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SHARP BOUNDS INVOLVING THE SÁNDOR-YANG MEANS IN TERMS OF OTHER BIVARIATE MEANS

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Abstract. In this paper, we present the best possible parameters $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4 \in [0,1]$ such that the double inequalities

$$\frac{(1+\alpha_1)G(a,b)+(1-\alpha_1)A(a,b)}{(1-\alpha_1)G(a,b)+(1+\alpha_1)A(a,b)} < \frac{X(a,b)}{A(a,b)} < \frac{(1+\beta_1)G(a,b)+(1-\beta_1)A(a,b)}{(1-\beta_1)G(a,b)+(1+\beta_1)A(a,b)},$$

$$\frac{(1+\alpha_2)Q(a,b)+(1-\alpha_2)A(a,b)}{(1-\alpha_2)Q(a,b)+(1+\alpha_2)A(a,b)} < \frac{R_{QA}(a,b)}{A(a,b)} < \frac{(1+\beta_2)Q(a,b)+(1-\beta_2)A(a,b)}{(1-\beta_2)Q(a,b)+(1+\beta_2)A(a,b)},$$

$$\frac{(1+\alpha_3)A(a,b)+(1-\alpha_3)G(a,b)}{(1-\alpha_3)A(a,b)+(1+\alpha_3)G(a,b)} < \frac{I(a,b)}{G(a,b)} < \frac{(1+\beta_3)A(a,b)+(1-\beta_3)G(a,b)}{(1-\beta_3)A(a,b)+(1+\beta_3)G(a,b)},$$

$$\frac{(1+\alpha_4)A(a,b)+(1-\alpha_4)Q(a,b)}{(1-\alpha_4)A(a,b)+(1+\alpha_4)Q(a,b)} < \frac{R_{AQ}(a,b)}{Q(a,b)} < \frac{(1+\beta_4)A(a,b)+(1-\beta_4)Q(a,b)}{(1-\beta_4)A(a,b)+(1+\beta_4)Q(a,b)}$$

hold for all $a,b>0$ with $a\neq b$. Here $G(a,b)$, $A(a,b)$ and $Q(a,b)$ denote respectively the classical geometric, arithmetic and quadratic means of a and b, and $R_{GA}(a,b)=X(a,b)$, $R_{AG}(a,b)=I(a,b)$, $R_{QA}(a,b)$ and $R_{AQ}(a,b)$ are Sándor, identric and two Sándor -Yang means derived from the Schwab-Borchardt mean.

Keywords: Sándor-Yang mean; Schwab-Borchardt mean; geometric mean; arithmetic mean; quadratic mean.

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1. Introduction

For all $a,b>0$ with $a\neq b$, the Schwab-Borchardt mean $SB(a,b)$[1, 2, 3] is defined by

$$SB(a,b)=\begin{cases}\frac{\sqrt{b^2-a^2}}{\arccos(a/b)}, & if \quad a<b\ ,\\ \frac{\sqrt{a^2-b^2}}{\arccos h(a/b)}, & if \quad a>b\end{cases}$$

where $\arccos(x)$ and $\arccos h(x)=\log\left(x+\sqrt{x^2-1}\right)$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

Let $G(a,b)=\sqrt{ab}$, $A(a,b)=(a+b)/2$ and $Q(a,b)=\sqrt{(a^2+b^2)/2}$ are respectively the classical geometric, arithmetic, quadratic and contra-harmonic means of a and b. It is well known that the Schwab-Borchardt mean is strictly increasing in both a and b, non-symmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean. For example,

$P(a,b)=\frac{a-b}{2\arcsin[(a-b)/(a+b)]}=SB[G(a,b),A(a,b)]$ is the first Seiffert mean,
$T(a,b)=\frac{a-b}{2\arctan[(a-b)/(a+b)]}=SB[A(a,b),Q(a,b)]$ is the second Seiffert mean,
$M(a,b)=\frac{a-b}{2\arcsin h[(a-b)/(a+b)]}=SB[Q(a,b),A(a,b)]$ is the Neuman-Sándor mean,
$L(a,b)=\frac{a-b}{2\arctan h[(a-b)/(a+b)]}=SB[A(a,b),G(a,b)]$ is the logarithmic mean.

Yang[5] found a new mean (call Sándor-Yang mean) derived from the Schwab-Borchardt mean as follows:

$$R(a,b)=be^{a/SB(a,b)-1}$$

Let $R_{GA}(a,b)=R[G(a,b),A(a,b)]$, $R_{AG}(a,b)=R[A(a,b),G(a,b)]$, $R_{AQ}(a,b)=R[A(a,b),Q(a,b)]$, $R_{QA}(a,b)=R[Q(a,b),\ A(a,b)]$. Then the following explicit formulas for
$R_{GA}(a,b)$, $R_{AG}(a,b)$, $R_{AQ}(a,b)$ and $R_{QA}(a,b)$ are found by Yang[5]:

$$R_{GA}(a,b)=A(a,b)\,e^{G(a,b)/P(a,b)-1}=X(a,b)\,,R_{AQ}(a,b)=Q(a,b)\,e^{A(a,b)/T(a,b)-1},$$

$$R_{AG}(a,b)=G(a,b)\,e^{A(a,b)/L(a,b)-1}=I(a,b)\,,R_{QA}(a,b)=A(a,b)\,e^{Q(a,b)/M(a,b)-1},$$

where $X(a,b)$[6] and $I(a,b)$[7, 8] are respectively Sándor and identric means. Then it is that the inequalities (See [4], Theorem 4.1)

$$G(a,b)<X(a,b)<I(a,b)<A(a,b)<R_{AQ}(a,b)<R_{QA}(a,b)<Q(a,b)$$

holds for all $a,b>0$ with $a\neq b$.

