Available online at http://scik.org
Adv. Inequal. Appl. 2019, 2019:1
https://doi.org/10.28919/aia/3847
ISSN: 2050-7461

FIXED POINT RESULTS FOR RATIONAL α-GERAGHTY CONTRACTIVE MAPPINGS

KOTI N.V.V. VARA PRASAD AND A. K. SINGH*

Department of Pure & Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.), India

Abstract. In this paper, we introduce a notion of rational α-Geraghty contractive mapping in the setting of metric space and establish some fixed point theorems for such maps and give a suitable example to illustrate our results. Also, we discuss application into ordinary differential equations.

Keywords: fixed point; rational α-Geraghty contractive mapping; complete metric space.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

It is well known that contractive mapping principle, formulated and proved in the Ph.D. dissertation of Banach in 1920, which was published in 1922 , is one of the most important theorems in nonlinear analysis. A number of authors have improved, generalized and extend this basic result either by defining a new contractive mapping in the context of a complete metric space or by investigating the existing contractive mappings in various spaces.

*Corresponding author
E-mail address: awnish.singh85@gmail.com
Received August 14, 2018

In 1973, Geraghty [6] introduced an auxiliary function by the generalization of Banach contraction principle in complete metric space. Later on, many researchers [2-3, 8, 11-12] characterized the results of Geraghty in different spaces. In particularly Amini-Harandi and Emami [1] characterized the results of Geraghty in the context of partial order metric space. Caballero et al.[4] discussed the existence of best proximity type Geraghty contraction. Recently, Samet et al.[14], established remarkable fixed point results by define the notion of $\alpha-\psi$ contraction mapping. Very recently, Karapinar and Samet[9] introduced the concept of generalized $\alpha-\psi$ contractive mapping and also established fixed point theorems for such mapping and listed some of the consequence of their main results.

The motivation of above development Cho et al.[5] introduced the concept of α-Geraghty contraction type map in the setting of metric space. After ward, Ovidiu Popescu[13] generalized the result obtained in [5] and established the existence and uniqueness of fixed point theorems of α-Geraghty contraction type map in complete metric space.

Definition 1.1. *Let $T : X \to X$ be a map and $\alpha : X \times X \to R$ be a function, then T is said to be*

(a) an α-`admissible` if $\alpha(x,y) \geq 1$ implies $\alpha(Tx,Ty) \geq 1$[14].

(b) an α-admissible map is said to be `triangular` α-`admissible` map, if $\alpha(x,z) \geq 1$ and $\alpha(z,y) \geq 1$ implies $\alpha(x,y) \geq 1$[10].

(c) an α-`orbital admissible` if $\alpha(x,Tx) \geq 1$ implies $\alpha(Tx,T^2x) \geq 1$[13].

(d) a `triangular` α-`orbital admissible` if $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$ implies $\alpha(x,Ty) \geq 1$[13].

Remark: It is very clear that every α -admissible mapping is an α-orbitally admissible mapping and also every triangular α -admissible mapping is a triangular α-orbital admissible mapping, its converse is not hold[13].

Lemma 1.1. [10] *Let $T : X \to X$ be a triangular α-admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $T_{n+1} = Tx_n$, then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \in \mathbb{N}$ with $n < m$.*

Let $\mathscr{T}$ be the family of all functions $\beta:[0,\infty)\to[0,1)$ which satisfies the condition $\lim_{n\to\infty}\beta(t_n)=1$ implies $\lim_{n\to\infty}t_n=0$.

By using the auxiliary function $\beta\in\mathscr{T}$, Geraghty[6] introduced an interesting contraction and investigated the existence and uniqueness of such mappings.

Theorem 1.2. [6] *Let (X,d) be a complete metric space and let T be a mapping on X. Suppose that there exists $\beta\in\mathscr{T}$ such that for all $x,y\in X$,*

$$d(Tx,Ty)\leq\beta(d(x,y))d(x,y).$$

Then T has a fixed point $z\in X$ and $\{T^n x\}$ converges to z.

Definition 1.3. [14] Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized α-Geraghty contraction type map if there exists $\beta\in\mathscr{T}$ such that for all $x,y\in X$,

$$\alpha(x,y)d(Tx,Ty)\leq\beta(M(x,y))M(x,y),$$

where $M(x,y)=max\{d(x,y),d(x,Tx),d(y,Ty)\}$.

The aim of this work: In this paper, we introduce the concept of rational α-Geraghty contractive mapping and establish some fixed point theorems in a complete metric space. We give an example to illustrate our result and consider an application of our result in the area of ordinary differential equations.

