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COINCIDENCE POINTS AND COMMON FIXED POINTS FOR THREE MAPPINGS WITH INTEGRAL TYPE IMPLICIT CONTRACTION CONDITIONS ON ORDERED METRIC SPACES

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Abstract. In this paper, by using the known class Ψ^* of 5-dimensional functions determined by the known classes $\{\mathscr{C}, \Phi_u\}$, we discuss the existence problems of coincidence points for three mappings of integral type with semi-implicit contraction conditions and obtain a common fixed point theorem for two mappings on ordered metric spaces. Finally, we give a sufficient condition under which there exists a unique common fixed point.

Keywords: class $\mathscr{C}$; class Ψ^*; class Φ_u; semi-implicit; sub-additive; common fixed point.

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1. Introduction and preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$ and

$\Phi = \{\phi : \phi : \mathbb{R}^+ \to \mathbb{R}^+ | \, \phi$ is Lebesgue integral, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^{\varepsilon} \phi(t)dt > 0$ for each $\varepsilon > 0\}$

The famous Banach's contraction principle is as follows:

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Theorem 1.1([1]) Let f be a self mapping on a complete metric space (X,d) satisfying

$$d(fx,fy) \le cd(x,y),\ \forall\, x,y \in X, \tag{1.1}$$

where $c \in [0,1)$ is a constant. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

It is known that the Banach's contraction principle has a lot of generalizations and various applications in many directions, see, for examples, [2-12] and the references cited therein. Especially, in 1962, Rakotch[13] extended the Banach's contraction principle with replacing the contraction constant c in (1.1) by a contraction function γ and obtained the next theorem:

Theorem 1.2([13]) Let f be a self-mapping on a complete metric space (X,d) satisfying

$$d(fx,fy) \le \gamma(d(x,y))d(x,y),\ \forall\, x,y \in X, \tag{1.2}$$

where $\gamma : \mathbb{R}^+ \to [0,1)$ is a monotonically decreasing function. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

In 2002, Branciari[14] gave an integral version of Theorem 1.1 as follows:

Theorem 1.3([14]) Let f be a self-mapping on a complete metric space (X,d) satisfying

$$\int_0^{d(fx,fy)} \phi(t)dt \le c\int_0^{d(x,y)} \phi(t)dt,\ \forall\, x,y \in X, \tag{1.3}$$

where $c \in (0,1)$ is a constant and $\phi \in \Phi$. Then f has a unique fixed point $\hat{x} \in X$ such that $\lim_{n\to\infty} f^n x = \hat{x}$ for each $x \in X$.

The further generalizations of Branciari' theorem were given with replacing the contraction constant c in (1.3) by two or three contraction functions in [15-16].

The following concept of class $\mathscr{C}$ of functions $F : [0,\infty)^2 \to \mathbb{R}$ was introduced in [17-18].

Definition 1.1([17-18]) A continuous mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called $\mathscr{C}$-*class* function if it satisfies following axioms:

(i)$F(s,t) \le s$;

(ii) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.

Definition 1.2([18]) Let Φ_u be a set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(i) φ is continuous;

(ii) $\varphi(t) > 0$ for all $t > 0$ and $\varphi(0) \geq 0$.

Definition 1.3([18]) Let $\psi \in \Psi^*$ if and only if $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ is a continuous and nondecreasing function about the 4th and 5th variables and the following conditions hold:

(i) there exists $F_1 \in \mathscr{C}, \varphi_1 \in \Phi_u$, such that $u \leq \psi(v,u,v,0,u+v)$ implies $u \leq F_1(v, \varphi_1(v))$;

(ii) there exists $F_2 \in \mathscr{C}, \varphi_2 \in \Phi_u$, such that $u \leq \psi(v,v,u,u+v,0)$ implies $u \leq F_2(v, \varphi_2(v))$;

(iii) $\psi(t,0,0,t,t) < t$, $\psi(0,t,0,0,t) < t$ and $\psi(0,0,t,t,0) < t$ for all $t > 0$.