In recent years,the Sándor-Yang type means have been the subject on intensive research. In particular, many remarkable inequalities for the Sándor-Yang type means can be found in the literature [9, 10, 13, 14, 15, 16, 17, 19, 20].

Alzer and Qiu[11] prove that the inequality

$$\alpha A(a,b)+(1-\alpha)G(a,b)<I(a,b)<\beta A(a,b)+(1-\beta)G(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 2/3,\beta\geq 2/e=0.7357\cdots$.

In[12], Qian et al. present the best possible parameters $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4\in(0,1)$ such that the double inequalities

$$\alpha_1 A(a,b)+(1-\alpha_1)H(a,b)<X(a,b)<\beta_1 A(a,b)+(1-\beta_1)H(a,b),$$

$$\alpha_2 A(a,b)+(1-\alpha_2)G(a,b)<X(a,b)<\beta_2 A(a,b)+(1-\beta_2)G(a,b),$$

$$H[\alpha_3 a+(1-\alpha_3)b,\alpha_3 b+(1-\alpha_3)a]<X(a,b)<H[\beta_3 a+(1-\beta_3)b,\beta_3 b+(1-\beta_3)a],$$

$$G[\alpha_4 a+(1-\alpha_4)b,\alpha_4 b+(1-\alpha_4)a]<X(a,b)<G[\beta_4 a+(1-\beta_4)b,\beta_4 b+(1-\beta_4)a]$$

hold for all $a,b>0$ with $a\neq b$,where $H(a,b)=2ab/(a+b)$ is harmonic mean of a and b.

In[18, 19], the authors established the following sharp inequalities

$$M_\alpha(a,b)<R_{QA}(a,b)<M_\beta(a,b),$$

$$M_\lambda(a,b)<R_{AQ}(a,b)<M_\mu(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq\log 2/\left[1+\log 2-\sqrt{2}\log\left(1+\sqrt{2}\right)\right]=1.5517\cdots,\beta\geq 5/3,\lambda\leq 4\log 2/[4+2\log 2-\pi]=1.2351\cdots$ and $\mu\geq 4/3$.Where $M_p(a,b)=\left[(a^p+b^p)/2\right]^{1/p}$ and $M_0(a,b)=\sqrt{ab}$ is the pth power mean of a and b.

The main purpose of this paper is to present the best possible parameters $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2,\beta_3,\beta_4\in[0,1]$ such that the double inequalities

$$\frac{(1+\alpha_1)G(a,b)+(1-\alpha_1)A(a,b)}{(1-\alpha_1)G(a,b)+(1+\alpha_1)A(a,b)}<\frac{X(a,b)}{A(a,b)}<\frac{(1+\beta_1)G(a,b)+(1-\beta_1)A(a,b)}{(1-\beta_1)G(a,b)+(1+\beta_1)A(a,b)},$$

$$\frac{(1+\alpha_2)Q(a,b)+(1-\alpha_2)A(a,b)}{(1-\alpha_2)Q(a,b)+(1+\alpha_2)A(a,b)}<\frac{R_{QA}(a,b)}{A(a,b)}<\frac{(1+\beta_2)Q(a,b)+(1-\beta_2)A(a,b)}{(1-\beta_2)Q(a,b)+(1+\beta_2)A(a,b)},$$

$$\frac{(1+\alpha_3)A(a,b)+(1-\alpha_3)G(a,b)}{(1-\alpha_3)A(a,b)+(1+\alpha_3)G(a,b)}<\frac{I(a,b)}{G(a,b)}<\frac{(1+\beta_3)A(a,b)+(1-\beta_3)G(a,b)}{(1-\beta_3)A(a,b)+(1+\beta_3)G(a,b)},$$

$$\frac{(1+\alpha_4)A(a,b)+(1-\alpha_4)Q(a,b)}{(1-\alpha_4)A(a,b)+(1+\alpha_4)Q(a,b)}<\frac{R_{AQ}(a,b)}{Q(a,b)}<\frac{(1+\beta_4)A(a,b)+(1-\beta_4)Q(a,b)}{(1-\beta_4)A(a,b)+(1+\beta_4)Q(a,b)}$$

hold for all $a,b>0$ with $a\neq b$.

2. Lemmas

In order to prove our main results we need two lemmas, which we present in this section.

Lemmas 2.1. Let $p\in(0,1)$ and

$$\begin{aligned}\phi(x)=&\left(1-p^2\right)^2x^4+2\left(-2p^4+p^3-p+2\right)x^3+2(1-p)^2\left(3p^2+2p+3\right)x^2\\&+2\left(-2p^4+5p^3-5p+2\right)x+p^4-4p^3-2p^2-4p+1.\end{aligned}\tag{2.1}$$

Then the following statements are true:

(1) If $p=2/3$, then $\phi(x)<0$ for all $x\in(0,1)$ and $\phi(x)>0$ for all $x\in\left(1,\sqrt{2}\right)$;

(2) If $p=(e-1)/(e+1)=0.4621\cdots$, then there exists $\lambda_1\,(=0.5736\cdots)\in(0,1)$ such that $\phi(x)<0$ for $x\in(0,\lambda_1)$ and $\phi(x)>0$ for $x\in(\lambda_1,1)$;

(3) If $p=\left(3+2\sqrt{2}\right)\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}-e\right]/\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}+e\right]=0.7145\cdots$, then there exists $\lambda_2\,(=1.1126\cdots)\in\left(1,\sqrt{2}\right)$ such that $\phi(x)<0$ for $x\in(1,\lambda_2)$ and $\phi(x)>0$ for $x\in\left(\lambda_2,\sqrt{2}\right)$.

