Available online at http://scik.org
Adv. Inequal. Appl. 2019, 2019:3
https://doi.org/10.28919/aia/3915
ISSN: 2050-7461

ON A CHARACTERIZATION OF COMPACT FRAMES BY NETS

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Abstract. In this paper we analyze the property of compactness and almost compactness in frames using nets. By introducing and investigating net convergence in frames, we characterize almost compact frames and regular compact frames as frames where every net clusters. With the aid of the results derived, we also present here a characterization of minimal Hausdorff frames through nets.

Keywords: clustering; convergence; derived nets; derived filters; almost compact frames; compact frames; minimal Hausdorff frames.

2010 AMS Subject Classification: 06D22, 54A20, 54D30, 54D10, 54D15.

1. Introduction

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Received September 30, 2018.

Several chattels of topological spaces can be portrayed by means of its open sets without using the points of the underlying set. This motivates the development of Frame theory and hence it also has the name point-free topology.

Compactness in classical topology is the generalization of the Heine-Borel Property of closed sets. Point-free definition of compactness is the direct translation of its topological counterpart introduced by Alexandrov and Urysohn [1] in 1929 and it states that a frame L is compact if every cover of L has a finite subcover. A compact topological space can also be characterized as a space where every net clusters. We shall prove in this paper that such a characterization can also be given in point-free context.

Study of nets in topology was initiated by E. H. Moore and H. L. Smith [9] and has been developed by J. L. Kelley [8] where as the concept of filter was introduced by H. Cartan [4] [3]. One can approach convergence in a topological space using nets as well as filters. Each has its own advantage over the other.

Convergence in frames using filters was introduced by Banaschewski and Pultr [2] in 1990. Later in 1995, S.S. Hong [5] gave another definition of convergence in terms of cover of a frame. Our purpose with this paper is to analyze the property of compactness in frames using nets. We start by proving several basic properties of net convergence and clustering. Then by defining derived nets and derived filters, we establish a relation between nets and filters in frame context. With the aid of these results, we prove that a regular frame is compact if and only if every net in it clusters and present a characterization of minimal Hausdorff frames via nets.

2. Preliminaries

For general notions and facts concerning frames, we refer to Stone Spaces [7] and Frames and Locales [11]. A *frame* L is a complete lattice which satisfies the infinite distributive law : $x \wedge \bigvee S = \bigvee\{x \wedge s : s \in S\}$ for all $x \in L$, $S \subseteq L$. We use the notation 0_L and 1_L, respectively, for the least and the greatest element of the frame L . For each $a \in L$, $a^* = \bigvee\{x \in L : x \wedge a = 0_L\}$ denotes the *pseudocomplement* of a. An element $a \in L$ is said to be *dense* if $a^* = 0_L$

A *cover* of a frame L is a subset C of L with $\bigvee C = 1_L$. L is called a *compact frame* if every cover of L has a finite subcover and is *almost compact* if whenever $\bigvee\{x_i : i \in I\} = 1_L$ then there exist a finite set $K \subseteq I$ such that $(\bigvee\{x_i : i \in K\})^* = 0_L$. A frame L is said to be *regular* if every element k in L can be written as $k = \bigvee\{x : x \prec k\}$ where $x \prec k \Rightarrow x^* \vee k = 1$

The reader is referred to the papers [5] for the background material on filter convergence, and to [6] on minimal Hausdorff frames.

3. Nets in frames

We now turn to the definition of clustering and convergence in frames motivated from S.S.Hong [5] and prove in this section some of the basic properties of net convergence in frames.

Definition 3.1. A net in a frame is a function $s : D \to L$ where D is a directed set. $T : E \to L$ is said to be a subnet of S if there exist a function $N : E \to D$ such that $T = S \circ N$ and for any $n \in D$ there exist a $p \in E$ such that $\forall m \in E, m \geq p \Rightarrow N(m) \geq n$ in D.

Definition 3.2. A net $s : D \to L$ is convergent if for any cover C of L there exist a $g \in C$ and $m \in D$ such that $\bigvee_{n \geq m} s(n) < g$.

Definition 3.3. A net $s : D \to L$ is clustered if for any cover C of L there exist a $g \in C$ satisfying the condition that $a \wedge g \neq 0_L$ for each $a \in L$ with the property that $\bigvee_{n \geq m} s(n) < a$ for some $m \in D$.