Ansari and Piao et al[18] discussed and obtained several common fixed point theorems for two mappings of integral type with semi-implicit contractive conditions determined by functions $F \in \mathscr{C}$ and $\varphi \in \Phi_u$ and $\psi \in \Psi^*$ in metric spaces. These results generalized and improved many common fixed point theorems for mappings with integral type.

The aim of this paper is to discuss the existence problems of coincidence points and common fixed points for three and two mappings respectively with integral type and semi-implicit contractive conditions on ordered metric spaces, and finally give a sufficient condition under which there exists a unique common fixed point. The obtained results further extent and improve the corresponding various known conclusions.

To do this, we give some known definitions and lemmas.

Lemma 1.1([19]) Let (X,d) be a metric space and $\{x_n\}$ a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ for each $k \in \mathbb{N}$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and the following result hold

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon.$$

Remark 1.1 Under the conditions of Lemma 1.9, we easily obtian the following result:

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon.$$

Definition 1.4 Let X be a nonempty set. Then $(X,d,\preceq)$ is called an ordered metric space if

(i) (X,d) is a metric space;

(ii) $(X,\preceq)$ is a partially ordered set.

Definition 1.5([20-21]) Let $(X,\preceq)$ be a partially ordered set and $f,g,h:X\to X$ three given mappings such that $fX\subset hX$. We say that (f,g) is partially weakly increasing with respect to h if and only if for all $x\in X$ and $y\in h^{-1}(fx)$, we have

$$fx\preceq gy.$$

Definition 1.6([21]) Let $(X,d,\preceq)$ be an ordered metric space. We say that X is regular if the following hypothesis holds: if $\{x_n\}$ is a non-decreasing sequence in X with respect to $\preceq$ such that $x_n\to x$ as $n\to\infty$, then $x_n\preceq x$ for all $n\in\mathbb{N}$; equivalently, if $\{y_n\}$ is a non-increasing sequence in X with respect to $\preceq$ such that $y_n\to y$ as $n\to\infty$, then $y_n\succeq y$ for all $n\in\mathbb{N}$.

Lemma 1.2([15]) Let $\phi\in\Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty}r_n=a$. Then

$$\lim_{n\to\infty}\int_0^{r_n}\phi(t)dt=\int_0^{a}\phi(t)dt.$$

Lemma 1.3([15]) Let $\phi\in\Phi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then

$$\lim_{n\to\infty}\int_0^{r_n}\phi(t)dt=0\Longleftrightarrow\lim_{n\to\infty}r_n=0.$$

Definition 1.7([22]) $\phi\in\Phi$ is called to be sub-additive if, for all $a,b\in\mathbb{R}^+$,

$$\int_0^{a+b}\phi(t)dt\le\int_0^{a}\phi(t)dt+\int_0^{b}\phi(t)dt.$$

Definition 1.8 $\phi\in\Phi$ is called to be strictly increasing about integral type if, for any $x,y\in[0,\infty)$ with $x<y$,

$$\int_0^{x}\phi(t)dt<\int_0^{y}\phi(t)dt.$$

It is easy to check that the function $\phi:\mathbb{R}^+\to\mathbb{R}^+$, $\phi(t)=\frac{1}{1+t}$ for each $t\in\mathbb{R}^+$ is a sub-additive and strictly increasing function about integral type.

Definition 1.9 Let $X\neq\emptyset$ and $f,g,T:X\to X$ three mappings. If there exist $v,u\in X$ such that $fv=gv=Tv=u$, then v is called a coincidence point of $\{f,g,T\}$ and u is called a point of coincidence of $\{f,g,T\}$.

Let $C(f,g,T)$ be the set of all points of coincidence of $\{f,g,T\}$.

2. Coincidence points and Common fixed points

The following result is the main coincidence point theorem.

