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REVERSIBLE FUZZY POSETS

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Abstract. In this paper we introduce the concept of reversible fuzzy partial ordered set (Reversible F-Poset) as a generalization of reversible Posets. Some useful characterization theorem for reversible F-Posets are introduced, also the connection between reversible F-Posets and reversible Posets are also studied. Some class of reversible Posets are also studied.

Keywords: fuzzy relations; fuzzy posets; poset isomorphism; strict extension; reversible F-posets.

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1. Introduction

The study of fuzzy relations was started by Zadeh [16] in 1971. In that celebrated paper the author introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy orderings.The concept of fuzzy order was introduced by generalizing

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the notion of reflexivity, antisymmetry and transitivity, there by facilitating the derivation of known results in various areas and stimulating the discovery of new ones. Fuzzy orderings have broad utility. They can be applied, for example, when expressing our preferences with a set of alternatives.Since then many notions and results from the theory of ordered sets have been extended to the fuzzy ordered sets. In [14] Venugopalan introduced a definition of fuzzy ordered set and he call it as *foset*,and presented an example on the set of positive integers. He extended this concept to obtain a fuzzy lattice in which he defined a (fuzzy) relation as a generalization of equivalence. The notion of a multichain in a fuzzy ordered set is defined in [1]. In [13] , Seselja and Tepavcevic presented a survey on representations of ordered structures by fuzzy sets. An order relation and a ranking method for type-2 fuzzy values are proposed in [11]. See also [3, 5, 9, 10, 15].

Moreover, thinking of fuzzy ordering as mathematical concept like *'Approximately smaller or equal' or 'approximately greater or equal'* one immediately observes that there is an inherent component of indistinguishability. This concept can be generalized to allow for various degrees or strengths of association or interaction between the elements. Degrees of association can be represented by membership grades in fuzzy relation in the same way as degrees of set membership are represented in fuzzy sets. In fact, crisp relations can be viewed as a restricted case of fuzzy relations.

The idea of reversibility was initially introduced and studied in Topological spaces by A. Willansky and M. Rajagopalan in 1965 [12]. They proved that *'A topological space is reversible if and only if it is maximal or minimal with respect to some topological property'*. The same concept of reversibility in partially ordered sets was introduced by Michal Kukiela in [8]. We extend this idea of reversibility in Fuzzy Posets (F-Posets) and study some class of reversible F-Posets.Some characterization theorems for reversible fuzzy posets, reversible fuzzy posets and its relation with crisp notion are also derived. We also obtain some results analogous to those in reversible posets [8].

2. Preliminaries

A *fuzzy binary relation* $\mathscr{A}$ on a nonempty crisp set X is a fuzzy subset of the Cartesian product $X \times X$, i.e a mapping $\mathscr{A}:X \times X \longrightarrow [0,1]$. For any pair $(x,y) \in X \times X$, $\mathscr{A}(x,y)$ has to be thought of as the degree to which the alternative $x \in X$ is at least as good as the alternative $y \in X$. A fuzzy relation $\mathscr{A}$ is is *reflexive* if $\mathscr{A}(x,x) = 1$ for all x in X, *antisymmetric* if $\mathscr{A}(x,y) > 0$ and $\mathscr{A}(y,x) > 0 \Rightarrow x = y$, *transitive* if $\mathscr{A}(x,z) \geq \sup_{x \in X} \min\{\mathscr{A}(x,y), \mathscr{A}(y,z)\}$ [16].
A *fuzzy preorder* $\mathscr{A}$ on X is a fuzzy subset of $X \times X$, which is reflexive and transitive. A non empty set X with fuzzy preorder $\mathscr{A}$ defined on it is called a fuzzy *preordered set* and we denote it by $(X, \mathscr{A})$. A fuzzy preordered set is called *fuzzy ordered* if $\mathscr{A}$ is antisymmetric, in this case $(X, \mathscr{A})$ is called a *fuzzy Poset (F-Poset).*A fuzzy preorder $\mathscr{A}$ is said to be *linear (total)* if for all $x \neq y$ we have $\mathscr{A}(x,y) \neq \mathscr{A}(y,x)$. A fuzzy subset on which fuzzy preorder is linear is called a *fuzzy chain*.
For a subset $A \subset X$ an *upper bound* is an element $x \in X$ satisfying $\mathscr{A}(y,x) \neq \mathscr{A}(x,y)$ for all $y \in A$, similarly we can define *lower bounds* of A. A greatest element of A is an $x \in A$ satisfying $\mathscr{A}(y,x) \neq \mathscr{A}(x,y)$ for all $y \in A$. Least elements are defined in the obvious fashion [2, 4].
In concrete category an isomorphism is necessarily a bijective morphism. It makes sense to ask whether in a given category the reverse implication also holds. In many of the cases the answer is yes, but in general, though it is not so: in the category of Posets and order preserving maps consider any bijection from two element antichain to a two element chain [8]. Two posets are *isomorphic* if there exist an order preserving (Homomorphism) bijective map between them such that whose inverse is also order preserving. Similarly two fuzzy posets $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are isomorphic if there exist a function f from X into Y such that f is a bijection, order preserving i.e $\mathscr{A}(x,y) \leq \mathscr{B}(f(x), f(y))$ for all $x, y \in X$ and f^{-1} is also order preserving [14].