2. Main Results

Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A self map T on X is called a ***rational α-Geraghty contractive mapping*** if there exists $\beta\in\mathscr{T}$ such that for all $x,y\in X$,

$$\alpha(x,y)d(Tx,Ty)\leq\beta(M(x,y))M(x,y) \tag{2.1}$$

where $M(x,y)=\max\left\{d(x,y),\dfrac{d(x,Tx)d(y,Ty)}{1+d(x,y)},\dfrac{d(y,Tx)d(x,Ty)}{1+d(Tx,Ty)}\right\}$

Theorem 2.1. *Let (X,d) be a complete metric space , $\alpha:X\times X\to\mathbb{R}$ be a function, and let T be a self map on X and satisfying (2.1). Suppose that the following conditions are satisfied :*

(1) *T is a triangular α-orbital admissible mapping;*

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*

(3) *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_1\}$ converges to z.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. A sequence $\{x_n\}$ in X defined by $x_{n+1} = Tx_n$ for $n \geq 1$ and T is rational α-Geraghty contractive mapping. If $x_{n(0)} = x_{n(0)+1}$ for some $n(0) \geq 1$, then it is easy to observe that x_{n_0} is a fixed point of T. Hence, we suppose that $x_n \neq x_{n+1}$ for all $n \geq 1$. By above Lemma 1.1 we have $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$. Then we consider

$$\begin{aligned} d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) &\leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) \\ &\leq \beta(M(x_n, x_{n+1})) M(x_n, x_{n+1}), \end{aligned} \tag{2.2}$$

where

(2.3)

$$\begin{aligned} M(x_n, x_{n+1}) &= max\left\{ d(x_n, x_{n+1}), \frac{d(x_n, Tx_n) d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \frac{d(x_{n+1}, Tx_n) d(x_n, Tx_{n+1})}{1 + d(Tx_n, Tx_{n+1})} \right\} \\ &= max\left\{ d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, \frac{d(x_{n+1}, x_{n+1}) d(x_n, x_{n+2})}{1 + d(x_{n+1}, x_{n+2})} \right\} \\ &\leq max\left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \right\} \end{aligned}$$

Since $\frac{d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \leq 1$.

If $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$, then from (2.1) we get

$$d(x_{n+1}, x_{n+2}) \leq \beta(d(x_{n+1}, x_{n+2})) d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_{n+2}), \ (since\ \beta \in \mathscr{T})$$

which is a contradiction. Then we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}).$$

Thus, the sequence $\{d(x_n, x_{n+1})\}$ is positive and decreasing. Therefore, there exists $r \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = r$. We will show that $r = 0$.

Suppose, $r > 0$. Then we have

$$\frac{d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1})} \leq \beta(M(x_n, x_{n+1})) < 1.$$

On taking limits as $\lim_{n\to\infty}\beta(M(x_n,x_{n+1}))=1$ implies $\lim_{n\to\infty}M(x_n,x_{n+1})=0$, and then $\lim_{n\to\infty}d(x_n,x_{n+1})=0$, a contradiction.

Now, we have to show that $\{x_n\}$ is a Cauchy sequence. Suppose assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon>0$ such that, for all $k\geq 1$, there exists $m(k)>n(k)>k$ with $d(x_{n(k)},x_{m(k)})\geq\varepsilon$. Let $m(k)$ be the smallest positive number satisfying the conditions above. Hence, we have $d(x_{n(k)},x_{m(k)-1})<\varepsilon$. Therefore, we get

$$\varepsilon\leq d(x_{n(k)},x_{m(k)})\leq d(x_{n(k)},x_{m(k)-1})+d(x_{m(k)-1},x_{m(k)})<\varepsilon+d(x_{m(k)-1},x_{m(k)}).$$

On taking limits $k\to\infty$, we have

$$\lim_{k\to\infty}d(x_{n(k)},x_{m(k)})=\varepsilon. \tag{2.4}$$

By the using triangular property, we have

$$|d(x_{n(k)},x_{m(k)-1})-d(x_{n(k)},x_{m(k)})|\leq d(x_{m(k)},x_{m(k)-1}),$$

Taking limits as $k\to\infty$ and (2.4), we have

$$\lim_{k\to\infty}d(x_{n(k)},x_{m(k)-1})=\varepsilon. \tag{2.5}$$

With using triangular property and using (2.4), we get

$$\lim_{k\to\infty}d(x_{m(k)},x_{n(k)-1})=\varepsilon \text{ and } \lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)-1})=\varepsilon.$$