Theorem 2.1 Let $(X,d,\preceq)$ be an ordered metric space such that X is regular and $T,f,g:X\to X$ three mappings. Suppose that for each $x,y\in X$ with Tx and Ty being comparable,

$$\begin{aligned}&\int_0^{d(fx,gy)}\phi(t)dt\\ \leq&\psi\Big(\int_0^{d(Tx,Ty)}\phi(t)dt,\int_0^{d(Tx,fx)}\phi(t)dt,\int_0^{d(Ty,gy)}\phi(t)dt,\int_0^{d(Tx,gy)}\phi(t)dt,\int_0^{d(Ty,fx)}\phi(t)dt\Big),\end{aligned}\tag{2.1}$$

where $\phi\in\Phi$ and $\psi\in\Psi^*$. If

(i) ϕ is sub-additive and strictly increasing about the integral type;

(ii) $fX\cup gX\subset TX$;

(iii) (f,g) and (g,f) are both partially weakly increasing with respect to T;

(iv) one of $\{TX,fX,gX\}$ is complete.

Then $C(T,f,g)\neq\emptyset$, that is, there exist $u,v\in X$ such that $Tv=fv=gv=u\in TX$. Furthermore, either $C(T,f,g)$ is singleton or any different two elements in $C(T,f,g)$ are not comparable.

Proof. Let $x_0\in X$ be an arbitrary point. Since $fx_0\in fX\subset TX$, there exists $x_1\in X$ such that $fx_0=Tx_1$; Since $gx_1\in gX\subset TX$, there exists $x_2\in X$ such that $gx_1=Tx_2$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$y_{2n}=fx_{2n}=Tx_{2n+1},y_{2n+1}=gx_{2n+1}=Tx_{2n+2},\forall n=0,1,2,\cdots.\tag{2.2}$$

From (2.2), we obtain

$$x_{2n+1}\in T^{-1}(fx_{2n}),x_{2n+2}\in T^{-1}(gx_{2n+1}),\forall n=0,1,2,\cdots,$$

hence using (iii), we have

$$fx_{2n}\preceq gx_{2n+1},\ gx_{2n+1}\preceq fx_{2n+2},\forall n=0,1,2,\cdots,$$

that is,

$$y_{2n}=fx_{2n}\preceq gx_{2n+1}=y_{2n+1}\preceq fx_{2n+2}=y_{2n+2},\forall n=0,1,2,\cdots.$$

Therefore,

$$y_0=Tx_1\preceq y_1=Tx_2\preceq y_2=Tx_3\preceq\cdots,\tag{2.3}$$

Let $d_n = d(y_n, y_{n+1})$ for all $n = 0, 1, 2, \cdots$. Since $Tx_n \preceq Tx_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$, using (2.1) and (i), we have

$$\begin{aligned}
&\int_0^{d_{2n}} \phi(t)dt = \int_0^{d(y_{2n},y_{2n+1})} \phi(t)dt = \int_0^{d(fx_{2n},gx_{2n+1})} \phi(t)dt \\
&\leq \psi\Big(\int_0^{d(Tx_{2n},Tx_{2n+1})} \phi(t)dt, \int_0^{d(Tx_{2n},fx_{2n})} \phi(t)dt, \int_0^{d(Tx_{2n+1},gx_{2n+1})} \phi(t)dt, \\
&\qquad\qquad \int_0^{d(Tx_{2n},gx_{2n+1})} \phi(t)dt, \int_0^{d(Tx_{2n+1},fx_{2n})} \phi(t)dt\Big) \\
&= \psi\Big(\int_0^{d_{2n-1}} \phi(t)dt, \int_0^{d_{2n-1}} \phi(t)dt, \int_0^{d_{2n}} \phi(t)dt, \int_0^{d(y_{2n-1},y_{2n+1})} \phi(t)dt, 0\Big) \\
&\leq \psi\Big(\int_0^{d_{2n-1}} \phi(t)dt, \int_0^{d_{2n-1}} \phi(t)dt, \int_0^{d_{2n}} \phi(t)dt, \int_0^{d_{2n-1}+d_{2n+1}} \phi(t)dt, 0\Big) \\
&\leq \psi\Big(\int_0^{d_{2n-1}} \phi(t)dt, \int_0^{d_{2n-1}} \phi(t)dt, \int_0^{d_{2n}} \phi(t)dt, \int_0^{d_{2n-1}} \phi(t)dt + \int_0^{d_{2n}} \phi(t)dt, 0\Big).
\end{aligned}$$