Proof For part (1), if $p=2/3$, then (2.1) becomes

$$\phi(x)=\frac{1}{81}(x-1)\left(25x^3+225x^2+327x+287\right)\tag{2.2}$$

Therefore, part (1) follows easily from (2.2).

For part (2), if $p=(e-1)/(e+1)=0.4621\cdots$, then simple computations lead to

$$-2p^4+p^3-p+2=1.5453...>0,\tag{2.3}$$

$$-2p^4+5p^3-5p+2=0.0916...>0,\tag{2.4}$$

$$\phi(0)=p^4-4p^3-2p^2-4p+1=-1.6247\cdots<0,\tag{2.5}$$

$$\phi(1)=8(2-3p)=4.9091\cdots>0.\tag{2.6}$$

It follows from (2.3) and (2.4) that

$$\phi'(x) = 4\left(1-p^2\right)^2 x^3 + 6\left(-2p^4+p^3-p+2\right)x^2 + 4(1-p)^2\left(3p^2+2p+3\right)x$$
$$+2\left(-2p^4+5p^3-5p+2\right) > 0 \tag{2.7}$$

for $x \in (0,1)$.

Therefore, part (2) follows easily from (2.5) and (2.6) together with (2.7).

For part (3), if $p = \left(3+2\sqrt{2}\right)\left[\left(1+\sqrt{2}\right)^{\sqrt{2}} - e\right] / \left[\left(1+\sqrt{2}\right)^{\sqrt{2}} + e\right] = 0.7145\cdots$, then numerical computations lead to

$$-2p^4+p^3-p+2 = 1.1287\cdots > 0, \tag{2.8}$$

$$-2p^4+5p^3-5p+2 = -0.2699\cdots < 0, \tag{2.9}$$

$$\phi(0) = p^4-4p^3-2p^2-4p+1 = -4.0785\cdots < 0, \tag{2.10}$$

$$\phi\left(\sqrt{2}\right) = (-12\sqrt{2}+17)p^4 + (14\sqrt{2}-20)p^3 - 2p^2$$
$$+(-14\sqrt{2}-20)p + 17 + 12\sqrt{2} = 4.4433\cdots > 0 \tag{2.11}$$

and

$$\phi'(x) = 4\left(1-p^2\right)^2 x^3 + 6\left(-2p^4+p^3-p+2\right)x^2 + 4(1-p)^2\left(3p^2+2p+3\right)x$$
$$+2\left(-2p^4+5p^3-5p+2\right). \tag{2.12}$$

It follows from (2.12)

$$\phi'(x) > 4\left(1-p^2\right)^2 + 6\left(-2p^4+p^3-p+2\right) + 4(1-p)^2\left(3p^2+2p+3\right)$$
$$+2\left(-2p^4+5p^3-5p+2\right) = 32(1-p) > 0 \tag{2.13}$$

Therefore, part (3) follows from (2.10), (2.11) and (2.13).

Lemmas 2.2. Let $p \in (0,1)$ and

$$\varphi(x) = \left(p^4-4p^3-2p^2-4p+1\right)x^4 + 2\left(-2p^4+5p^3-5p+2\right)x^3$$
$$+2(1-p)^2\left(3p^2+2p+3\right)x^2 + 2\left(-2p^4+p^3-p+2\right)x + \left(1-p^2\right)^2. \tag{2.14}$$

Then the following statements are true:

(1) If $p=2/3$, then $\phi(x)>0$ for all $x\in(0,1)$ and $\phi(x)<0$ for all $x\in\left(1,\sqrt{2}\right)$;
(2) If $p=1$, then $\phi(x)<0$ for all $x\in\left(1,\sqrt{2}\right)$;
(3) If $p=\left(3+2\sqrt{2}\right)\left(1-e^{\frac{\pi}{4}-1}\right)/\left(1+e^{\frac{\pi}{4}-1}\right)=0.6230\cdots$, then there exists $\lambda_3(=1.1054\cdots)\in\left(1,\sqrt{2}\right)$ such that $\phi(x)>0$ for $x\in(1,\lambda_3)$ and $\phi(x)<0$ for $x\in\left(\lambda_3,\sqrt{2}\right)$.

Proof For part (1), if $p=2/3$, then (2.14) lead to

$$\varphi(x)=-\frac{1}{81}(x-1)\left(287x^3+327x^2+225x+25\right). \tag{2.15}$$

Therefore, part (1) follows easily from (2.15).

For part (2), if $p=1$, then (2.14) lead to

$$\varphi(x)=-8x^4<0. \tag{2.16}$$

Therefore, part (2) follows easily from (2.16).

For part (3), If $p=\left(3+2\sqrt{2}\right)\left(1-e^{\frac{\pi}{4}-1}\right)/\left(1+e^{\frac{\pi}{4}-1}\right)=0.6230\cdots$, then numerical computations lead to

$$p^4-4p^3-2p^2-4p+1=-3.0848\cdots<0, \tag{2.17}$$

$$-2p^4+5p^3-5p+2=-0.2072\cdots<0, \tag{2.18}$$

$$-2p^4+p^3-p+2=1.3175\cdots>0, \tag{2.19}$$

$$\varphi(1)=8(2-3p)=1.0478\cdots>0, \tag{2.20}$$

$$\varphi\left(\sqrt{2}\right)=-6.3354\cdots<0, \tag{2.21}$$

and

$$\begin{aligned}\varphi'(x)=&4\left(p^4-4p^3-2p^2-4p+1\right)x^3+6\left(-2p^4+5p^3-5p+2\right)x^2\\&+4(1-p)^2\left(3p^2+2p+3\right)x+2\left(-2p^4+p^3-p+2\right).\end{aligned} \tag{2.22}$$

It follows from (2.17)-(2.19) and (2.22) that

$$\begin{aligned}\varphi'(x) &< 4\left(p^4-4p^3-2p^2-4p+1\right)x+6\left(-2p^4+5p^3-5p+2\right)x\\ &+4(1-p)^2\left(3p^2+2p+3\right)x+2\left(-2p^4+p^3-p+2\right)x\\ &=32(1-2p)x<0 \end{aligned}\tag{2.23}$$

for $x\in\left(1,\sqrt{2}\right)$.

