}\right]} = \sum_{d^r|n,d>p} \mu_r(d^r)\left(2^{\left[\frac{m}{p^r}\right]} - 1\right)$$

$$\geq - \sum_{d^r|n,d>p} \left(2^{\left[\frac{m}{d^r}\right]} - 1\right)$$

$$\geq \left(2^{\left[\frac{m}{p^r}\right]} - 1\right) \sum_{d^r|n} 1$$

$$\geq -m\left(2^{\left[\frac{m}{p^r}\right]} - 1\right)$$

$$\Rightarrow 2^m + 2^{\left[\frac{m}{d^r}\right]} - \Phi^{(r)}([1,m],n) \leq m\left(2^{\left[\frac{m}{p^r}\right]} - 1\right)$$

which proves (i).

(ii) The number of subsets of cardinality k of $\{1,2,...,m\}$ consisting of multiples of p^r is

$$\binom{\left[\frac{m}{p^r}\right]}{k}.$$

Hence

$$\Phi_k^{(r)}([1,m],n) \leq \binom{m}{k} - \binom{\left[\frac{m}{p^r}\right]}{k}$$

By theorem (2)

$$\Phi_k^{(r)}([1,m],n) = \sum_{d^r|n} \mu_r(d^r)\binom{\left[\frac{m}{d^r}\right]}{k}$$

$$= \mu_r(1)\binom{m}{k} + \mu_r(p^r)\binom{\left[\frac{m}{p^r}\right]}{k} + \sum_{d^r|n,d>p} \mu_r(d^r)\binom{\left[\frac{m}{d^r}\right]}{k}$$

$$= \binom{m}{k} - \binom{\left[\frac{m}{p^r}\right]}{k} + \sum_{d^r|n,d>p} \mu_r(d^r)\binom{\left[\frac{m}{d^r}\right]}{k}$$

$$\geq \binom{m}{k} - \binom{\left[\frac{m}{p^r}\right]}{k} - \sum_{d^r|n,d>p} \mu_r(d^r)\binom{\left[\frac{m}{d^r}\right]}{k}$$

$$\geq \binom{m}{k} - \binom{\left[\frac{m}{p^r}\right]}{k} - m\binom{\left[\frac{m}{p^r}\right]}{k}$$

$$\therefore 0 \leq \binom{m}{k} - \binom{\left[\frac{m}{p^r}\right]}{k} - \Phi_k^{(r)}([1,m],n) \leq m\binom{\left[\frac{m}{p^r}\right]}{k}$$

which proves (ii).

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] Melvyn B. Nathanson, Affine invariants, relatively prime sets, and a phi function for subsets of $\{1,2,...,n\}$, Integer, 7 (2007), Article ID A01.

[2] Melvyn B. Nathanson, Asymptotic Estimates for Phi Functions for Subsets of $\{M,M+1,...,N\}$, Integers, 7(2007), Article ID A54.

[3] Mohamed El Bachraoui, The Number of relatively prime subsets and phi functions for $\{m,m+1,...,n\}$, Integer, 7(2007), Article ID A43.

[4] G.Kamala,G.Lalitha, The Number of r-Relatively Prime Subsets and Their Phi Functions for $\{m,m+1,...,n\}$, Ann. Pure Appl. Math. 14(3)(2017), 581-592.

[5] G. Kamala, G. Lalitha, r-Relatively Prime Sets and r-Generalization of Phi Functions for Subsets of $\{1,2,...,n\}$, Ann. Pure Appl. Math. 14(3)(2017), 497-508.

[6] G. Lalitha, G. Kamala, r-Generalization of Phi Functions for Subsets of $\{m,m+1,...,n\}$, Int. J. Appl. Eng. Res. 13 (18) (2018), 13428-13438.