By Lemma 1.1, we have $\alpha(x_{n(k)},x_{m(k)-1})\geq 1$, and by using (2.2)-(2.5),

$$\begin{aligned} d(x_{n(k)},x_{m(k)})&=d(Tx_{n(k)-1},Tx_{m(k)-1})\leq\alpha(x_{n(k)-1},x_{m(k)-1})d(Tx_{n(k)-1},Tx_{m(k)-1})\\ &\leq\beta(M(x_{n(k)-1},x_{m(k)-1}))M(x_{n(k)-1},x_{m(k)-1}),\end{aligned} \tag{2.6}$$

where

$$M(x_{n(k)-1},x_{m(k)-1}) = max\left\{d(x_{n(k)-1},x_{m(k)-1}), \frac{d(x_{n(k)-1},Tx_{n(k)-1}),d(x_{m(k)-1},Tx_{m(k)-1})}{1+d(x_{n(k)-1},x_{m(k)-1})},\right.$$
$$\left.\frac{d(x_{m(k)-1},Tx_{n(k)-1}),d(x_{n(k)-1},Tx_{m(k)-1})}{1+d(Tx_{n(k)-1},Tx_{m(k)-1})}\right\}$$
$$= max\left\{d(x_{n(k)-1},x_{m(k)-1}), \frac{d(x_{n(k)-1},x_{n(k)})d(x_{m(k)-1},x_{m(k)})}{1+d(x_{n(k)-1},x_{m(k)-1})},\right.$$
$$\left.\frac{d(x_{m(k)-1},x_{n(k)})d(x_{n(k)-1},x_{m(k)})}{1+d(x_{n(k)},x_{m(k)})}\right\}$$
$$\leq max\left\{d(\varepsilon, 0, \frac{\varepsilon^2}{1+\varepsilon})\right\}.$$

Clearly, we deduce that

$$\lim_{k\to\infty} M(x_{n(k)-1},x_{m(k)-1}) = \varepsilon. \tag{2.7}$$

From (2.6), we have

$$\frac{d(x_{n(k)},x_{m(k)})}{M(x_{n(k)-1},x_{m(k)-1})} \leq \beta(M(x_{n(k)-1},x_{m(k)-1})) < 1.$$

Letting $k \to \infty$, we conclude that

$$\lim_{k\to\infty} \beta(M(x_{n(k)-1},x_{m(k)-1})) = 1 \text{ implies that } \lim_{k\to\infty} M(x_{n(k)-1},x_{m(k)-1}) = 0.$$

Hence, $\varepsilon = 0$, which is a contradiction. Thus, we get that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, it follows that there exists $z = \lim_{n\to\infty} x_n \in X$. By the continuity of T, we get $\lim_{n\to\infty} Tx_n = Tz$, and so $z = Tz$, which means that z is a fixed point of T. □

With the inspiration of the paper[5], in the following theorem, we replace the continuity of the operator T by a suitable condition.

Theorem 2.2. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and T be a self map on X and satisfying (2.1). Suppose that the following conditions are satisfied :*

(1) *T is a triangular α-orbital admissible mapping;*

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*

(3) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1, \forall n$ and $\lim_{n\to\infty} x_n = x(\in X)$, then there exists a subsequence $\{x_n(k)\}$ of $\{x_n\}$such that $\alpha(x_{n(k)}, z) \geq 1, \forall k$.*

Then T has a fixed point $z \in X$ and $\{T^n x_1\}$ converges to z.

Proof. From the proof of Theorem 2.1, we conclude that the sequence $\{x_n\}$ define by $x_{n+1} = Tx_n$ for $n \geq 1$. converges to $z \in X$. From the condition (3) we deduce that there exists a sub sequence $\{x_{n_k}\}$ of x_n such that $\alpha(x_{n_k}, z) \geq 1, \forall k$. From (2.1), we have

$$d(x_{n(k)+1}, Tz) = d(Tx_{n(k)}, Tz) \leq \alpha(x_{n(k)}, z) d(Tx_{n(k)}, Tz)$$
$$\leq \beta(M(x_{n(k)}, z)) M(x_{n(k)}, z),$$

where

$$M(x_{n(k)}, z) = max\left\{d(x_{n(k)}, z), \frac{d(x_{n(k)}, Tx_{n(k)}) d(z, Tz)}{1 + d(x_{n(k)}, z)}, \frac{d(x_{n(k)}, Tz) d(z, Tx_{n(k)})}{1 + d(Tx_{n(k)}, Tz)}\right\}$$
$$= max\left\{d(x_{n(k)}, z), \frac{d(x_{n(k)}, x_{n(k)+1}) d(z, Tz)}{1 + d(x_{n(k)}, z)}, \frac{d(x_{n(k)}, Tz) d(z, x_{n(k)+1})}{1 + d(x_{n(k)+1}, Tz)}\right\}.$$