Therefore, part (3) follows from (2.20), (2.21) and (2.23).

3. Main results

Theorem 3.1. The double inequality

$$A\frac{(1+\alpha_1)G+(1-\alpha_1)A}{(1-\alpha_1)G+(1+\alpha_1)A}<X(a,b)<A\frac{(1+\beta_1)G+(1-\beta_1)A}{(1-\beta_1)G+(1+\beta_1)A}\tag{3.1}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\geq 2/3$, $\beta_1\leq(e-1)/(e+1)=0.4621\cdots$.

Proof Since $X(a,b)$, $G(a,b)$ and $A(a,b)$ are symmetric and homogenous of degree 1, we assume that $a>b$. Let $v=(a-b)/(a+b)\in(0,1)$,$x=\sqrt{1-v^2}\in(0,1)$ and $p\in[0,1]$. Then (3.1) can be rewritten as

$$\beta_1<\frac{(A+G)\left[A-X(a,b)\right]}{(A-G)\left[A+X(a,b)\right]}<\alpha_1\tag{3.2}$$

$$\frac{(A+G)\left[A-X(a,b)\right]}{(A-G)\left[A+X(a,b)\right]}=\frac{\left(1+\sqrt{1-v^2}\right)\left(1-e^{\sqrt{1-v^2}\arcsin(v)/v-1}\right)}{\left(1-\sqrt{1-v^2}\right)\left(1+e^{\sqrt{1-v^2}\arcsin(v)/v-1}\right)}\tag{3.3}$$

$$\begin{aligned}&\log X(a,b)-\log A\frac{(1+p)G+(1-p)A}{(1-p)G+(1+p)A}\\ &=\frac{\sqrt{1-v^2}\arcsin(v)}{v}-\log\frac{(1+p)\sqrt{1-v^2}+1-p}{(1-p)\sqrt{1-v^2}+1+p}-1\\ &=\frac{x\arcsin\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}-\log\frac{(1+p)x+1-p}{(1-p)x+1+p}-1.\end{aligned}\tag{3.4}$$

Let

$$F(x)=\frac{x\arcsin\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}-\log\frac{(1+p)x+1-p}{(1-p)x+1+p}-1\tag{3.5}$$

Then simple computations lead to

$$F\left(0^{+}\right)=\log\left(\frac{1+p}{1-p}\right)-1, F\left(1^{-}\right)=0, \tag{3.6}$$

$$F'(x)=\frac{1}{\left(1-x^{2}\right)^{3/2}}f(x), \tag{3.7}$$

where

$$f(x)=\arcsin\left(\sqrt{1-x^{2}}\right)-\frac{\sqrt{1-x^{2}}\left[\left(1-p^{2}\right)x^{3}+2(1-p)^{2}x^{2}+\left(1-p^{2}\right)x+4p\right]}{\left[(1-p)x+1+p\right]\left[(1+p)x+1-p\right]} \tag{3.8}$$

$$f\left(0^{+}\right)=\frac{\pi}{2}-\frac{4p}{1-p^{2}}, f\left(1^{-}\right)=0 \tag{3.9}$$

$$f'(x)=-\frac{2\sqrt{1-x^{2}}}{\left[(1-p)x+1+p\right]^{2}\left[(1+p)x+1-p\right]^{2}}\phi(x) \tag{3.10}$$

where $\phi(x)$ is defined Lemma 2.1.

We divide the proof into two cases.

- **Case 1** If $p=2/3$. Then (3.4)-(3.7), (3.9) and (3.10) together with Lemma 2.1(1) lead to the conclusion that

$$X(a,b)>A\frac{5G+A}{G+5A} \tag{3.11}$$

- **Case 2** If $p=(e-1)/(e+1)$.Then from (3.6) and (3.9) together with numerical computations we get

$$F\left(0^{+}\right)=0, f\left(0^{+}\right)=\frac{\pi}{2}-\frac{e^{2}-1}{e}=-0.7796\cdots<0. \tag{3.12}$$

Let $\lambda_1=0.5736\cdots$ be the number given in Lemma 2.1(2).

We divide the discussion into two subcases.

- **subcase 1** $x\in(0,\lambda_1]$.Then Lemma 2.1(2) and (3.10) lead to the conclusion that $f(x)$ is Strictly increasing on the interval $(0,\lambda_1]$.
- **subcase 2** $x\in[\lambda_1,1)$.Then Lemma 2.1(2) and (3.10) lead to the conclusion that $f(x)$ is Strictly decreasing on the interval $x\in[\lambda_1,1)$, with (3.9) imply that $f(x)>0$.

Then from (3.12) and Subcase1 we know that there exists $x_0\in(0,\lambda_1)$ such that $f(x)<0$ for $x\in(0,x_0]$ and $f(x)>0$ for $[x_0,\lambda_1]$.

Thus, $f(x)<0$ for $x\in(0,x_0]$ and $f(x)>0$ for $x\in[x_0,1)$.

With (3.7) we know that $F(x)$ is strictly decreasing on $(0,x_0]$ and strictly increasing on $[x_0,1)$.