3. Reversible F-Posets

Recall that a topological space X is reversible [10] if the only continuous self-bijections $f : X \rightarrow X$ are the homeomorphisms and a poset L is reversible if every order preserving self-bijections on L is a poset isomorphism [8].
We introduce a new definition called strict extension of a fuzzy relation $\mathscr{A}$ as defined below.

Definition 3.1. A relation $\mathscr{A}^*$ is said to be an *extension* of the fuzzy relation $\mathscr{A}$ on X, if $\mathscr{A}^*$ contains all relations in $\mathscr{A}$ and at least one other than those in $\mathscr{A}$, which is not derivable from the fuzzy relation $\mathscr{A}$. We denote $\mathscr{A} \subset \mathscr{A}^*$, if $\mathscr{A}^*$ is a strict extension of $\mathscr{A}$

Example 3.2. Let $X = \{1,2,3,4\}$, $\mathscr{A} = \{\frac{(1,1)}{.8}, \frac{(1,2)}{.5}, \frac{(1,3)}{.6}\}$ and $\mathscr{A}^* = \{\frac{(1,1)}{1}, \frac{(1,2)}{.7}, \frac{(1,3)}{.6}, \frac{(2,2)}{.9}\}$ then $\mathscr{A} \subset \mathscr{A}^*$.

Definition 3.3. A fuzzy poset $(X,\mathscr{A})$ is *reversible (Reversible F-poset)* if every order preserving self-bijection on X is a poset isomorphism.

Remark 3.4. The following theorem characterizes reversible fuzzy Posets.

Theorem 3.5. *A fuzzy poset $(X,\mathscr{A})$ is reversible if and only if there is no strict extension $\mathscr{A}^*$ of $\mathscr{A}$ such that $(X,\mathscr{A})$ is isomorphic to $(X,\mathscr{A}^*)$,equivalently, if $\mathscr{A}$ is not a strict extension of any order $\mathscr{A}_*$ such that $(X,\mathscr{A})$ is isomorphic to $(X,\mathscr{A}_*)$.*

Proof. Suppose fuzzy poset $(X,\mathscr{A})$ is reversible. If $\mathscr{A}^*$ is a strict extension of $(X,\mathscr{A})$ such that $(X,\mathscr{A})$ and $\mathscr{A}_*$ are isomorphic.Let $f:(X,\mathscr{A}) \to (X,\mathscr{A}^*)$ be the isomorphism, then one of the poset can be embedded in to the other as a sub poset and they are isomorphic through f. Then the restriction of f defined by $f^{|}:(X,\mathscr{A}) \to (X,\mathscr{A})$ is a bijection but not an isomorphism in general. Which is a contradiction for $(X,\mathscr{A})$ is reversible.