Suppose that $d(z, Tz) > 0$, for k large enough, we have

$$\lim_{k \to \infty} M(x_{n(k)}, z) = d(z, Tz).$$

Since $\frac{d(x_{n(k)+1}, Tz)}{M(x_{n(k)}, z)} \leq \beta(M(x_{(n(k)}, z))$ for all k, letting $k \to \infty$, we have

$$\lim_{k \to \infty} \beta(M(x_{n(k)}, z)) = 1 \text{ implies } \lim_{k \to \infty} M(x_{n(k)}, z) = 0.$$

Hence $d(z, Tz) = 0$, a contradiction. Therefore $Tz = z$. □

For the uniqueness part of a fixed point of rational α-Geraghty contractive mapping. We consider the following hypothesis.

(A) for all x, y are two fixed points of T, there exists $w \in X$ such that $\alpha(x, w) \geq 1$ and $\alpha(y, w) \geq 1$.

Theorem 2.3. *Adding additional hypothesis (A) in Theorem 2.1(resp. Theorem 2.2), we obtain that z is the unique fixed point of T.*

Proof. From Theorem 2.1(resp. Theorem 2.2), we obtain that z is a fixed point of T. Now we show that z is a unique common fixed point of T. Suppose z' is another fixed point of T i.e., $(z \neq z')$. By the condition (A), there exists $w \in X$ such that

$$\alpha(z, w) \geq 1, \ \alpha(z', w) \geq 1.$$

Since T is satisfying the condition (2.1) of Theorem 2.1(resp. Theorem 2.2), we get

$$d(z,T^{n+1}w) \leq \alpha(z,T^n w)d(Tz,T^{n+1}w) \leq \beta(M(z,T^n w))\, M(z,T^n w), \forall n \geq 1$$

where

$$M(z,T^n w) = max\left\{d(z,T^n w), \frac{d(z,Tz)d(T^n w,T^{n+1}w)}{1+d(z,T^n w)}, \frac{d(T^n w,Tz)d(z,T^{n+1}w)}{1+d(Tz,T^{n+1}w)}\right\}$$
$$\leq max\left\{d(z,T^n w), d(T^n w,T^{n+1}w)\right\} = d(z,T^n w).$$

By Theorem 2.1(resp. Theorem 2.2), we deduce that the sequence $\{T^n w\}$ converge to z_* of T. Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} M(z,T^n w) = d(z,z_*)$. If we suppose that $z \neq z_*$, then we obtain $\frac{d(z,T^{n+1}w)}{M(z,T^n w)} \leq \beta\ (M(z,T^n w))$, from above inequality and taking limits as $n \to \infty$, we have $\lim\limits_{n\to\infty} \beta\ M(z,T^n w) = 1$ implies $\lim\limits_{n\to\infty} M(z,T^n w) = 0$, this leads to $d(z,z_*) = 0$, a contradiction. Hence $z = z_*$. Similarly, by using the same process, we get $z' = z_*$. Therefore T have unique fixed point point in X. □

Example 2.4. Let $X = [0,\infty)$, $d : X \times X \to \mathbb{R}$, $d(x,y) = |x-y|$, $T : X \to X$ such that $Tx = \frac{x}{4}$ if $x \in [0,3]$, $Tx = 0$ if $x > 3$ and $\alpha : X \times X \to R$, $\alpha(x,y) = 1$ if $xy \geq 0$ and $\alpha(x,y) = 0$ otherwise. Since X is a complete metric space. Obviously T satisfying an α- orbital admissible mapping and also triangular α-orbital admissible mapping. Take $\beta(t) = \frac{1}{1+t}$ when $t > 0$.
Case I : If $x,y \in (0,3)$, we have $\beta(M(x,y))M(x,y) - \alpha(x,y)d(Tx,Ty) = \frac{|x-y|}{1+|x-y|} - \frac{1}{4}|x-y| \geq 0$.
Case II : If $x = 0, y = 3$, we obtain that $\beta(M(x,y))M(x,y) - \alpha(x,y)d(Tx,Ty) = \frac{3}{4} - \frac{3}{4} \geq 0$.
Case III : If $x = 0, y \in (0,3)$, we given that $\beta(M(x,y))M(x,y) - \alpha(x,y)d(Tx,Ty) \geq 0$.
Rest of all possible cases vanishes easily (in these cases $\alpha(x,y) = 0$). Therefore all the hypotheis of Theorem 2.1 satisfied and 0 is unique fixed point of T.

