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COMMON FIXED POINT THEOREM FOR TWO SELFMAPS OF AN S-METRIC SPACE WITH RATIONAL INEQUALITY

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Abstract: A common fixed point theorem for two self maps of an S-metric space with rational inequality is proved in the present paper.

Keywords: S-metric space; fixed point; associated sequence of a point relative to two self maps; compatible mappings.

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1. INTRODUCTION

As a generalization of metric spaces B.C.Dhage [1,2] introduced D- metric spaces. Later many researchers proved many of their results are not valid. As a probable modification to D- metric spaces, Shaban Sedghi, Nabi Shobe and Haiyun Zhou [4] introduced D^*- metric spaces. In 2006, Zead Mustafa and Brailey Sims [6] have initiated G- metric spaces, while Shaban Sedghi, Nabi Shobe and Abdelkrim Aliouche [5] considered S-mertic spaces in 2012. Of these three generalizations, the S-metric space generated interest in researchers. The notion of commutativity is generalized by Gerald Jungck [3] by initiating compatibility.

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The purpose of this paper is to prove a common fixed point theorem for two self maps of an S-metric space with rational inequality.

2. PRELIMINARIES

Definition 2.1 [5]: Let X be a non empty set. By an $S-$metric we mean a function $S: X^3 \to [0,\infty)$ which satisfies the following conditions for all $x,y,z,w \in X$

(a) $S(x,y,z) \geq 0$

(b) $S(x,y,z) = 0$ if and only if $x = y = z$.

(c) $S(x,y,z) \leq S(x,x,w) + S(y,y,w) + S(z,z,w)$

Also, the pair (X,S) is called a S-metric space.

Example 2.2: Let $X = \mathbb{R}$ and $S: \mathbb{R}^3 \to [0,\infty)$ be defined by $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in X$, then (X,S) is a S- metric space.

Remark 2.3:It was shown in ([5], Lemma 2.5) that $S(x,x,y) = S(y,y,x)$ for all $x,y \in X$.

Definition 2.4: Let (X,S) be a $S-$metric space. A sequence $\{x_n\}$ in X is said to Convergent, if there is a $x \in X$ such that $S(x_n,x_n,x) \to 0$ as $n \to \infty$; that is, for each $\varepsilon > 0$,there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,we have $S(x_n,x_n,x) < \varepsilon$ and we write in this case that $\lim_{n\to\infty} x_n = x$.

Definition 2.5: Let (X,S) be a $S-$metric space. A sequence $\{x_n\}$ in X is said to be a **Cauchy sequence,** if for each $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq n_0$.

It is easy to see that (in fact proved in [5], Lemma 2.10 and Lemma 2.11), if $\{x_n\}$ converges to x in (X,S) then x is unique and $\{x_n\}$ is a Cauchy sequence in (X,S). However, a Cauchy sequence in (X,S) need not be convergent as shown in the following example

Example 2.6 : Let $X=(0,1]$ and $S(x,y,z)=|x-y|+|y-z|+|z-x|$ for $x,y,z\in X$, so that (X,S) is a $S-$metric space. Taking $x_n=\frac{1}{n}$ for $n=1,2,3,...$ then $S(x_n,x_n,x_m)=2\left|\frac{1}{n}-\frac{1}{m}\right|$ so that $S(x_n,x_n,x_m)\to 0$ as $n,m\to\infty$ proving that $\{x_n\}$ is a Cauchy sequence in (X,S) but $\{x_n\}$ does not converge to any point in X

Definition 2.7: Let (X,S) be an S - metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$ then $\lim_{n\to\infty}S(x_n,x_n,y_n)=S(x,x,y)$. Then we say that $S(x,y,z)$ is continuous in x and y.

Definition 2.8 : If g and f are selfmaps of a S - metric space (X,S) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty}gx_n=\lim_{n\to\infty}fx_n=t$ for some $t\in X$ we have $\lim_{n\to\infty}S(gfx_n,gfx_n,fgx_n)=0$ then g and f are said to be compatible.

Trivially commuting self maps of a S -metric space are compatible but not conversely. As an example we have the following.

Example 2.9: Let $X=[0,1]$ with $S(x,y,z)=|x-y|+|y-z|+|z-x|$ for $x,y,z\in X$. Then S is a S - metric on X. Define $g:X\to X, f:X\to X$ by $gx=\frac{x^2}{2}$ and $fx=\frac{x^2}{3}$ for $x\in X$.

we now prove that g,f are compatible.