Therefore,

$$X(a,b) < A\frac{eG+A}{G+eA} \tag{3.13}$$

follows from (3.4)-(3.6) and (3.12) together with the piecewise monotonicity of $F(x)$.

Note that

$$\lim_{v\to 0^+}\frac{\left(1+\sqrt{1-v^2}\right)\left(1-e^{\sqrt{1-v^2}\arcsin(v)/v-1}\right)}{\left(1-\sqrt{1-v^2}\right)\left(1+e^{\sqrt{1-v^2}\arcsin(v)/v-1}\right)} = \frac{2}{3}, \tag{3.14}$$

$$\lim_{v\to 1^-}\frac{\left(1+\sqrt{1-v^2}\right)\left(1-e^{\sqrt{1-v^2}\arcsin(v)/v-1}\right)}{\left(1-\sqrt{1-v^2}\right)\left(1+e^{\sqrt{1-v^2}\arcsin(v)/v-1}\right)} = \frac{e-1}{e+1} = 0.4621\cdots. \tag{3.15}$$

In conclusion, Theorem 2.1 follows form (3.3), (3.11) and (3.13)-(3.15) together with that fact that inequality (3.1) is equivalent to (3.2).

Theorem 3.2. The double inequality

$$A\frac{(1+\alpha_2)Q+(1-\alpha_2)A}{(1-\alpha_2)Q+(1+\alpha_2)A} < R_{QA}(a,b) < A\frac{(1+\beta_2)Q+(1-\beta_2)A}{(1-\beta_2)Q+(1+\beta_2)A} \tag{3.16}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2 \leq 2/3$, $\beta_2 \geq \left(3+2\sqrt{2}\right)\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}-e\right]/\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}+e\right] = 0.7145\cdots$.

Proof Since $R_{QA}(a,b)$,$A(a,b)$ and $Q(a,b)$ are symmetric and homogenous of degree 1, we assume that $a>b$. Let $v=(a-b)/(a+b)\in(0,1)$,$x=\sqrt{1+v^2}\in\left(1,\sqrt{2}\right)$ and $p\in[0,1]$. Then (3.16) can be rewritten as

$$\alpha_2 < \frac{(Q+A)(R_{QA}(a,b)-A)}{(Q-A)(R_{QA}(a,b)+A)} < \beta_2 \tag{3.17}$$

$$\frac{(Q+A)(R_{QA}(a,b)-A)}{(Q-A)(R_{QA}(a,b)+A)} = \frac{\left(\sqrt{1+v^2}+1\right)\left(e^{\sqrt{1+v^2}\operatorname{arcsin}h(v)/v-1}-1\right)}{\left(\sqrt{1+v^2}-1\right)\left(e^{\sqrt{1+v^2}\operatorname{arcsin}h(v)/v-1}+1\right)} \tag{3.18}$$

$$\begin{aligned}
&\log R_{QA}(a,b) - \log A\frac{(1+p)Q+(1-p)A}{(1-p)Q+(1+p)A} \\
&= \frac{\sqrt{1+v^2}\operatorname{arcsinh}(v)}{v} - \log\frac{(1+p)\sqrt{1+v^2}+1-p}{(1-p)\sqrt{1+v^2}+1+p} - 1 \\
&= \frac{x\operatorname{arcsinh}\left(\sqrt{x^2-1}\right)}{\sqrt{x^2-1}} - \log\frac{(1+p)x+1-p}{(1-p)x+1+p} - 1.
\end{aligned} \tag{3.19}$$

Let

$$G(x)=\frac{x\operatorname{arcsinh}\left(\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}-\log\frac{(1+p)x+1-p}{(1-p)x+1+p}-1. \tag{3.20}$$

Then simple computations lead to

$$G\left(1^{+}\right)=0,G\left(\sqrt{2}^{-}\right)=\sqrt{2}\log\left(1+\sqrt{2}\right)-\log\frac{(1+p)\sqrt{2}+1-p}{(1-p)\sqrt{2}+1+p}-1, \tag{3.21}$$

$$G'(x)=\frac{1}{(x^2-1)^{3/2}}g(x), \tag{3.22}$$

where

$$g(x)=\frac{\sqrt{x^2-1}\left[\left(1-p^2\right)x^3+2(1-p)^2x^2+\left(1-p^2\right)x+4p\right]}{[(1-p)x+1+p][(1+p)x+1-p]}-\operatorname{arcsinh}\left(\sqrt{x^2-1}\right) \tag{3.23}$$

$$g\left(1^{+}\right)=0,g\left(\sqrt{2}^{-}\right)=\frac{\left(4-3\sqrt{2}\right)p^2-4p+4+3\sqrt{2}}{\left(2\sqrt{2}-3\right)p^2+3+2\sqrt{2}}-\log\left(1+\sqrt{2}\right), \tag{3.24}$$

$$g'(x)=\frac{2\sqrt{x^2-1}}{[(1-p)x+1+p]^2[(1+p)x+1-p]^2}\phi(x) \tag{3.25}$$

where $\phi(x)$ is defined Lemma 2.1.

We divide the proof into two cases.

- **Case 1** If $p=2/3$. Then (3.19)-(3.22), (3.24) and (3.25) together with Lemma 2.1(1) lead to the conclusion that

$$R_{QA}(a,b)>A\frac{5Q+A}{Q+5A} \tag{3.26}$$

- **Case 2** If $p=\left(3+2\sqrt{2}\right)\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}-e\right]/\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}+e\right]=0.7145\cdots$.Then from (3.21) and (3.24) together with numerical computations we get

$$G\left(\sqrt{2}^{-}\right)=0,g\left(\sqrt{2}^{-}\right)=0.0349\cdots. \tag{3.27}$$

Let $\lambda_2=1.1126\cdots$ be the number given in Lemma 2.1(3). We divide the discussion into two subcases.