Corollary 2.1. *Let (X,d) be a complete metric space , $\alpha : X \times X \to \mathbb{R}$ be a function, and T be self map on X and satisfying (2.1). Suppose that the following conditions are satisfied :*

(1) *T is a α-orbital admissible mapping;*

(2) *there exists $x_1 \in X$ such that $\alpha(x_1,Tx_1) \geq 1$;*

(3) *T is continuous.*

Then T has a fixed point $z \in X$ and $\{T^n x_1\}$ converges to z.

Corollary 2.2. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and T be self map on X and satisfying (2.1). Suppose that the following conditions are satisfied :*

(1) *T is a α-orbital admissible mapping;*

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*

(3) *T if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1, \forall n$ and $\lim_{n\to\infty} x_n = x(\in X)$ then there exists a subsequence $\{x_n(k)\}$ of $\{x_n\}$such that $\alpha(x_{n(k)}, z) \geq 1; \forall k$..*

Then T has a fixed point $z \in X$ and $\{T^n x_1\}$ converges to z.

3. α-orbital attractive mappings

Definition 3.1. [13] Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be α- `orbital attractive` if $\alpha(x, Tx) \geq 1$ implies $\alpha(x, y) \geq 1$ or $\alpha(y, Tx) \geq 1$, for every $y \in X$.

Theorem 3.2. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function, and T be a self map on X and satisfying (2.1). Suppose that the following conditions are satisfied :*

(1) *T is α-orbital admissible mapping;*

(2) *there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;*

(3) *T is α-orbital attractive, we have*

Then T has a fixed point $z \in X$ and $\{T^n x_1\}$ converges to z.

Proof. Proof is same lines up to (2.5) in Theorem 2.1.

Since $\alpha(x_{n(k)-1}, x_{n(k)}) \geq 1$, and T is α-orbital attractive, then we have

$$\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1 \text{ or } \alpha(x_{m(k-1)}, x_{n(k)}) \geq 1.$$

Now two cases are arises,

(a) there exists an infinite subset I of N such that $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$ for every $k \in I$,

(b) there exists an infinite subset J of N such that $\alpha(x_{m(k)-1}, x_{n(k)}) \geq 1$ for every $k \in J$.

Case (a):

$$
\begin{aligned}
d(x_{n(k)},x_{m(k)}) = d(Tx_{n(k)-1},Tx_{m(k)-1}) &\leq \alpha(x_{n(k)-1},x_{m(k)-1})d(Tx_{n(k)-1},Tx_{m(k)-1}) \\
&\leq \beta(M(x_{n(k)-1},x_{m(k)-1}))M(x_{n(k)-1},x_{m(k)-1}),
\end{aligned} \tag{3.1}
$$

where

$$
\begin{aligned}
M(x_{n(k)-1},x_{m(k)-1}) &= max\left\{d(x_{n(k)-1},x_{m(k)-1}), \frac{d(x_{n(k)-1},Tx_{n(k)-1}),d(x_{m(k)-1},Tx_{m(k)-1})}{1+d(x_{n(k)-1},x_{m(k)-1})},\right. \\
&\left.\frac{d(x_{n(k)-1},Tx_{n(k)-1}),d(x_{m(k)-1},Tx_{m(k)-1})}{1+d(Tx_{n(k)-1},Tx_{m(k)-1})}\right\} \\
&= max\left\{d(x_{n(k)-1},x_{m(k)-1}), \frac{d(x_{n(k)-1},x_{n(k)})d(x_{m(k)-1},x_{m(k)})}{1+d(x_{n(k)-1},x_{m(k)-1})},\right. \\
&\left.\frac{d(x_{n(k)-1},x_{n(k)})d(x_{m(k)-1},x_{m(k)})}{1+d(x_{n(k)},x_{m(k)})}\right\} \\
&\leq max\left\{d(x_{n(k)-1},x_{m(k)-1}), d(x_{n(k)},x_{n(k)-1})\right\} \\
&= d(x_{n(k)},x_{n(k)-1}).
\end{aligned}
$$

On taking $k \to \infty$, $k \in I$ and using (2.2)-(2.6) we deduce that

$$\lim_{k\to\infty, k\in I} M(x_{n(k)-1},x_{m(k)-1}) = \varepsilon.$$

Then from (3.1), we get

$$\frac{d(x_{n(k)},x_{m(k)})}{M(x_{n(k)-1},x_{m(k)-1})} \leq \beta(M(x_{n(k)-1},x_{m(k)-1})).$$

Letting $k \to \infty$, we conclude that $\lim_{n\to\infty} \beta(M(x_{n(k)-1},x_{m(k)-1})) = 1$, which yields that $\lim_{k\to\infty} \beta(M(x_{n(k)-1},x_{m(k)-1})) = 0$. Since $\beta \in \mathscr{T}$, then,$\varepsilon = 0$, which is a contradiction.