Let $\{x_n\}$ be a sequence in X with $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = t$ for some $t \in X$. Then $\lim_{n\to\infty} \frac{x_n^2}{2} = t = \lim_{n\to\infty} \frac{x_n^2}{3}$ so that $3t = 2t$ which shows that $t = 0$. Also since $gfx_n = \frac{x_n^4}{18}$ and then we have $\lim_{n\to\infty} S(gfx_n, gfx_n, fgx_n) = \lim_{n\to\infty} S\left(\frac{x_n^4}{18}, \frac{x_n^4}{18}, \frac{x_n^4}{12}\right) = \lim_{n\to\infty} \frac{x_n^4}{18} = 0$

Showing that (g, f) is a pair of compatible self maps. But $gf(1) = \frac{1}{18}$ and $fg(1) = \frac{1}{12}$ proves that $gf(1) \neq fg(1)$ showing that that g and f are not commutative

Definition 2.10: Let g and f be self maps of a $S-$metric space such that $g(X) \subseteq f(X)$. For any $x_0 \in X$, if $\{x_n\}$ is a sequence in X such that $fx_n = gx_{n-1}$ for $n \geq 0$. Then $\{x_n\}$ is called an associated sequence of x_0 relative to the two self maps g and f .

3. MAIN RESULTS

Before stating main Theorem, we prove an essential Lemma.

Lemma 3.1: Let f and g be compatible self maps of an S-metric space (X, S). Suppose $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$ and some sequence $\{x_n\}$ in X , then $\lim_{n\to\infty} gfx_n = fx$ if f is continuous.

Proof: suppose f and g are compatible mappings and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some $x \in X$. Then

(3.1.1) $\lim_{n\to\infty} S(fgx_n, fgx_n, gfx_n) = 0$

since f is continuous and $gx_n \to x$ as $n \to \infty$ we have

(3.1.2) $\lim_{n\to\infty} fgx_n = fx$.

From (3.1.1) and (3.1.2) we get

$\lim_{n\to\infty} S(fx, fx, gfx_n) = 0$ which imply $\lim_{n\to\infty} gfx_n = fx$.

Proving the lemma.

Theorem 3.2. Let f and g be self maps of a S-metric space (X, S) satisfying

(i) $g(X) \subseteq f(X)$

(ii) $S(gx, gx, gy) \leq \dfrac{\alpha S(fx, fx, gy)[1 + S(fx, fx, gx)]}{1 + S(fx, fx, fy)} + \beta S(fx, fx, fy)$

for all $x, y \in X$ where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$

(iii) one of f and g is continuous

(iv) f and g are compatible

(v) an associated sequence $\{x_n\}$ of a point $x_0 \in X$ relative to the self maps f and g is such that $\{fx_n\}$ converges to t for some point $t \in X$,

Then t is the common fixed point of f and g.

Proof: From (v), the associated sequence $\{x_n\}$ of x_0 relative to the selfmaps f and g such that $fx_n = gx_{n-1}$ for $n \geq 1$ and $fx_n \to t$ as $n \to \infty$ it follows that $gx_n \to t$ as $n \to \infty$

Case(i): If f is continuous, then we have by Lemma 3.1 that

(**3.2.1**) $\lim\limits_{n \to \infty} gfx_n = ft$

and also

(**3.2.2**) $\lim\limits_{n \to \infty} f^2 x_n = ft.$

Now from (ii) we get

$$S(gfx_n, gfx_n, gx_{n-1}) \leq \frac{S(f^2x_n, f^2x_n, gx_{n-1})[1 + S(f^2x_n, f^2x_n, gfx_n)]}{1 + S(f^2x_n, f^2x_n, fx_{n-1})} + \beta[S(f^2x_n, f^2x_n, fx_{n-1})]$$

Where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$

on letting $n \to \infty$ in the above inequality and using (3.2.1) and (3.2.2), we get

$$S(ft, ft,t) \le \frac{\alpha S(ft, ft,t)[1+S(ft, ft, ft)]}{1+S(ft, ft,t)} + \beta S(ft, ft,t)$$

i.e $$S(ft, ft,t) \le \frac{\alpha S(ft, ft,t)}{1+S(ft, ft,t)} + \beta S(ft, ft,t)$$

$$S(ft, ft,t) < (\alpha+\beta) S(ft, ft,t)$$

which implies $S(ft, ft,t) = 0$ and hence $ft = t$

Also from (ii), we get

$$S(gt, gt, gx_{n-1}) \le \frac{\alpha S(ft, ft, gx_{n-1})[1+S(ft, ft, gt)]}{1+S(ft, ft, fx_{n-1})} + \beta S(ft, ft, fx_{n-1})$$

where $\alpha, \beta \ge 0, \ \ \alpha+\beta < 1$

Letting $n \to \infty$ in the above inequality, we obtain

$$S(gt, gt, t) \le \frac{\alpha S(ft, ft,t)[1+S(ft, ft,t)]}{1+S(ft, ft,t)} + \beta S(ft, ft,t)$$

since $ft = t$, we get $S(gt, gt, t) = 0$ which implies $gt = t$, showing that t is a common fixed point of f and g.