Conversely assume there is no strict extension $\mathscr{A}^*$ of $\mathscr{A}$, such that $(X,\mathscr{A})$ is isomorphic to $(X,\mathscr{A}^*)$. Let $f:(X,\mathscr{A}) \to (X,\mathscr{A})$ is any order preserving bijection. Define a fuzzy relation $\mathscr{A}^*$ on X defined by $\mathscr{A}^*(x,y) > 0$ if and only if $\mathscr{A}(f(x),f(y)) > 0$. Then $\mathscr{A} \subset \mathscr{A}^*$, since $\mathscr{A}(x,y) > 0$, then $\mathscr{A}(f(x),f(y)) > 0$ hence $\mathscr{A}^*(x,y) > 0$, and $(X,\mathscr{A}^*)$ is a fuzzy poset on X.Then the map $f:(X,\mathscr{A}) \to (X,\mathscr{A}^*)$ is an isomorphism. Since f is order preserving bijection also its inverse is also order preserving, take $\mathscr{A}^*(x,y) > 0$, then $\mathscr{A}(f(x),f(y)) > 0$, then $\mathscr{A}(x^{|},y^{|}) > 0$, where $x^{|} = f^{-1}(x)$ and $y^{|} = f^{-1}(y)$. This is possible only if $\mathscr{A}^* = \mathscr{A}$. Hence $f:(X,\mathscr{A}) \to (X,\mathscr{A})$ is an isomorphism. This completes the proof.

Remark 3.6. Analogues proposition and proof are available in [8].

Proposition 3.7. *Every finite fuzzy Posets are reversible.*

Proof. Suppose $(X,\mathscr{A})$ is a finite fuzzy poset, with $|X| < \infty$. Assume $(X,\mathscr{A})$ is not reversible then by definition there exist a fuzzy partial order $\mathscr{A}^*$ such that $(X,\mathscr{A})$ is isomorphic to$(X,\mathscr{A}^*)$.Since $\mathscr{A}^*$ is a strict extension, there exist $x,y \in X$ such that $\mathscr{A}^*(x,y) > 0$ and

$\mathscr{A}(x,y)=0$, which is not possible since isomorphism preserves order. Thus $(X,\mathscr{A})$ is reversible. This completes the proof.

Lemma 3.8. *Given a fuzzy order $\mathscr{A}$ on a non empty set X, we define $\mathscr{A}_{op}:X\times X\to[0,1]$ by $\mathscr{A}_{op}(x,y)=\mathscr{A}(y,x)$,for all $x,y\in X$ then $\mathscr{A}_{op}$ is a fuzzy order on X* [4].

Proposition 3.9. *Let $(X,\mathscr{A})$ is a reversible F-poset then $\mathscr{A}_{op}$ is a reversible F-poset.*

Proof. Trivial.

Definition 3.10. A set X is *well fuzzy ordered* by a fuzzy relation $\mathscr{A}$ if every non empty subset of X has a least element [2].

Lemma 3.11. *Every well fuzzy ordered set is necessarily a fuzzy chain.*

Proof. Consider a well fuzzy ordered set $(X,\mathscr{A})$. Let $A=\{x,y\}\subseteq X$, since X is well fuzzy ordered by $\mathscr{A}$ then A has a least element. Thus $\mathscr{A}(y,x)\geq\mathscr{A}(x,y)$ or $\mathscr{A}(x,y)\geq\mathscr{A}(y,x)$. Hence $(X,\mathscr{A})$ is a fuzzy chain. This completes the proof.

Proposition 3.12. *Every well fuzzy ordered set is reversible.*

Proof. Let $(X,\mathscr{A})$ is a well fuzzy ordered set then by above lemma $(X,\mathscr{A})$ is a fuzzy chain. Since every order preserving self-bijection on well-ordered sets is an order isomorphism [2], it follows that every self-bijective order preserving maps on X is an isomorphism, hence $(X,\mathscr{A})$ is reversible.This completes the proof.

Proposition 3.13. *Every fuzzy chain is reversible.*

Proof. Let $(X,\mathscr{A})$ is a fuzzy chain and $f:(X,\mathscr{A})\to(X,\mathscr{A})$ is an order preserving self-bijection then clearly f is an isomorphism, since $f^{-1}:(X,\mathscr{A})\to(X,\mathscr{A})$ is also order preserving it follows that $(X,\mathscr{A})$ is reversible.This completes the proof.

Remark 3.14. The following theorem characterize the relationship between a reversible F-posets and its level set. Recall the theorem of Inheung Chon [4].

Theorem 3.15.

Let $\mathscr{B}$ be a fuzzy relation on X, and $\mathscr{B}_p=\{(x,y)\in X\times X;\mathscr{B}(x,y)\geq p\}$. Then $\mathscr{B}$ is a fuzzy partial order relation if and only if $\mathscr{B}_p$ is a partial order relation in $X\times X$ for all p such that $0<p\leq 1$ [4].