Case (b):

$$
\begin{aligned}
d(x_{m(k)},x_{n(k)+1}) = d(Tx_{m(k)-1},Tx_{n(k)}) &\leq \alpha d(x_{m(k)-1},x_{n(k)})d(Tx_{m(k)-1},Tx_{n(k)}) \\
&\leq \beta(M(x_{m(k)-1},x_{n(k)}))M(x_{m(k)-1},x_{n(k)}),
\end{aligned} \tag{3.2}
$$

where

$$M(x_{m(k)-1},x_{n(k)}) = max\left\{d(x_{m(k)-1},x_{n(k)}), \frac{d(x_{m(k)-1},Tx_{m(k)-1}),d(x_{n(k)},Tx_{n(k)})}{1+d(x_{m(k)-1},x_{n(k)})},\right.$$
$$\left.\frac{d(x_{m(k)-1},Tx_{m(k)-1}),d(x_{n(k)},Tx_{n(k)})}{1+d(Tx_{m(k)-1},Tx_{n(k)})}\right\}$$
$$= max\left\{d(x_{m(k)-1},x_{n(k)}), \frac{d(x_{m(k)-1},x_{m(k)})d(x_{n(k)},x_{n(k+1)})}{1+d(x_{m(k)-1},x_{n(k)})},\right.$$
$$\left.\frac{d(x_{m(k)-1},x_{n(k)})d(x_{m(k)-1},x_{m(k)})}{1+d(x_{n(k)},x_{m(k)})}\right\}$$
$$\leq max\left\{d(x_{m(k)-1},x_{n(k)}),d(x_{n(k)},x_{m(k)-1})\right\}$$
$$= d(x_{m(k)-1},x_{n(k)}).$$

On taking $k\to\infty$, $k\in J$ and using (2.2)-(2.6) we deduce that

$$\lim_{n\to\infty,\ k\in J} M(x_{m(k)-1},x_{n(k)}) = \varepsilon.$$

Then from (3.2), we have

$$\frac{d(x_{m(k)},x_{n(k)+1})}{M(x_{m(k)-1},x_{n(k)})} \leq \beta(M(x_{m(k)-1},x_{n(k)})).$$

Letting $k\to\infty$, we conclude that $\lim_{n\to\infty}\beta(M(x_{m(k)-1},x_{n(k)})) = 1$, which yields that $\lim_{k\to\infty}\beta(M(x_{m(k)-1},x_{n(k)})) = 0$. Since $\beta\in\mathscr{T}$, then,$\varepsilon = 0$, which is a contradiction. Thus, we get that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, it follows that there exists $z=\lim_{n\to\infty}x_n\in X$. By the continuity of T, we get $\lim_{n\to\infty}Tx_n = Tz$.

Finally we have to show that $z=Tz$. we assume that $z\neq Tz$.

Since T is α-orbital attractive, for every $n\geq 1$ such that

$$\alpha(x_n,z)\geq 1 \text{ or } \alpha(z,x_{n+1})\geq 1, \tag{3.3}$$

Hence, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_n,z)\geq 1$ or $\alpha(z,x_n)\geq 1$, for all $k\geq 1$.

In the first case of (3.3), we get

$$d(x_{n(k)+1},Tz) = d(Tx_{n(k)},Tz) \leq \alpha(x_{n(k)},z)d(Tx_{n(k)},Tz)$$
$$\leq \beta(M(x_{n(k)},z))M(x_{n(k)},z),$$

where

$$\begin{aligned} M(x_{n(k)},z) &= max\left\{d(x_{n(k)},z), \frac{d(x_{n(k)},Tx_{n(k)})d(z,Tz)}{1+d(x_{n(k)},z)}, \frac{d(x_{n(k)},Tx_{n(k)})d(z,Tz)}{1+d(Tx_{n(k)},Tz)}\right\} \\ &= max\left\{d(x_{n(k)},z, \frac{d(x_{n(k)},x_{n(k)+1})d(z,Tz)}{1+d(x_{n(k)},z)}, \frac{d(x_{n(k)},z)d(z,Tz)}{1+d(x_{n(k)+1},Tz)}\right\} \\ &\leq max\left\{d(x_{n(k)},z), d(z,Tz)\right\} = d(z,Tz). \end{aligned}$$