We divide the discussion into two subcases.

- **subcase 1** $x\in(1,\lambda_2]$.Then Lemma 2.1(3) and (3.24) and (3.25) imply that

$$g(x)<0.$$

- **subcase 2** $x \in \left[\lambda_2, \sqrt{2}\right)$.Then Lemma 2.1(3) and (3.25) lead to the conclusion that $g(x)$ is strictly increasing on the interval $\left[\lambda_2, \sqrt{2}\right)$. Then from (3.27) and Subcase 1 we know that there exists $x_1 \in \left[\lambda_2, \sqrt{2}\right)$ such that $g(x) < 0$ for $x \in [\lambda_2, x_1)$ and $g(x) > 0$ for $\left(x_1, \sqrt{2}\right)$.

It follows from Subcase 1 and 2 together with (3.22) that $G(x)$ is strictly decreasing on $(1, x_1]$ and strictly increasing on $\left[x_1, \sqrt{2}\right)$. Therefore,

$$R_{QA}(a,b) < A\frac{(1+p)Q+(1-p)A}{(1-p)Q+(1+p)A} \tag{3.28}$$

follows from (3.19)-(3.21) and (3.27) together with the piecewise monotonicity of $G(x)$.

Note that

$$\lim_{v\to 0^+} \frac{\left(\sqrt{1+v^2}+1\right)\left(e^{\sqrt{1+v^2}\operatorname{arcsin}h(v)/v-1}-1\right)}{\left(\sqrt{1+v^2}-1\right)\left(e^{\sqrt{1+v^2}\operatorname{arcsin}h(v)/v-1}+1\right)} = \frac{2}{3}, \tag{3.29}$$

$$\begin{aligned} &\lim_{v\to 1^-} \frac{\left(\sqrt{1+v^2}+1\right)\left(e^{\sqrt{1+v^2}\operatorname{arcsin}h(v)/v-1}-1\right)}{\left(\sqrt{1+v^2}-1\right)\left(e^{\sqrt{1+v^2}\operatorname{arcsin}h(v)/v-1}+1\right)} \\ &= \frac{\left(3+2\sqrt{2}\right)\left[\left(1+\sqrt{2}\right)^{\sqrt{2}}-e\right]}{\left(1+\sqrt{2}\right)^{\sqrt{2}}+e} = 0.7145\cdots. \end{aligned} \tag{3.30}$$

Therefore, Theorem 2.2 follows form (3.18), (3.26) and (3.28)-(3.30) together with that fact that inequality (3.16) is equivalent to (3.17).

Theorem 3.3. The double inequality

$$G\frac{(1+\alpha_3)A+(1-\alpha_3)G}{(1-\alpha_3)A+(1+\alpha_3)G} < I(a,b) < G\frac{(1+\beta_3)A+(1-\beta_3)G}{(1-\beta_3)A+(1+\beta_3)G} \tag{3.31}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 2/3$, $\beta_3 \geq 1$.

Proof Since $I(a,b)$,$G(a,b)$ and $A(a,b)$ are symmetric and homogenous of degree 1, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$,$x = \sqrt{1-v^2} \in (0,1)$ and $p \in [0,1]$. Then

(3.31) can be rewritten as

$$\alpha_3 < \frac{[A+G]\,[I(a,b)-G]}{[A-G]\,[I(a,b)+G]} < \beta_3, \tag{3.32}$$

$$\frac{[A+G]\,[I(a,b)-G]}{[A-G]\,[I(a,b)+G]} = \frac{\left(1+\sqrt{1-v^2}\right)\left(e^{\arctan h(v)/v-1}-1\right)}{\left(1-\sqrt{1-v^2}\right)\left(e^{\arctan h(v)/v-1}+1\right)}. \tag{3.33}$$

$$\begin{aligned}
&\log I(a,b) - \log G(a,b)\,\frac{(1+p)A(a,b)+(1-p)\,G(a,b)}{(1-p)A(a,b)+(1+p)\,G(a,b)} \\
&= \frac{\arctan h(v)}{v} - \log\frac{(1-p)\sqrt{1-v^2}+1+p}{(1+p)\sqrt{1-v^2}+1-p} - 1 \\
&= \frac{\arctan h\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} - \log\frac{(1-p)x+1+p}{(1+p)x+1-p} - 1.
\end{aligned} \tag{3.34}$$

Let

$$H(x) = \frac{\arctan h\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}} - \log\frac{(1-p)x+1+p}{(1+p)x+1-p} - 1 \tag{3.35}$$

Then simple computations lead to

$$H\left(1^-\right) = 0, \tag{3.36}$$

$$H'(x) = \frac{x}{(1-x^2)^{3/2}}h(x), \tag{3.37}$$

where

$$h(x) = \arctan h\left(\sqrt{1-x^2}\right) - \frac{\sqrt{1-x^2}\left[4px^3+\left(1-p^2\right)x^2+2(1-p)^2x+1-p^2\right]}{x^2\left[(1-p)x+1+p\right]\left[(1+p)x+1-p\right]} \tag{3.38}$$

$$h\left(1^-\right) = 0, \tag{3.39}$$

$$h'(x) = \frac{2\sqrt{1-x^2}}{x^3[(1-p)x+1+p]^2[(1+p)x+1-p]^2}\varphi(x) \tag{3.40}$$

where $\varphi(x)$ is defined Lemma 2.2.

We divide the proof into two cases.