Proposition 3.1. *Every converging net in a frame clusters.*

Proof. Let $s : D \to L$ be a converging net in the frame L which is not eventually zero and let C be a cover of L. If s is not clustering then there exist a p in L with $\bigvee_{n \geq m_1} s(n) < p$ for some $m_1 \in D$ and $p \wedge g = 0_L$ for all g in C. Since s converges there exist a $g_1 \in C$ such that $\bigvee_{n \geq m_2} s(n) < g_1$ for some $m_2 \in D$. Now $(\bigvee_{n \geq m_1} s(n)) \wedge (\bigvee_{n \geq m_2} s(n)) < p \wedge g_1 = 0_L$. Thus $\bigvee_{n \geq m} s(n) = 0$ where $m = m_1 \vee m_2$. This implies that s is an eventually zero net which is a contradiction. This completes the proof.

Proposition 3.2. *Every net in a 2 point frame converges.*

Proof. Let $s : D \to L$ be a net in $L = \{0, 1\}$ and let C be a cover of L. Then C should contain 1_L. Hence s converges. This completes the proof.

Remark 3.1. Converse of the above theorem is not always true unlike the spatial situation. For example, every net in a bounded chain L, which is obviously a frame, converges.

Proposition 3.3. *If a net in a frame L converges then so does every subnet of it.*

Proof. Let $s : D \to L$ be a converging net in the frame L and let $T : E \to L$ be a subnet of s. Let C be any arbitrary cover of L. Since s is convergent there exist an element $g \in C$ and $m \in D$ such that $\bigvee_{n \geq m} s(n) < g$. Since T is a subnet of s there exist a function $N : E \to D$ such that $T = s \circ N$ and a $p \in D$ such that $N(n) \geq m$ for all $n \geq p$. Thus $\bigvee_{N(n) \geq m} s(N(n)) < g$. ie. $\bigvee_{n \geq p} T(n) < g$. Hence T converges in L. This completes the proof.

Next we give a necessary and sufficient condition for a net to be clustered in a frame.

Theorem 3.4. *A net $s : D \to L$ clusters if and only if*

$$\bigvee \{x^* : x \in L \; and \; \bigvee_{n \geq m} s(n) < x \; for \; some \; m \in D\} \neq 1_L.$$

Proof. Let $s : D \to L$ be a net in L which clusters in L and let $Q = \{x^* : x \in L \; and \; \bigvee_{n \geq m} s(n) < x \; for \; some \; m \in D\}$. If possible assume that $\bigvee Q = 1_L$. Then Q is a cover for L. Since s clusters in L, there exist a $g \in L$ such that $\bigvee_{n \geq m} s(n) < g$ and $a \wedge g^* \neq 0_L$ for those $a \in L$ having the property $\bigvee_{n \geq m} s(n) < a$. Hence $g \wedge g^* \neq 0_L$. This is a contradiction.Hence $\bigvee Q \neq 1_L$.

Conversely let $\bigvee Q \neq 1_L$. If possible let s is not clustering. Then there exist a cover C of$L$ such that for each $g \in C$ there exist a $a_g \in L$ with $\bigvee_{n \geq m} s(n) < a_g$ such that $a_g \wedge g = 0_L$. ie. $g \leq a_g^*$. ie.$\bigvee Q = 1_L$. This is a contradiction. This completes the proof.

Definition 3.4.

A net $s : D \to L$ in a frame L is said to be *eventually prime* if $s(k) \leq \bigvee P$ for some $k \in D$ and $P \subseteq L$, then there exist a $p \in P$ and $m \in D$ such that $\bigvee_{n \geq m} s(n) < p$.

Proposition 3.5. *Every eventually prime net converges in a frame.*

Proof. Let $s : D \to L$ be an eventually prime net in L and let C be a cover of L. Then $s(n) \leq 1_L = \bigvee C$. As s is eventually prime , there exist a $g \in C$ and $m \in D$ such that $\bigvee_{n \geq m} s(n) < g$. Hence s converges in L. This completes the proof.