Since

$$\frac{d(x_{n(k)},Tx_{n(k)})}{M(x_{n(k)},z)} \leq \beta(M(x_{n(k)},z)),$$

we have

$$\beta(M(x_{n(k)},z)) = 1.$$

Since $\beta \in \mathscr{T}$, we have

$$\lim_{k\to\infty} M(x_{n(k)},z) = 0,$$

which is a contradiction. With the same procedure we will get $z = Tz$ easily in the second case of (3.3) also.

Uniqueness: Suppose w and z are two distinct fixed points of T, from the hypothesis, we get

$$\alpha(x_n,w) \geq 1 \text{ or } \alpha(w,x_{n+1}) \geq 1. \tag{3.4}$$

Hence, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},w) \geq 1$ or $\alpha(w,x_{n(k)}) \geq 1$, for all $k \geq 1$. Then, choose any one in (3.4)(follow same procedure if we choose second one in (3.4)), we get

$$\begin{aligned} d(x_{n(k)+1},Tw) = d(Tx_{n(k)},Tw) &\leq \alpha(x_{n(k)},w)d(Tx_{n(k)},Tw) \\ &\leq \beta(M(x_{n(k)},w))\, M(x_{n(k)},w), \end{aligned}$$

where

$$\begin{aligned} M(x_{n(k)},w) &= max\left\{d(x_{n(k)},w), \frac{d(x_{n(k)},Tx_{n(k)})d(w,Tw)}{1+d(x_{n(k)},w)}, \frac{d(w,Tx_{n(k)})d(x_{n(k)},Tw)}{1+d(Tx_{n(k)},Tw)}\right\} \\ &= max\left\{d(x_{n(k)},w), \frac{d(x_{n(k)},x_{n(k)+1})d(w,Tw)}{1+d(x_{n(k)},w)}, \frac{d(w,x_{n(k)+1})d(x_{n(k)},Tw)}{1+d(x_{n(k)+1},Tw)}\right\} \\ &\leq max\left\{d(x_{n(k)},w), d(w,Tw)\right\} = d(x_{n(k)},w). \end{aligned}$$

Since

$$\frac{d(x_{n(k)}, Tx_{n(k)})}{M(x_{n(k)}, w)} \leq \beta(M(x_{n(k)}, w)),$$

we have

$$\lim_{k\to\infty} \beta(M(x_{n(k)}, w)) = 1.$$

Since $\beta \in \mathscr{T}$, we have

$$\lim_{k\to\infty} M(x_{n(k)}, w) = 0,$$

a contradiction. Therefore T has a unique fixed point in X. . □

4. Application in ordinary differential equations

We consider the following boundary value problem to Fredholm integral equation;

$$\frac{d^2y}{dt^2} + y = \begin{cases} f(t, y(t)) & \text{if } t \in [0, \frac{\pi}{2}] \\ y(0) = y(\frac{\pi}{2}) = 0, \end{cases}$$

where $f : [0, \frac{\pi}{2}] \times \mathbb{R} \to \mathbb{R}$ is continuous function. The Green's function of the given boundary condition is given by

$$G(x,t) = \begin{cases} cost\, sinx & \text{if } 0 \leq x < t \\ sint\, cosx & \text{if } t < x \leq \frac{\pi}{2}, \end{cases}$$

Let C(I) be the space of all continuous function defined in I, where $I = [0, \frac{\pi}{2}]$, and suppose that

$$d(x,y) = sup_{t\in I}|x(t) - y(t)|,$$

for all $x, y \in C(I)$. Then $(C(I), d)$ is a complete metric space.