- **Case 1** If $p = 2/3$. Then (3.34)-(3.37), (3.39) and (3.40) together with Lemma 2.2(1) lead to the conclusion that

$$I(a,b) > G\frac{5A+G}{A+5G} \tag{3.41}$$

- **Case 2** If $p=1$. Then from Lemma 2.2(2) and (3.34)-(3.37) together with (3.39)-(3.40) we know that

$$I(a,b)<A(a,b). \tag{3.42}$$

$$\lim_{v\to 0^+}\frac{\left(1+\sqrt{1-v^2}\right)\left(e^{\operatorname{arctan}h(v)/v-1}-1\right)}{\left(1-\sqrt{1-v^2}\right)\left(e^{\operatorname{arctan}h(v)/v-1}+1\right)}=\frac{2}{3}, \tag{3.43}$$

$$\lim_{v\to 1^-}\frac{\left(1+\sqrt{1-v^2}\right)\left(e^{\operatorname{arctan}h(v)/v-1}-1\right)}{\left(1-\sqrt{1-v^2}\right)\left(e^{\operatorname{arctan}h(v)/v-1}+1\right)}=1. \tag{3.44}$$

Therefore, Theorem 2.3 follows for (3.33), (3.41) and (3.42)-(3.44) together with that fact that inequality (3.31) is equivalent to (3.32).

Theorem 3.4. The double inequality

$$Q\frac{(1+\alpha_4)A+(1-\alpha_4)Q}{(1-\alpha_4)A+(1+\alpha_4)Q}<R_{AQ}(a,b)<Q\frac{(1+\beta_4)A+(1-\beta_4)Q}{(1-\beta_4)A+(1+\beta_4)Q} \tag{3.45}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_4\geq 2/3$, $\beta_4\leq\left(3+2\sqrt{2}\right)\left(1-e^{\pi/4-1}\right)/\left(1+e^{\pi/4-1}\right)=0.6230\cdots$.

Proof Since $R_{AQ}(a,b)$,$A(a,b)$ and $Q(a,b)$ are symmetric and homogenous of degree 1, we assume that $a>b$. Let $v=(a-b)/(a+b)\in(0,1)$,$x=\sqrt{1+v^2}\in\left(1,\sqrt{2}\right)$ and $p\in[0,1]$. Then (3.45) can be rewritten as

$$\beta_4<\frac{[Q+A]\,[Q-R_{AQ}(a,b)]}{[Q-A]\,[Q+R_{AQ}(a,b)]}<\alpha_4, \tag{3.46}$$

$$\frac{[Q+A]\,[Q-R_{AQ}(a,b)]}{[Q-A]\,[Q+R_{AQ}(a,b)]}=\frac{\left(\sqrt{1+v^2}+1\right)\left(1-e^{\arctan(v)/v-1}\right)}{\left(\sqrt{1+v^2}-1\right)\left(1+e^{\arctan(v)/v-1}\right)}. \tag{3.47}$$

$$\begin{aligned}&\log R_{AQ}(a,b)-\log Q\frac{(1+p)A+(1-p)Q}{(1-p)A+(1+p)Q}\\&=\frac{\arctan(v)}{v}-\log\frac{(1-p)\sqrt{1+v^2}+1+p}{(1+p)\sqrt{1+v^2}+1-p}-1\\&=\frac{\arctan\left(\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}-\log\frac{(1-p)x+1+p}{(1+p)x+1-p}-1.\end{aligned} \tag{3.48}$$

Let

$$J(x)=\frac{\arctan\left(\sqrt{x^2-1}\right)}{\sqrt{x^2-1}}-\log\frac{(1-p)x+1+p}{(1+p)x+1-p}-1 \tag{3.49}$$

Then simple computations lead to

$$J\left(1^+\right)=0, J\left(\sqrt{2}^-\right)=\frac{\pi}{4}-\log\frac{(1-p)\sqrt{2}+1+p}{(1+p)\sqrt{2}+1-p}-1, \tag{3.50}$$

$$J'(x)=\frac{x}{\left(x^2-1\right)^{3/2}}J_1(x), \tag{3.51}$$

where

$$J_1(x)=\frac{\sqrt{x^2-1}\left[4px^3+\left(1-p^2\right)x^2+2(1-p)^2x+\left(1-p^2\right)\right]}{x^2\left[(1-p)x+1+p\right]\left[(1+p)x+1-p\right]}-\arctan\left(\sqrt{x^2-1}\right), \tag{3.52}$$

$$J_1\left(1^+\right)=0, J_1\left(\sqrt{2}^-\right)=\frac{2\sqrt{2}p}{\left(2\sqrt{2}-3\right)p^2+2\sqrt{2}+3}-\frac{\pi-2}{4}, \tag{3.53}$$

$$J_1'(x)=-\frac{2\sqrt{x^2-1}}{x^3[(1-p)x+1+p]^2[(1+p)x+1-p]^2}\varphi(x) \tag{3.54}$$

where $\varphi(x)$ is defined Lemma 2.2.

We divide the proof into two cases.

- **Case 1** If $p=2/3$. Then (3.48)-(3.51) and (3.53) together with Lemma 2.2(1) lead to the conclusion that

$$R_{AQ}(a,b)>Q\frac{5A+Q}{A+5Q}. \tag{3.55}$$

- **Case 2** If $p=\left(3+2\sqrt{2}\right)\left(1-e^{\pi/4-1}\right)/\left(1+e^{\pi/4-1}\right)=0.6230\cdots$. Then from (3.50) and (3.53) together with numerical computations we get

$$J\left(\sqrt{2}^-\right)=0, J_1\left(\sqrt{2}^-\right)=0.0204\cdots. \tag{3.56}$$

Let $\lambda_3=1.1054\cdots$ be the number given in Lemma 2.1(3).