Case(ii) : Now suppose that g is a continuous , then we have by Lemma 3.1 , that

(**3.2.3**) $\lim_{n\to\infty} fgx_n = gt$

(3.2.4) $\lim_{n\to\infty} g^2 x_n = gt$

Now from (ii), we get

(3.2.5) $$S(g^2x_n, g^2x_n, gx_{n-1}) \le \frac{S(fgx_n, fgx_n, gx_{n-1})[1+S(fgx_n, fgx_n, g^2x_n)]}{1+S(fgx_n, fgx_n, fx_{n-1})} + \beta[S(fgx_n, fgx_n, fx_{n-1})]$$

where $\alpha, \beta \ge 0, \ \ \alpha+\beta < 1$

on letting $n \to \infty$ in the above inequality and using (3.2.3) and (3.2.4), we get

$$S(gt,gt,t) \le \frac{\alpha S(gt,gt,t)[1+S(gt,gt,gt)]}{1+S(gt,gt,t)} + \beta S(gt,gt,t)$$
$$< (\alpha+\beta) S(gt,gt,t)$$

which implies $S(gt,gt,t)=0$ and hence $gt=t$

(since $\alpha+\beta<1,\ 1+S(gt,gt,t)>1 \Rightarrow \frac{1}{1+S(gt,gt,t)}<1$)

From (i), we can find $w\in X$ such that $gt = fw$. Now from (ii) we have

$$S(g^2x_n, g^2x_n, gw) \le \frac{S(fgx_n, fgx_n, gw)[1+S(fgx_n, fgx_n, g^2x_n)]}{1+S(fgx_n, fgx_n, fw)} + \beta[S(fgx_n, fgx_n, fw)]$$

where $\alpha,\beta\ge 0,\ \ \alpha+\beta<1$

Letting $n\to\infty$ in the above inequality and using (3.2.3) and (3.2.4), we obtain

$$S(gt,gt,gw) \le \frac{\alpha S(gt,gt,gw)[1+S(gt,gt,gt)]}{1+S(gt,gt,fw)} + \beta S(gt,gt,fw)$$

since $t=gt=fw$, we obtain

$$S(gt,gt,gw) \le \frac{\alpha S(gt,gt,gw)(1)}{1}.$$

That is, $S(gt,gt,gw) \le \alpha . S(gt,gt,gw)$

which implies that $gt=gw$ since $\alpha\in(0,1)$

thus $t=gt=gw=fw.$

Now put $y_n = w$ for $n=0,1,2,3...$ then $fy_n \to fw$ and $gy_n \to gw$ as $n\to\infty$ since $fw=gw$ and f,g are compatible

$\lim_{n\to\infty} S(fgx_n, fgx_n, gfx_n)=0$ giving $S(fgw, fgw, gfw)=0$ which implies that $fgw=gfw$ since $fw=gw=t$ we get $ft=gt$ and since $gt=t$, it follows that $ft=gt=t$, showing that t is a common fixed point of f and g.

Finally to prove the uniqueness of common fixed point f and g.

Suppose $u = fu = gu$ and $v = fv = gv$ for some $u, v \in X$

From (ii) , we get

$$S(u,u,v) = S(gu, gu, gv) \le \frac{\alpha\, S(fu, fu, gv)[1 + S(fu, fu, gu)]}{1 + S(fu, fu, fv)} + \beta\, S(fu, fu, fv)$$

where $\alpha, \beta \ge 0,\ \alpha + \beta < 1$

$$S(u,u,v) \le \frac{\alpha . S(u,u,v)[1 + S(u,u,u)]}{1 + S(u,u,v)} + \beta S(u,u,v)$$

$$= \frac{[\alpha + \beta] S(u,u,v)}{1 + S(u,u,v)}$$

which implies that $S(u,u,v) = 0$ since $\frac{S(u,u,v)}{1 + S(u,u,v)} < 1$ and hence $u = v$, proving the theorem completely.

3.3 Example: Let $X = [0,1)$ and $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in X$ and $d(x, y) = |x - y|$, then (X, S) is a S-metric space. Define $f : X \to X$ and $g : X \to X$ by $f(x) = x$, $g(x) = \frac{x}{2}$ for all $x \in X$ Then $g(X) = [0, \frac{1}{2}) \subset [0,1) = f(X)$, clearly $fg = gf$, so that f and g are compatible. Also an associated sequence of $x_0 = 0$ relative to the self maps f and g is given by $x_n = 0$ for $n \ge 0$ and since $\{fx_n\}$ is a constant sequence converging to ‘0’, which is a point in X taking $\alpha = 0, \beta = \frac{1}{2}$ then f and g satisfy the inequality (ii).

Thus the conditions (iii) and (v) of Theorem 3.2 are satisfied.

Hence by Theorem 3.2, ‘0’ is the unique common fixed point of f and g.

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