Suppose that $\eta : [0, \frac{\pi}{2}] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\eta(a,b) \geq 0$ for all $a, b \in \mathbb{R}$, we have

$$|f(t,a) - f(t,b)| \leq \log(|a-b| + 1),$$

now we define the mapping $U : C[0, \frac{\pi}{2}] \to C[0, \frac{\pi}{2}]$ by above associated Green's function

$$Uy(t) = \int_0^{\frac{\pi}{2}} G(x,t) f(t, y(t)) dt.$$

Let $x,y \in C(I)$ such that $\eta(x(t),y(t)) \geq 0$ for all $t \in I$.

$$\begin{aligned} d(Ux,Uy) = |Ux(t) - Uy(t)| &= |\int_0^{\frac{\pi}{2}} [G(x,t)f(t,x(t)) - G(x,t)f(t,y(t))]dt| \\ &\leq \int_0^{\frac{\pi}{2}} |G(x,t)| \log(|x(t) - y(t)| + 1)dt \\ &\leq \log(|x(t) - y(t)| + 1)[|\int_0^x G(x,t)dt + \int_x^{\frac{\pi}{2}} G(x,t)dt|] \\ &\leq \log(|x(t) - y(t)| + 1)[|\int_0^x cosx\, sint dt + \int_x^{\frac{\pi}{2}} sinx\, cost dt|] \\ &\leq \log(|x(t) - y(t)| + 1) sup_{t \in I}[sinx - 1] \\ &\leq \log(M(x,y) + 1), \end{aligned}$$

where $M(x,y) = max\{d(x,y), \frac{d(x,Ux)d(y,Uy)}{1+d(x,y)}, \frac{d(y,Ux)d(x,Uy)}{1+d(Ux,Uy)}\}$, We define $\alpha : C[0,\frac{\pi}{2}] \times C[0,\frac{\pi}{2}] \to \mathbb{R}$,

$$\alpha(x,y) = \begin{cases} 1 & \text{if } \eta(x(t),y(t)) \geq 0 \text{ for all } t \in I \\ 0, & \text{otherwise.} \end{cases}$$

Then, for all $x,y \in C(I)$, we have

$$\alpha(x,y)d(Tx,Ty) \leq \beta(M(x,y))M(x,y),$$

Thus, $\alpha(x,y) = 1$ and $\alpha(y,z) = 1$ implies $\alpha(x,z) = 1$ for all $x,y,z \in C(I)$. If $\alpha(x,y) = 1$ for all $x,y \in C(I)$, then $\eta(x(t),y(t)) \geq 0$, we obtain that $\eta(Ux(t),Uy(t)) \geq 0$, and so $\alpha(Ux,Uy) = 1$. Obviously, T is triangular α-admissible. let z be a lower solution of equation (2.1). In [7] we have derive $z = Uz$.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Amini-Harandi and H. Emami, A fixed point theorem for contraction type maps in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., TMA. 72 (2010) 2238-2242.

[2] M. Asadi, E. Karapinar and A. Kumar, $\alpha - \psi$-Geraghty contractions on generalized metric spaces, J. Inequal. Appl. 2014 (2014), Article ID 423.

[3] H. Aydi, E. Karapinar, I. Erhan and P. Salimi, Best proximity points of generalized almost ψ-Geraghty contractive non-self mappings, Fixed Point Theory Appl. 2014 (2014), Article ID 32.

[4] J. Caballero, J. Harjani and K. Sadarangani, A best proximity point theorem for Geraghty-contractions. Fixed Point Theory Appl. 2012 (2012), Article ID 231.

[5] SH. Cho, JS. Bae and E. Karapinar, Fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. (2013).

[6] M. Geraghty, On contractive mapping, Proc. Amer. Math. Soc. 40 (1973) 604-608.

[7] ME. Gordji, M. Ramezani, YJ. Cho and S. Pirbavafa, A generalization of Geraghty's theorem in partially ordered metric space and application to ordinary differential equations, Fixed Point Theory Appl. 2012 (2012), Article ID 74.

[8] J. Harjani and K. Sadarangani, Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal. TMA. 71 (2009) 3403-3410.

[9] E. Karapinar and B. Samet, Generalized $\alpha-\psi$ contractive type mappings and related fixed point theorems with applications, Abstr. Appl. Anal. 2012 (2012), Article ID 793486.

[10] E. Karapinar, P. Kumam and P. Salimi, On $\alpha-\psi$-Meir-Keeler contractive mappings, Fixed Point Theory Appl. 2013 (2013), Article ID 94.

[11] E. Karapinar, $\alpha-\psi$-Geraghty contraction type mappings and some related fixed point results, Filomat 28 (1) (2014), 37-48.

[12] E. Karapinar and B. Samet, A note on α-Geraghty type contractions, Fixed Point Theory Appl. 2014 (2014), Article ID 26.

[13] O. Popescu, Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces, Fixed Point Theory Appl. 2014 (2014), Article ID 190.

[14] B. Samet, C. Vetro and P. Vetro, Fixed point theorems for $\alpha-\psi$ contractive mappings, Nonlinear Anal. TMA. 75 (2012), 2154-2165.