Theorem 3.16. *A fuzzy poset $(X,\mathscr{A})$ is reversible if and only if $\mathscr{A}_p$ is a reversible partial order relation in $X \times X$ for all p such that $0 < p \leq 1$.*

Proof. Suppose $(X,\mathscr{A})$ is a reversible F-poset. Then by theorem 3.15 $\mathscr{A}_p$ is a partial order relation in $X \times X$ and hence $(X,\mathscr{A}_p)$ is a poset for $0 < p \leq 1$. Let $f : X \to X$ be an order preserving self-bijection, if f is an isomorphism then we are done, on the contrary suppose f is not an isomorphism then f^{-1} is not order preserving, then $\exists x^{|}, y^{|} \in X$ such that $(x^{|}, y^{|}) \in \mathscr{A}_p$ but $(x,y) \ni \mathscr{A}_p$ where $x = f^{-1}(x^{|}), y = f^{-1}(y^{|})$, then $\mathscr{A}(x^{|}, y^{|}) \geq p$ and $\mathscr{A}(f^{-1}(x^{|}), f^{-1}(y^{|})) = 0$ this is a contradiction to the assumption $(X,\mathscr{A})$ is a reversible F-poset.

Conversely assume $\mathscr{A}_p$ is a partial order relation in $X \times X$, and $(X,\mathscr{A}_p)$ is a reversible poset for $0 < p \leq 1$. Again by theorem 3.15 $\mathscr{A} : X \times X \to [0,1]$ is a fuzzy partial order relation in X. Let $f : X \to X$ be a fuzzy order preserving self-bijection, if f is an isomorphism then we are done, on the contrary suppose f is not an isomorphism then f^{-1} is not order preserving then by the same argument used above we can find $x^{|}, y^{|} \in X$ such that $\mathscr{A}(x^{|}, y^{|}) > 0$ but $\mathscr{A}(f^{-1}(x^{|}), f^{-1}(y^{|})) = 0$. Then $(x^{|}, y^{|}) \in \mathscr{A}_p$ for some $0 < p \leq 1$ and $(f^{-1}(x^{|}), f^{-1}(y^{|})) \ni \mathscr{A}_p$ for all $0 < p \leq 1$. Contradiction to the assumption $(X,\mathscr{A}_p)$ is a reversible poset. This completes the proof.

Corollary 3.17. *Let $(X,\mathscr{A})$ is a reversible F-poset then support of $\mathscr{A}$ is a reversible poset.*

Proof. Result follows from Theorem 3.16.This completes the proof.

Proposition 3.18. *Let X is a non empty set and $I = [0,1]$ be a complete lattice, $\mathscr{A} : I \times I \to [0,1]$ be a fuzzy order on I. Define $\mathscr{B} : I^X \times I^X \to [0,1]$ defined by $\mathscr{B}(\mu, \nu) = \wedge\{\mathscr{A}(\mu(x), \nu(x); x \in X\}$. Then $\mathscr{B}$ is a fuzzy order I^X and $(I^X, \mathscr{B})$ is a reversible F-poset.*

Proof. Since $\mathscr{A}$ is a fuzzy poset using this we immediately get $(I^X, \mathscr{B})$ is F-poset. Let $f : (I^X, \mathscr{B}) \to (I^X, \mathscr{B})$ is any order preserving self-bijection. Assume f^{-1} is not order preserving, then $\exists u^{|}, v^{|} \in I^X$ such that $\mathscr{B}(u^{|}, v^{|}) > 0$ but $\mathscr{B}(f^{-1}(u^{|}), f^{-1}(v^{|})) = 0$. Take $f^{-1}(u^{|}) = u$ and $f^{-1}(v^{|}) = v$ then we get $\mathscr{B}(u,v) = 0$ and $\mathscr{B}(f(u), f(v)) = \mathscr{B}(u^{|}, v^{|}) > 0$. A contradiction to the assumption f is order preserving. Hence $(I^X, \mathscr{B})$ is reversible as an F-poset. This completes the proof.

Remark 3.19. The following examples are non reversible fuzzy posets.