Proposition 3.6. *A net $s : D \to L$ in a frame L converges if and only if its restriction to an eventual subset of D converges.*

Proof. Let $s : D \to L$ converges in L and let E be an eventual subset of D. Then there exist a $m' \in D$ such that $n \geq m' \Rightarrow n \in E$. Let C be a cover of L. Since s converges there exist a $g \in C$

and $m \in D$ such that $\bigvee_{n \geq m} s(n) < g$. since D is directed there exist a $p \in D$ such that $p \geq m$ and $p \geq m'$.Thus $p \in E$ and $\bigvee_{n \geq p} s(n) < g$. Hence s restricted to E converges . Converse part is trivial. This completes the proof.

Proposition 3.7. *A net $s : D \to L$ in a frame L converges then its restriction to a cofinal subset of D converges.*

Proof. Let C be a cover of L. Since s converges there exist a $g \in C$ and $m \in D$ such that $\bigvee_{n \geq m} s(n) < g$. Let F be a cofinal subset of D. Then there exist a $p \in F$ such that $p \geq m$. Thus $\bigvee_{n \geq p} s(n) < g$. Hence s/F converges in L.

Next we define derived filters and derived nets in frame context and give relations between them.

Definition 3.5. Let F be a filter in L. Define a relation $\leq^d$ on F by $x \leq^d y$ if and only if $x \geq y$ in L. Then F will be a directed set with respect to the relation $\leq^d$. Define a net $s : F \to L$ with the property that $s(f) \leq f$ for each $f \in F$. Such a net is called *derived net* of F.

Definition 3.6. Let $s : D \to L$ be a net. Then derived filter F of s in L is defined as $F = \{l \in L :$ s is eventually less than $l\}$

Theorem 3.8. *A filter in a frame converges if and only if all its derived nets converge.*

Proof. Let F be a filter which converges in frame L and let $s : F \to L$ be any of its derived net. Let C be a cover of L. Since F converges there exist a $g \in C \cap F$. Now for all $k < g$, $s(k) < g$. Hence $\bigvee_{k<g} s(k) < g$. ie. $\bigvee_{k \geq^d g} s(k) < g$. Thus s converges in L.

Conversely, let all the derived nets of filter F converges in L. If F doesn't converge then there exist a cover C of L such that $C \cap F = \Phi$. Now consider the derived net $s : F \to L$ defined as $s(f) = f$ for each f in F. Then $s(f) \not< g$ for all g in C. This implies that s is not convergent in L. This is a contradiction. Hence F converges in L.

Theorem 3.9. *A net is convergent if and only if its derived filter converges.*

Proof. Let $s : D \to L$ be a net converging in L and let F be its derived filter. Let C be a cover of L. As s converges there exist a g in C and m in D such that $\bigvee_{n \geq m} s(n) < g$. Thus $s(n) < g \ \forall n \geq m$. Hence $g \in F$. Hence F converges in L. Converse part can be similarly proved.

4. Characterizations of compactness, almost compactness and Minimal Hausdorffness in frames using nets

In this section we characterize compactness, almost compactness and Minimal Hausdorffness using nets.

Theorem 4.1. *Every net in a compact frame clusters.*

Proof. Let L be a compact frame and let $s : D \to L$ be a net in L which is not eventually zero. If possible assume that s is not clustering. Then there exist a cover C of L such that, for each $g \in C$ and $a \wedge g = 0_L$ for those a in L with the property that $\bigvee_{n \geq m} s(n) < a$ for some m in D. Now let a_0 be an element in L such that $a_0 \wedge g = 0_L$ $\forall g \in C$ and $\bigvee_{n \geq m} s(n) < a_0$ for a $m \in D$. Hence

$$(\bigvee_{n \geq m} s(n)) \wedge g < a_0 \wedge g = 0_L, \forall g \in C$$

$$\Rightarrow g \leq [\bigvee_{n \geq m} s(n)]^*$$

$$\Rightarrow g^* \geq [\bigvee_{n \geq m} s(n)]^{**} \geq \bigvee_{n \geq m} s(n)$$

$$s(n) \leq g^*;\ \forall n \geq m, \forall g \in C$$

. Now since L is compact , C has a finite subcover ie. there exist $g_1, g_2, ..., g_k$ such that $g_1 \vee g_2 \vee ... \vee g_k = 1_L$ ie.$g_1^* \wedge g_2^* ... \wedge g_k^* = 0_L$. ie.$s(n) = 0_L$ $\forall n \geq m$. This contradicts our assumption. Hence every net in L clusters.