We divide the discussion into two subcases.

- **subcase 1** $x\in(1,\lambda_3]$. Then Lemma 2.2(2) and (3.53) and (3.54) imply that

$$J_1(x)<0. \tag{3.57}$$

- **subcase 2** $x \in \left[\lambda_3, \sqrt{2}\right)$. Then Lemma 2.2(2) and (3.54) lead to the conclusion that $J_1(x)$ is strictly increasing on the interval $\left[\lambda_3, \sqrt{2}\right)$. Then from (3.56) and Subcase 1 we know that there exists $x_2 \in \left[\lambda_3, \sqrt{2}\right)$ such that $J_1(x) < 0$ for $x \in [\lambda_3, x_2)$ and $J_1(x) > 0$ for $\left(x_2, \sqrt{2}\right)$.

It follows from Subcase 1 and 2 together with (3.51) that $J(x)$ is strictly decreasing on $(1, x_2]$ and strictly increasing on $\left[x_2, \sqrt{2}\right)$. Therefore,

$$R_{AQ}(a,b) < Q\frac{(1+p)A+(1-p)Q}{(1-p)A+(1+p)Q} \tag{3.58}$$

follows from (3.48)-(3.50) and (3.56) together with the piecewise monotonicity of $J(x)$.

Note that

$$\lim_{v\to 0^+} \frac{\left(\sqrt{1+v^2}+1\right)\left(1-e^{\arctan(v)/v-1}\right)}{\left(\sqrt{1+v^2}-1\right)\left(1+e^{\arctan(v)/v-1}\right)} = \frac{2}{3}, \tag{3.59}$$

$$\lim_{v\to 1^-} \frac{\left(\sqrt{1+v^2}+1\right)\left(1-e^{\arctan(v)/v-1}\right)}{\left(\sqrt{1+v^2}-1\right)\left(1+e^{\arctan(v)/v-1}\right)} = \frac{\left(3+2\sqrt{2}\right)\left(1-e^{\pi/4-1}\right)}{\left(1+e^{\pi/4-1}\right)} = 0.6230\cdots. \tag{3.60}$$

Therefore, Theorem 2.2 follows form (3.47), (3.55) and (3.57)-(3.59) together with that fact that inequality (3.45) is equivalent to (3.46).

Conflict of Interests

The authors declare that there is no conflict of interests.

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REFERENCES

[1] J L. Brenner, B C. Carlson, Homogeneous mean values: weights and asymptotes, J. Math. Anal. Appl., 123 (1987), 265-280.

[2] B C. Carlson, Algorithms involving arithmetic and geometric means, Am. Math. Mon., 78 (1971), 496-505.

[3] E.Neuman, J. Sándor, On the Schwab-Borchardt mean, Math. Pannon, 14(2) (2003), 253-266.

[4] E. Neuman, On a new family of bivariate means, J. Math. Inequal., 11(3) (2017), 673-681.

[5] Zh.-H. Yang, Three families of two-parameter means constructed by trigonometric functions, J. Inequal. Appl., 2013 (2013), 541.

[6] J.Sándor, On two new means of two variables, Notes Numb. Theory Discr. Math., 20 (2014), 1-9.

[7] J. Sándor, On the identric and logarithmic means, Aequationes Math., 40 (1990), 261-270.

[8] J. Sándor, On certain identities for means, Studia Univ. Babes, Bolyai, Math., 38(4)(1993), 7-14.

[9] J. Sándor, Two sharp inequalities for trigonometric and hyperbolic functions, Math. Inequal. Appl., 15(2) (2012), 409-413.

[10] P.S. Bullen, Handbook of means and their inequalities, Kluwer Acad, Publ., Dordrecht, 2003.

[11] H. Alzer and S. L. Qiu, Inequalities for means in two variables, Arch. (Basel) 80(2)(2003), 201-215.

[12] W.-M. Qian, Y.-M.Chu, and X.-H. Zhang, Sharp Bounds for Sándor Mean in Terms of Arithmetic, Geometricand Harmonic Means, J. Inequal. Appl., 2015(2015), 221.

[13] A. O. Pittenger. Inequalities between arithmetic and logarithmic means, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., 678-715(1980), 15-18.

[14] Zh.-H. Yang, L.-M.Wu, and Y.-M. Chu, Sharp power mean bounds for Sndor mean, Abstr. Appl. Anal., 2014(2014), Article ID 172867.

[15] S.-S. Zhou, W.-M.Qian, Y.-M.Chu and X.-H. Zhang, Sharp power-type Heronian mean bounds for the Sándor and Yang means, J. Inequal. Appl., 2015 (2015), 159.

[16] H. Alzer, Ungleichungen für mittelwerte, Arch. Math., 47(5), 422-426(1986).

[17] Y.-Q. Song, W.-F.Xia, X.-H.Shen and Y.-M. Chu, Bounds for the identric mean in terms of one-parameter mean, Appl. Math. Sci., 7(88)(2013), 4375-4386.

[18] T.-H. Zhao, W.-M. Qian and Y.-Q. Song, Optimal bounds for two Sándor-type means in terms of power means, J. Inequal. Appl., 2016 (2016), 64.

[19] Zh.-H. Yang and Y.-M Chu, Optimal evaluations for the Sndor-Yang mean by power mean, Math. Inequal. Appl., 19 (2016), 1031-1038.

[20] Yang Yue-ying and Qian Wei-mao, Two Optimal Inequalities Related to the Sándor- Yang Type Mean and One-parameter Mean, Commun. Math. Res., 32(4)(2016), 352-358, (in Chinese).