So by the condition (ii) of $\psi \in \Psi^*$, there exist $F_2 \in \mathscr{C}$ and $\varphi_2 \in \Phi_u$ such that

$$\int_0^{d_{2n}} \phi(t)dt \leq F_2\Big(\int_0^{d_{2n-1}} \phi(t)dt, \varphi_2(\int_0^{d_{2n-1}} \phi(t)dt)\Big). \tag{2.4}$$

Similarly,

$$\begin{aligned}
&\int_0^{d_{2n+1}} \phi(t)dt = \int_0^{d(y_{2n+1},y_{2n+2})} \phi(t)dt = \int_0^{d(fx_{2n+2},gx_{2n+1})} \phi(t)dt \\
&\leq \psi\Big(\int_0^{d(Tx_{2n+2},Tx_{2n+1})} \phi(t)dt, \int_0^{d(Tx_{2n+2},fx_{2n+2})} \phi(t)dt, \int_0^{d(Tx_{2n+1},gx_{2n+1})} \phi(t)dt, \\
&\qquad\qquad \int_0^{d(Tx_{2n+2},gx_{2n+1})} \phi(t)dt, \int_0^{d(Tx_{2n+1},fx_{2n+2})} \phi(t)dt\Big) \\
&= \psi\Big(\int_0^{d_{2n}} \phi(t)dt, \int_0^{d_{2n+1}} \phi(t)dt, \int_0^{d_{2n}} \phi(t)dt, 0, \int_0^{d(y_{2n},y_{2n+2})} \phi(t)dt\Big) \\
&\leq \psi\Big(\int_0^{d_{2n}} \phi(t)dt, \int_0^{d_{2n+1}} \phi(t)dt, \int_0^{d_{2n}} \phi(t)dt, 0, \int_0^{d_{2n}+d_{2n+1}} \phi(t)dt\Big) \\
&\leq \psi\Big(\int_0^{d_{2n}} \phi(t)dt, \int_0^{d_{2n+1}} \phi(t)dt, \int_0^{d_{2n}} \phi(t)dt, 0, \int_0^{d_{2n}} \phi(t)dt + \int_0^{d_{2n+1}} \phi(t)dt\Big).
\end{aligned}$$

So by the condition (i) of $\psi \in \Psi^*$, there exist $F_1 \in \mathscr{C}$ and $\varphi_1 \in \Phi_u$ such that

$$\int_0^{d_{2n+1}} \phi(t)dt \leq F_1\Big(\int_0^{d_{2n}} \phi(t)dt, \varphi_1(\int_0^{d_{2n}} \phi(t)dt)\Big). \tag{2.5}$$

Combing (2.4)-(2.5) and using the property of F_1 and F_2, we have

$$\int_0^{d_{n+1}} \phi(t)dt \leq \int_0^{d_n} \phi(t)dt, \ \forall n \in \mathbb{N}. \tag{2.6}$$

If there exists $n_0 \in \mathbb{N}$ such that $d_{n_0+1} > d_{n_0}$, then by the strictly increasing property of ϕ about integral type, we obtain

$$\int_0^{d_{n_0+1}} \phi(t)dt > \int_0^{d_{n_0}} \phi(t)dt,$$

which is a contradiction to (2.6), hence we have

$$d_{n+1} \leq d_n, \forall\, n = 0, 1, 2, \cdots . \tag{2.7}$$

(2.7) implies that there is $u \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = u$. From Lemma 1.2 and (2.5), we obtain

$$\int_0^u \phi(t)dt = \lim_{n\to\infty} \int_0^{d_{2n+1}} \phi(t)dt \leq F_1(\lim_{n\to\infty} \int_0^{d_{2n}} \phi(t)dt, \lim_{n\to\infty} \varphi_1(\int_0^{d_{2n}} \phi(t)dt))$$
$$= F_1(\int_0^u \phi(t)dt, \varphi_1(\int_0^u \phi(t)dt)).$$