Definition 4.1. An element $l \in L$ is called a *tail dominating element* of a net $s : D \to L$ if there exist a $m \in D$ such that $\bigvee_{n \geq m} s(n) < l$. The collection of all tail dominating elements of a net s is denoted by $T(s)$.

Theorem 4.2. *If L is a frame in which every clustering net converges then L is almost compact.*

Proof. Let L be a frame in which every clustering net converges. If L is not almost compact then there exist a cover C of L such that $C \subseteq \{l \in L : l^* \neq 0_L\}$. Now consider a net $s : D \to L$ in L with $T(s)$ contains only dense elements of L. Then by theorem 3.4, s clusters but it has a empty intersection with cover C. ie. s is not convergent. This is a contradiction. Hence L is almost compact.

Theorem 4.3. *A frame L is almost compact if and only if every net in L clusters.*

Proof. Let L be almost compact and let $s : D \to L$ be a net in L which is not eventually zero. If s is not clustering then $\bigvee Q = 1_L$ where $Q = \{x^* : x \in L \, and \, \bigvee_{n \geq m} s(n) < x \, for \, some \, m \in D\}$. then there exist a finite subset $K = \{x_1^*, x_2^*, ..., x_n^*\} \subseteq Q$ such that $(\vee_{i=1}^k x_i^*)^* = 0_L$ ie. $\bigwedge_{i=1}^k x_i^{**} = 0_L$. Hence $\bigwedge_{i=1}^k x_i = 0_L$. As $K \subseteq Q$, for each $x_i \in L$ there exist $m_i \in D$ such that $\bigvee_{n \geq m_i} s(n) < x_i, i = 1, 2, .., k$. Thus $\bigvee_{n \geq m} s(n) < x_1 \wedge x_2 \wedge ... \wedge x_k = 0_L$ where $m = m_1 \vee m_2 \vee ... \vee m_k$. This implies that s is an eventually zero net, which is a contradiction. Hence s clusters.

If L is not almost compact, then there exist a proper filter F in L such that $\bigvee\{a^* : a \in F\} = 1_L$. ie. F is not clustering. Hence its derived net is not clustering. This is a contradiction.

Next we analyze Minimal Hausdorff frames using nets.

Corollary 4.4. *A regular frame L is compact if and only if every net in L clusters.*

Proof. From [10], a regular frame is almost compact if and only if it is compact. Hence the proof follows from the above theorem.

Lemma 4.5. *If L is an Hausdorff frame with a clustering net which is not converging then L is not a minimal Hausdorff frame.*

Proof. Let $s : D \to L$ be a net in L which clusters but is not converging. Let F be its derived filter. If F is not clustering then there exist a cover K of L such that for each $g \in L$ there exist a $a_g \in F$ such that $g \wedge a_g = 0_L$. As s clusters there exist a $g_0 \in K$ such that $g_0 \wedge a_{g_0} \neq 0_L$ for those a for which $\bigvee_{n \geq m} s(n) < a \, for \, some \, m \in D$. Corresponding to this g_0 there exist a $a_{g_0} \in F$ such that $g_0 \wedge a_{g_0} = 0_L$. This is a contradiction. Hence F clusters. Also by theorem 3.9, F is not convergent . Thus L contains a clustered filter that is not convergent. In [6] it is proved that *if L is any Hausdorff frame which contains a clustered filter that is not convergent then there exists a proper subframe of L which is Hausdorff*. Thus there exists a proper subframe of L which is Hausdorff. Hence L is not a minimal Hausdorff frame..

Theorem 4.6. *In a minimal Hausdorff frame, every clustering net converges.*

Proof. If there exist a net $s : D \to L$ in a frame L which clusters but is not converging, then by lemma 4.5, L is not a minimal Hausdorff frame.

Corollary 4.7. *Every net in a minimal Hausdorff frame converges.*

Proof. The proof follows from theorems 4.6 and 4.3.

Corollary 4.8. *A minimal Hausdorff frame is almost compact.*

Proof. The proof follows from theorems 4.6 and 4.2.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgment

The first author wishes to thank University Grants Commission, India, for the award of teacher fellowship under XII plan period.(FIP/12th Plan/KLMG008 TF 08).

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