Example 3.20. Consider example due to B. Schro der, On $\mathbf{Z}$ define two non-trivial fuzzy comparabilities $\mathscr{A}$ and $\mathscr{A}^*$ as follows, $\mathscr{A}(n,n) = 1$ for all $n \in \mathbf{Z}$ and $\mathscr{A}(n,m) > 0$ if n is even and m is very right of n *(fuzzy spatial relation)*[6] and $\mid n-m \mid$ near to 1 i.e $\mathscr{A}(n,m) = 1; \mid n-m \mid = 1$, $\mathscr{A}(n,m) = .6; \mid n-m \mid \leq 3$, $\mathscr{A}(n,m) = .4; \mid n-m \mid \leq 6$, $\mathscr{A}(n,m) = .2; \mid n-m \mid \leq 12$, $\mathscr{A}(n,m) = 0; \mid n-m \mid > 12$ OR $n = 4k$ and $m = 4k+5$ for some positive integer k OR $n = 4k$ and $m = 4k+11$ for some positive integer k. $\mathscr{A}^*$ is defined as same as that of $\mathscr{A}$ with $\mathscr{A}^*(0,11) = 0.8$ is added, then $\mathscr{A} \subset \mathscr{A}^*$. Define $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) = n+4$ then f is an order preserving isomorphism. Hence ($\mathbf{Z}$,$\mathscr{A}$) is not reversible by Theorem 3.5.

Remark 3.21. From theorem 3.16 we have the following trivial example.

Example 3.22. Every non reversible poset $(X, \leq)$ can be identified as a non reversible fuzzy poset, the corresponding fuzzy partial order is defined as $\mathscr{A}(x,y) = 1$ if and only if $x \leq y$.

4. Artinian and Noetherian F-Posets

Definition 4.1. A sequence $\{x_n\}$ in a fuzzy poset $(X, \mathscr{A})$ is *increasing* provided $\mathscr{A}(x_n, x_{n+1}) \geq \mathscr{A}(x_{n+1}, x_n)$ [2].

Definition 4.2. An infinite *ascending sequence* in a fuzzy poset $(X, \mathscr{A})$ is an infinite sequence $\{x_n\}$ in X such that $\mathscr{A}(x_n, x_{n+1}) \geq \mathscr{A}(x_{n+1}, x_n)$. Similarly an infinite *descending* sequence in $(X, \mathscr{A})$ is an infinite sequence $\{x_n\}$ in X such that $\mathscr{A}(x_{n+1}, x_n) \geq \mathscr{A}(x_n, x_{n+1})$.

Definition 4.3. An F-poset with no infinite descending sequence is called *Artinian* F-posets and an F-poset with no infinite ascending sequence is called *Noetherian* F-posets.

Remark 4.4. Every finite F-Posets are Artinian as well as Noetherian.

Lemma 4.5. *An F-poset $(X, \mathscr{A})$ is well fuzzy ordered if and only if it is fuzzy Artinian.*

Proof. Suppose $(X, \mathscr{A})$ is well fuzzy ordered, assume $(X, \mathscr{A})$ is not fuzzy Artinian .Let $\{x_n\}$ in be an infinite descending sequence in X. Let $T = \{x_n; n \in \mathbf{N}\}$, then T has no minimal element which is not possible since $(X, \mathscr{A})$ is well fuzzy ordered, so $(X, \mathscr{A})$is Artinian.

Conversely assume $(X, \mathscr{A})$ is an Artinian F-poset then there is no infinite descending sequence in X.Let $K \subset X$ with no minimal element. . Let $x_0 \in K$ be arbitrary, since x_0 is not a minimal element then $\exists x_1 \in K$ such that $\mathscr{A}(x_1, x_0) \leq \mathscr{A}(x_0, x_1)$, similarly if x_1 is not a minimal element we can find an x_2 in K such that $\mathscr{A}(x_2,x_1) \leq \mathscr{A}(x_1,x_2)$ and so on, thus we build an infinite descending sequence $\{x_n\}$ in X, a contradiction, hence $(X,\mathscr{A})$ must be well fuzzy ordered. This completes the proof.

Proposition 4.6. *Every Artinian F-posets are reversible.*

Proof. Using lemma 4.5 we proved every Artinian F-posets are well fuzzy ordered also from proposition 3.11 the theorem follows. This completes the proof.

Corollary 4.7. *Every Noetherian F-posets are reversible.*

Proof. Result follows from proposition 3.9 and proposition 4.6. This completes the proof.

Remark 4.8. Next proposition is due to Michal Kukiela [8].