So $\int_0^u \phi(t)dt = 0$ or $\varphi_1(\int_0^u \phi(t)dt) = 0$ by the property of F_1, which implies that $\int_0^u \phi(t)dt = 0$. Therefore, $u = 0$, i.e.,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d_n = 0. \tag{2.8}$$

We claim that $\{y_n\}$ is Cauchy. Otherwise, by Lemma 1.1 and Remark 1.1 and (2.8), there exists $\varepsilon > 0$ such that for each $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k)$ such that the parity of $m(k)$ and $n(k)$ is different and the following result holds

$$\lim_{k\to\infty} d(y_{m(k)}, y_{n(k)}) = \lim_{k\to\infty} d(y_{m(k)}, y_{n(k)-1}) = \lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)-1}) = \lim_{k\to\infty} d(y_{m(k)-1}, y_{n(k)}) = \varepsilon. \tag{2.9}$$

If $m(k)$ is even and $n(k)$ is odd, then by Lemma 1.2, (2.1), (2.8), (2.9) and (iii) of Definition 1.3,

$$\begin{aligned}
0 &< \int_0^{\varepsilon} \phi(t)dt \\
&= \lim_{k\to\infty} \int_0^{d(y_{m(k)},y_{n(k)})} \phi(t)dt \\
&= \lim_{k\to\infty} \int_0^{d(fx_{m(k)},gx_{n(k)})} \phi(t)dt \\
&\le \lim_{k\to\infty} \psi\Big(\int_0^{d(Tx_{m(k)},Tx_{n(k)})} \phi(t)dt, \int_0^{d(Tx_{m(k)},fx_{m(k)})} \phi(t)dt, \int_0^{d(Tx_{n(k)},gx_{n(k)})} \phi(t)dt, \\
&\qquad \int_0^{d(Tx_{m(k)},gx_{n(k)})} \phi(t)dt, \int_0^{d(fx_{m(k)},Tx_{n(k)})} \phi(t)dt\Big) \\
&= \lim_{k\to\infty} \psi\Big(\int_0^{d(y_{m(k)-1},y_{n(k)-1})} \phi(t)dt, \int_0^{d(y_{m(k)-1},y_{m(k)})} \phi(t)dt, \int_0^{d(y_{n(k)-1},y_{n(k)})} \phi(t)dt, \\
&\quad \int_0^{d(x_{m(k)-1},y_{n(k)})} \phi(t)dt, \int_0^{d(y_{m(k)},y_{n(k)-1})} \phi(t)dt\Big) \\
&= \psi\Big(\int_0^{\varepsilon} \phi(t)dt, 0, 0, \int_0^{\varepsilon} \phi(t)dt, \int_0^{\varepsilon} \phi(t)dt\Big) \\
&< \int_0^{\varepsilon} \phi(t)dt.
\end{aligned}$$

This is a contradiction. Similarly, we obtain the same contradiction for the case that $m(k)$ is odd and $n(k)$ is even. Hence, $\{y_n\}$ is a Cauchy sequence.

If TX is complete, then $y_n \in TX$ for all $n \in \mathbb{N} \cup \{0\}$ implies that there exist $u, v \in X$ such that $y_n \to u = Tv$ as $n \to \infty$.

If fX is complete, then $y_{2n} = fx_{2n} \in fX$ for all $n = 0, 1, 2, \cdots$ implies that there exist $u, v_1 \in X$ such that $y_{2n} \to u = fv_1$ as $n \to \infty$, therefore there exists $v \in X$ such that $y_{2n} \to u = fv_1 = Tv$ by the condition $fX \subset TX$. On the other hand, (2.8) and

$$d(y_{2n+1}, u) \le d((y_{2n+1}, y_{2n}) + d(y_{2n}, u)$$

imply that $y_{2n+1} \to u = fv_1$ as $n \to \infty$, hence $y_n \to u = Tv$ as $n \to \infty$.

Similarly, we can obtain the same result for the case of gX being complete. Hence we can assume that $y_n \to u = Tv$ as $n \to \infty$ for any case in the condition (iv).

Since X is regular, using (2.3) and $y_n \to u$, we obtain $y_n \preceq u$ for all $n = 0,1,2,\cdots$, hence $Tx_{n+1} = y_n \preceq u = Tv$ for all $n = 0,1,2,\cdots$. By Lemma 1.2, (2.1), (2.8) and $y_n \to u$,

$$\begin{aligned}
&\int_0^{d(u,gv)} \phi(t)dt \\
&= \lim_{n\to\infty} \int_0^{d(y_{2n},gv)} \phi(t)dt \\
&= \lim_{n\to\infty} \int_0^{d(fx_{2n},gv)} \phi(t)dt \\
&\leq \lim \psi\Big(\int_0^{d(Tx_{2n},Tv)} \phi(t)dt, \int_0^{d(Tx_{2n},fx_{2n})} \phi(t)dt, \int_0^{d(Tv,gv)} \phi(t)dt, \\
&\qquad\qquad \int_0^{d(Tx_{2n},gv)} \phi(t)dt, \int_0^{d(Tv,fx_{2n})} \phi(t)dt\Big) \\
&= \lim \psi\Big(\int_0^{d(y_{2n-1},u)} \phi(t)dt, \int_0^{d(y_{2n-1},y_{2n})} \phi(t)dt, \int_0^{d(u,gv)} \phi(t)dt, \\
&\qquad\qquad \int_0^{d(y_{2n-1},gv)} \phi(t)dt, \int_0^{d(u,y_{2n})} \phi(t)dt\Big) \\
&= \psi\Big(0,0, \int_0^{d(u,gv)} \phi(t)dt, \int_0^{d(u,gv)} \phi(t)dt, 0\Big).
\end{aligned}$$

Hence $\int_0^{d(u,gv)} \phi(t)dt = 0$ by the condition (iii) of Definition 1.3, therefore $u = gv$. Similarly, we can obtain $u = fv$, hence $u = Tv = fv = gv$. This complete $C(f,g,T) \neq \emptyset$.

If $C(f,g,T)$ is not singleton and there exist two different element u and u_1 in $C(f,g,T)$ such that they are comparable, then there exists $v, v_1 \in X$ such that $u = fv = gv = Tv$ and $u_1 = fv_1 = gv_1 = Tv_1$ and Tv and Tv_1 are comparable, hence by (2.1),

$$\begin{aligned}
&\int_0^{d(u,u_1)} \phi(t)dt \\
&= \int_0^{d(fv,gv_1)} \phi(t)dt \\
&\leq \psi\Big(\int_0^{d(Tv,Tv_1)} \phi(t)dt, \int_0^{d(Tv,fv)} \phi(t)dt, \int_0^{d(Tv_1,gv_1)} \phi(t)dt, \int_0^{d(Tv,gv_1)} \phi(t)dt, \int_0^{d(Tv_1,fv)} \phi(t)dt\Big) \\
&= \psi\Big(\int_0^{d(u,u_1)} \phi(t)dt, 0, 0, \int_0^{d(u,u_1)} \phi(t)dt, \int_0^{d(u,u_1)} \phi(t)dt\Big).
\end{aligned}$$

Hence $\int_0^{d(u,u_1)} \phi(t) = 0$ by (iii) of Definition 1.3, which implies $u = u_1$. This is a contradiction, hence any different two elements in $C(f,g,T)$ are not comparable.

Using Theorem 2.1, we obtain a common fixed point theorem for two self-mappings.