Proposition 4.9. *Every well fuzzy ordered set is hereditarily reversible (i.e. all its subposets are reversible).*

Proof. Clearly every subposets of a well fuzzy ordered set is well fuzzy ordered since if no, then the given F-poset also is not well fuzzy ordered. Then again from proposition 3.11 the result follows.This completes the proof.

5. Direct Product and Direct sum of F-Posets and their Reversibility

Definition 5.1. Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be fuzzy posets then fuzzy direct sum of X,Y is the fuzzy poset $X \sqcup Y$ Or $(X+Y)$ where the set is the union of X and Y and the fuzzy ordering $\mathscr{A} \bigoplus \mathscr{B}$ is given by $\mathscr{A} \bigoplus \mathscr{B}(x,y) > 0$ in $X \sqcup Y$ if either $x,y \in X$ and $\mathscr{A}(x,y) > 0$ OR $x,y \in Y$ and $\mathscr{B}(x,y) > 0$.

Definition 5.2. Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be fuzzy posets then the direct product $X \times Y = XY$ of X and Y is defined by $(XY, \mathscr{A} \times \mathscr{B})$ where $X \times Y : XY \to [0,1]$ is a fuzzy relation defined by $\mathscr{A} \times \mathscr{B}((x_1,y_1),(x_2,y_2)) = min\{\mathscr{A}(x_1,x_2),\mathscr{B}(y_1,y_2)\}$.

Remark 5.3. In [4] it is proved that the direct product of two fuzzy posets is again a fuzzy poset with respect to the relation defined in 5.2. Now we have to check the reversibility of these types of new F-posets.

Remark 5.4. In [7] Jayaprasad P.N, T.P Johnson proved that direct product of reversible *frames* is again reversible. As analogues we proved the following theorem for reversible F-posets.

Theorem 5.5. *Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be two reversible fuzzy posets with least elements $0_X, 0_Y$ then the direct product $(XY,\mathscr{A}\times\mathscr{B})$ is a reversible F-poset.*

Proof. Given $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be reversible fuzzy posets. Assume $(XY,\mathscr{A}\times\mathscr{B})$ is not reversible as F-poset, then by theorem 3.5 there exist a strict extension $\mathscr{A}^*\times\mathscr{B}^*$ of $\mathscr{A}\times\mathscr{B}$ such that $(XY,\mathscr{A}\times\mathscr{B})$ is isomorphic to $(XY,\mathscr{A}^*\times\mathscr{B}^*)$ as fuzzy posets. Let $f:(XY,\mathscr{A}\times\mathscr{B})\to(XY,\mathscr{A}^*\times\mathscr{B}^*)$ be the fuzzy poset isomorphism. Take $\mathscr{A}^*$,$\mathscr{B}^*$ the component fuzzy partial order of $\mathscr{A}^*\times\mathscr{B}^*$ such that $\mathscr{A}^*\times\mathscr{B}^*((x_1,y_1),(x_2,y_2)) = min\{\mathscr{A}^*(x_1,x_2),\mathscr{B}^*(y_1,y_2)\}$. Similarly $\mathscr{A}$ and $\mathscr{B}$ be the component fuzzy partial order relation of $\mathscr{A}\times\mathscr{B}$ such that $\mathscr{A}\times\mathscr{B}((x_1,y_1),(x_2,y_2)) = min\{\mathscr{A}(x_1,x_2),\mathscr{B}(y_1,y_2)\}$. Then $(X,\mathscr{A}^*)$ and $(Y,\mathscr{B}^*)$ are extensions of $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ with at least one of them a strict extension otherwise $\mathscr{A}^*\times\mathscr{B}^*$ of $\mathscr{A}\times\mathscr{B}$ would not be a strict extension.