Theorem 2.2 Let $(X,d,\preceq)$ be an ordered metric space such that X is regular and $f,g: X \to X$ two mappings. Suppose that for each $x,y \in X$ with x and y being comparable,

$$\begin{aligned}&\int_0^{d(fx,gy)} \phi(t)dt \\ &\le \psi\Big(\int_0^{d(x,y)} \phi(t)dt, \int_0^{d(x,fx)} \phi(t)dt, \int_0^{d(y,gy)} \phi(t)dt, \int_0^{d(x,gy)} \phi(t)dt, \int_0^{d(y,fx)} \phi(t)dt\Big),\end{aligned} \tag{2.10}$$

where $\phi \in \Phi$ and $\psi \in \Psi^*$. If

(i) ϕ is sub-additive and strictly increasing about the integral type;

(ii) $x \preceq gx$ for all $x \in fX$ and $x \preceq fx$ for all $x \in gX$;

(iii) fX or gX is complete.

Then f,g have a common fixed point v, that is, there exist $v \in X$ such that $v = fv = gv$.

Proof. Let $T = 1_X$, then $fx \in 1_X^{-1}(fx)$ and $gx \in 1_X^{-1}(gx)$ and (ii) imply that (f,g) and (g,f) are both partially weakly increasing with respect to 1_X, hence there exists $v \in X$ such that $fv = gv = Tv = v$ by Theorem 2.1, hence v is a common fixed point of $\{f,g,T\}$.

Now, we give a sufficient condition under which there exists a unique common fixed point in Theorem 2.2

Theorem 2.3 Suppose that all of the conditions of Theorem 2.2 are satisfied. Furthermore, if the following conditions hold:

(iv) for each $x,y \in X$, there exists $z \in fX \cup gX$ such that $\{z,x\}$ and $\{z,y\}$ are both comparable pair respectively;

(v) for any two elements $u,v \in X$ with $u \preceq v$, $f^n u \preceq v$ and $g^n u \preceq v$ for all $n \in \mathbb{N}$;

(vi) $\psi \in \Psi^*$ satisfies that there exist $F_3 \in \mathscr{C}$ and $\varphi_3 \in \Phi_u$ such that $u \le \psi(v,0,u+v,u,v)$ implies $u \le F_3(v,\varphi_3(v))$ and ψ is also non-decreasing about the 3th variable;

(vii) $fgX = gfX$.

Then f and g have a unique common fixed point.

Proof. We have proved that f and g have a common fixed point $v \in X$ in Theorem 2.2. Suppose u is also a common fixed point of f and g.

Case 1. If u and v are comparable, then by (2.10),

$$\begin{aligned}&\int_0^{d(u,v)}\phi(t)dt\\ =&\int_0^{d(fu,gv)}\phi(t)dt\\ \leq&\psi\Big(\int_0^{d(u,v)}\phi(t)dt,\int_0^{d(u,fu)}\phi(t)dt,\int_0^{d(v,gv)}\phi(t)dt,\int_0^{d(u,gv)}\phi(t)dt,\int_0^{d(v,fu)}\phi(t)dt\Big)\\ =&\psi\Big(\int_0^{d(u,v)}\phi(t)dt,0,0,\int_0^{d(u,v)}\phi(t)dt,\int_0^{d(v,u)}\phi(t)dt\Big)\end{aligned}$$

hence by (iii) of Definition 1.3,

$$\int_0^{d(u,v)}\phi(t)dt=0,$$

therefore $u=v$.

Case 2. If u and v are not comparable, then $u\neq v$ and there exist $w\in fX\cup gX$ such that w and u are comparable and w and v are also comparable by (iv). In this case, $w\neq u$ and $w\neq v$.

Without loss of generality, let $w\in fX$, then $w\preceq gw$ by (ii) in Theorem 2.2. Since $gw\in gfX=fgX\subset fX$ by (vii), $gw\preceq ggw=g^2w$ by (ii) in Theorem 2.2 again. Repeating this process, we obtain

$$w\preceq gw\preceq g^2w\preceq\cdots\preceq g^nw,\forall n=1,2,\cdots.$$

If $u\preceq w$, then using the above conclusion and (v), we obtain that for any $n=1,2,\cdots$,