Define $g:(X,\mathscr{A})\to(X,\mathscr{A}^*)$ by $g(x)=y$ where y is chosen so that $f(x,0_Y)=(y,0_Y)$ as f preserves minimal element. Since f is an isomorphism then g is one-one and onto (bijection). Take $x_1,x_2\in X$ such that $\mathscr{A}(x_1,x_2)>0$ then $\mathscr{A}\times\mathscr{B}((x_1,0_Y),(x_2,0_Y)) = min\{\mathscr{A}(x_1,x_2),1\}>0$. Taking the image of x_1,x_2 under g and applying f to $\mathscr{A}\times\mathscr{B}((x_1,0_Y),(x_2,0_Y))>0$, we get $\mathscr{A}^*\times\mathscr{B}^*((y_1,0_Y),(y_2,0_Y))>0$, hence $\mathscr{A}^*(y_1,y_2)>0$, so g is order preserving. Again take $y_1,y_2\in X$ such that $\mathscr{A}^*(y_1,y_2)>0$ then $\mathscr{A}^*\times\mathscr{B}^*((y_1,0_Y),(y_2,0_Y))>0$. Then there exist $x_1,x_2\in X$ such that $f(x_1,0_Y)=(y_1,0_Y)$ and $f((x_2,0_Y)=(y_2,0_Y)$ where $g(x_1)=y_1, g(x_2)=y_2$. Since f^{-1} is order preserving then $\mathscr{A}\times\mathscr{B}((x_1,0_Y),(x_2,0_Y))>0$ in $(XY,\mathscr{A}\times\mathscr{B})$, hence $\mathscr{A}(x_1,x_2)>0$ in $(X,\mathscr{A})$. Thus $\mathscr{A}(g^{-1}(y_1),g^{-1}(y_2))>0$, so g^{-1} is order preserving. Hence $g:(X,\mathscr{A})\to(X,\mathscr{A}^*)$ is a fuzzy isomorphism. In a similar way we can construct $g^{|}:(Y,\mathscr{B})\to(Y,\mathscr{B}^*)$. Thus either or both $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ are not reversible by Theorem 3.3, which is a contradiction. This completes the proof.

Corollary 5.6. *Cartesian product of finite number of reversible F-posets is reversible.*

Proof. Proof immediately follows from theorem 5.5. This completes the proof.

Remark 5.7. By a similar method as we use in Theorem 5.4 we prove the following proposition.

Proposition 5.8. *Let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be reversible fuzzy posets then fuzzy direct sum of X,Y, $(X\sqcup Y,\mathscr{A}\bigoplus\mathscr{B})$ is a reversible F-poset.*

Proof. Assume $(X\sqcup Y,\mathscr{A}\bigoplus\mathscr{B})$ is a not reversible F-poset. Then by theorem 3.5, 3 there exist a strict extension $\mathscr{A}^*\bigoplus\mathscr{B}^*$ of $\mathscr{A}\bigoplus\mathscr{B}$ such that $(X\sqcup Y,\mathscr{A}\bigoplus\mathscr{B})$ is isomorphic as an F-poset to $(X\sqcup Y,\mathscr{A}^*\bigoplus\mathscr{B}^*)$, and $\varphi:(X\sqcup Y,\mathscr{A}\bigoplus\mathscr{B})\to(X\sqcup Y,\mathscr{A}^*\bigoplus\mathscr{B}^*)$ be the fuzzy isomorphism. Take $\mathscr{A}^*,\mathscr{B}^*$ the component fuzzy partial order of $\mathscr{A}^*\bigoplus\mathscr{B}^*$ such that $\mathscr{A}^*\bigoplus\mathscr{B}^*(x,y)>0$ if and only if and $\mathscr{A}^*(x,y)>0$ in $(X,\mathscr{A}^*)$ OR and $\mathscr{B}^*(x,y)>0$ in $(Y,\mathscr{B}^*)$. Similarly $\mathscr{A}$ and $\mathscr{B}$ be the component fuzzy partial order relation of $\mathscr{A}\bigoplus\mathscr{B}$ such that $\mathscr{A}\bigoplus\mathscr{B}(x,y)>0$ if and only if $\mathscr{A}(x,y)>0$ in $(X,\mathscr{A})$ OR $\mathscr{B}(x,y)>0$ in $(Y,\mathscr{B})$. Then $(X,\mathscr{A}^*)$ and $(Y,\mathscr{B}^*)$ are extensions of $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ with at least one of them a strict extension otherwise $\mathscr{A}^*\bigoplus\mathscr{B}^*$ of $\mathscr{A}\bigoplus\mathscr{B}$ would not be a strict extension.

Define $f:(X,\mathscr{A})\to(X,\mathscr{A}^*)$ by $f(x)=y$, y is chosen so that $\varphi(x)=y$ i.e. $\varphi\mid_X=f$, then f is one one and onto. Let $x_1,x_2\in X$ such that $\mathscr{A}(x_1,x_2)>0$ then $\mathscr{A}\bigoplus\mathscr{B}(x_1,x_2)>0$, since φ is an isomorphism then $\mathscr{A}^*\bigoplus\mathscr{B}^*(\varphi(x_1),\varphi(x_2))>0$ implies $\mathscr{A}^*((f(x_1),f(x_2))>0$ hence f preserves order.

Now we prove f^{-1} is order preserving. Take $y_1,y_2\in X$ such that $\mathscr{A}^*(y_1,y_2)>0$,where $y_1=f(x_1)$ and $y_2=f(x_2)$ then $\mathscr{A}^*\bigoplus\mathscr{B}^*(y_1,y_2)>0$. Since φ^{-1} is order preserving there exist $x_1,x_2\in X$ such that $\mathscr{A}\bigoplus\mathscr{B}(x_1,x_2)>0$, then $\mathscr{A}(x_1,x_2)>0$ i.e. $\mathscr{A}(f^{-1}(y_1),f^{-1}(y_2))>0$, hence f^{-1} is order preserving. Thus we obtain an isomorphism $f:(X,\mathscr{A})\to(X,\mathscr{A}^*)$. Similarly we can construct an isomorphism between $(Y,\mathscr{B})$ and $(Y,\mathscr{B}^*)$. Thus either or both $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ are not reversible by Theorem 3.3, which is a contradiction. This completes the proof.

6. Concluding Remarks

In this paper we generalize the idea of reversible posets introduced by Michal Kukiela [8]. In our paper we obtain some class of reversible posets and some general results about reversible F-posets, results analogues to [8] are also obtained. Direct sum and Cartesian product of reversible product are also studied.

In topological context a reversible space has some features that it is maximal or minimal with respect to some topological property. Very similar way we can say that reversible posets are more stable. We can extend this reversible ideas in fuzzy spatial relations structures and fuzzy ordered structures, also modelling complex relations using fuzzy order and study in the context reversibility will definitely get more strong results.

Conflict of Interests

The authors declare that there is no conflict of interests.

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References

[1] D. Adnadjevic, Dimension of fuzzy ordered sets, Fuzzy Sets Syst., 67 (1994), 349-357.

[2] Ismat Beg,On fuzzy order relations, J. Nonlinear Sci. Appl, 5 (2012), 357-378.

[3] U. Bodenhofer, Representations and constructions of similarity-based fuzzy orderings, Fuzzy Sets Syst., 137 (2003), 113-137.

[4] Inheung Chon, Fuzzy partial order relations and fuzzy lattices, Korean J. Math.,17 (4) (2009), 361-374.

[5] J. C. Fodor and S. Ovchinnikov, On aggregation of T-transitivity fuzzy binary relations, Fuzzy Sets Syst., 72 (1995), 135-145.

[6] Celine Hudelot, Jamal Atif, Isbelle Bloch,Fuzzy spatial relation ontology for image interpretation, Fuzzy sets syst., 159 (2008), 1929-1951.

[7] P.N Jayaprasad, T.P Johnson, Reversible Frames, J. Adv.stud. Topol, 3 (2) (2012), 7-13.

[8] Michal Kukiela, Reversible and bijectievly related posets, Order, 26 (2009), 119-124.

[9] S. Kundu, Similarity relations, fuzzy linear orders and fuzzy partial orders, Fuzzy Sets Syst., 109 (2000), 419-428.

[10] H. Lai and D. Zhang, Fuzzy preorder and fuzzy topology, Fuzzy Sets Syst., 157 (2006), 1865-1885.

[11] S. Lee, K. H Lee and D. Lee, Order relation for type-2 fuzzy values, J. Tsinghua Sci. Technol., 8 (2003), 30-36.

[12] M.Rajagopalan and A.Willansky, Reversible Topological spaces, J. Aust. Math. Soc., 61 (1966), 129-138.

[13] B. Seselja and A. Tepavcevic, Representing ordered structures by fuzzy sets: An overview, Fuzzy Sets Syst., 136 (2003), 21-39.

[14] P. Venugopalan,Fuzzy ordered sets, Fuzzy Sets and Systems,46,(1992), 221-226.

[15] Wei Yao, An approach to the fuzzification of complete lattices, J. Intell. Fuzzy Syst., 26 (2014), 2239-2249.

[16] L. A. Zadeh, Similarity relation and fuzzy ordering, Inf. Sci., 1 (3) (1971), 177-200.