$$f^nu\preceq w\preceq gw\preceq g^2w\preceq\cdots\preceq g^nw,$$

hence f^nu and g^nw are comparable for all $n=0,1,2,\cdots$. By (2.10),

$$\begin{aligned}&\int_0^{d(u,g^nw)}\phi(t)dt\\ =&\int_0^{d(ff^{n-1}u,gg^{n-1}w)}\phi(t)dt\\ \leq&\psi\Big(\int_0^{d(f^{n-1}u,g^{n-1}w)}\phi(t)dt,\int_0^{d(f^{n-1}u,ff^{n-1}u)}\phi(t)dt,\int_0^{d(g^{n-1}w,gg^{n-1}w)}\phi(t)dt,\\ &\qquad\int_0^{d(f^{n-1}u,gg^{n-1}w)}\phi(t)dt,\int_0^{d(ff^{n-1}u,g^{n-1}w)}\phi(t)dt\Big)\\ =&\psi\Big(\int_0^{d(u,g^{n-1}w)}\phi(t)dt,0,\int_0^{d(g^{n-1}w,g^nw)}\phi(t)dt,\int_0^{d(u,g^nw)}\phi(t)dt,\int_0^{d(u,g^{n-1}w)}\phi(t)dt\Big)\\ \leq&\psi\Big(\int_0^{d(u,g^{n-1}w)}\phi(t)dt,0,\int_0^{d(u,g^{n-1}w)}\phi(t)dt+\int_0^{d(u,g^{n-1}w)}\phi(t)dt,\int_0^{d(u,g^nw)}\phi(t)dt,\int_0^{d(u,g^{n-1}w)}\phi(t)dt\Big),\end{aligned}$$

hence by (vi), there exist $F_3 \in \mathscr{C}$ and $\varphi_3 \in \Phi_u$ such that

$$\int_0^{d(u,g^n w)} \phi(t)dt \le F_3\Big(\int_0^{d(u,g^{n-1}w)} \phi(t)dt, \varphi_3(\int_0^{d(u,g^{n-1}w)} \phi(t)dt)\Big). \tag{2.11}$$

So by (i) in Definition 1.1,

$$\int_0^{d(u,g^n w)} \phi(t)dt \le \int_0^{d(u,g^{n-1}w)} \phi(t)dt,$$

hence by the strictly increasing property about the integral type of ϕ, we obtain

$$d(u,g^n w) \le d(u,g^{n-1}w), \forall n = 1,2,\cdots. \tag{2.12}$$

From (2.12), we know that $\{d(u,g^n w)\}_{n=1}^{\infty}$ is a nondecreasing and non-negative real sequence, hence there exists $M(u,w) \ge 0$ such that

$$\lim_{n\to\infty} d(u,g^n w) = M(u,w). \tag{2.13}$$

Letting $n \to \infty$ in (2.11) and using (2.13), we obtain

$$\int_0^{M(u,w)} \phi(t)dt \le F_3\Big(\int_0^{M(u,w)} \phi(t)dt, \varphi_3(\int_0^{M(u,w)} \phi(t)dt)\Big), \tag{2.14}$$

hence we obtain $\int_0^{M(u,w)} \phi(t)dt = 0$ or $\varphi_3(\int_0^{M(u,w)} \phi(t)dt) = 0$, therefore

$$\lim_{n\to\infty} d(u,g^n w) = M(u,w) = 0.$$

This implies that

$$\lim_{n\to\infty} g^n w = u. \tag{2.15}$$

If $w \preceq u$, then since $u = gu \in gX$, we also obtain similarly

$$g^n w \preceq u \preceq fu \preceq f^2 u \preceq \cdots \preceq f^n u, \ ,\forall n = 1,2,\cdots,$$

which shows that $g^n w$ and $f^n u$ are comparable for all $n = 0,1,2,\cdots$. Hence similarly, we also obtain (2.15). Therefore (2.15) always holds for two comparable elements u and w. Since w and v are also comparable, we also obtain

$$\lim_{n\to\infty} g^n w = v. \tag{2.16}$$

Hence (2.15) and (2.16) implies that $u = v$, which is a contradiction. Therefore f and g have a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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