Available online at http://scik.org
Adv. Inequal. Appl. 2019, 2019:10
https://doi.org/10.28919/aia/4067
ISSN: 2050-7461

GENERALIZED CONTINUOUS WAVELET TRANSFORM ON LOCALLY COMPACT ABELIAN GROUP

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Abstract: In this paper the mother wavelet on locally compact abelian groups is defined. The continuous wavelet transforms (CWT) and some of its basic properties are obtained. Its inversion formula, the Parseval relation and associated convolution are also studied.

Keywords: continuous wavelet transform; locally compact abelian groups; Fourier transform.

2010 AMS Subject Classification: 42C40, 65T60, 44A35,65R10

1. INTRODUCTION

Most of the spaces that we are interested in end up being topological groups. In this section we define the terms topology and group so that we can work with them. In addition, many of the topological spaces we work with are spaces of functions. In order to integrate and otherwise

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Received March 19, 2019

analyze function spaces, we introduce the Haar measure, which is a translation-invariant measure.

A set S becomes a group if an operator, say $+$, can be defined such that

- $x+(y+z)=(x+y)+z \quad \forall \; x,y,z\in S$
- There exists an element 0 ,such that $x+0=0+x=x \quad \forall \; x\in S$
- For each $x\in S$ there exists an inverse element $x^{-1}=-x$,

  such that $x+(-x)=(-x)+x=0$

In addition, S is a commutative group if it is also true that

- $x+y=y+x \quad \forall \; x,y\in S$

Given a set S , a topology T is a set of subsets on S that

- Contains S and the empty set ϕ
- Is closed under finite intersections and infinite unions of subsets.

S is a topological group if it has a group operation and a topology such that the maps $\alpha: G\times G\to G$ and $\beta: G\to G$ are continuous, where $\alpha(x,y)=x+y$ and $\beta(x)=x^{-1}$.

If S is locally compact, that is, every point in S is contained in a compact neighborhood, and its group operation is commutative, then we call it a locally compact abelian (LCA) group.

In order to define the Fourier transform on LCA groups, we must be able to integrate over these groups. This is done with respect to the Haar Measure.

Given a topological space X , we define the Borel set as a set of subsets of X that

- Contains all subsets of the topology on X
- Is closed under complements, countable unions, and countable intersections of subsets
- Is the smallest set of subsets that meets these condition

A measure μ on X is a function on the Borel sets where

- $\mu(E)=\sum\mu(E_i)$ if $E\subset X$ and $E=\cup_{i\in I}E_i$,where E_i is a countable pairwise disjoint set

- $\mu(E)$ is finite for all $E \subset X$ where the closure of E is compact.

A measure μ is regular if for all Borel sets E we have $\mu(E) = \inf_{K \supset E} \mu(K) = \sup_{K \subset E} \mu(K)$.

μ is invariant if $\mu(x+E) = \mu(E) \quad \forall \ x \in X$.

Let $M(X)$ be the space of all complex-valued regular measures on X where

$\|\mu\| = |\mu(S)|$ is finite.

A Haar measure is a measure which is nonnegative, regular, and invariant. In fact, Haar measures are unique up to a scalar, so we can call it the Haar measure. That is, if m_1, m_2 are both non negative, regular, translation invariant measures on S ,then there exists $\lambda \geq 0$ such that $m_1 = \lambda m_2$ .The corresponding integral is called the Haar integral, which is translation invariant. That is, integrals over a set E and $x+E$ are equivalent.

Given a LCA group G , we define an $L^p(G)$ space to be the space of all complex valued functions f on G such that the integral

$$\int_G |f|^p \, d\mu$$

exists with respect to the Haar measure.

$L^p(G)$ Becomes an algebra under convolution, which is an important characteristic later on.

Definition 1.1. A complex function γ on a LCA group G is called a character of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the functional equation $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $(x,y) \in G$ is satisfied **.** The set of all continuous characters of G form a group Γ, the dual group of G . Now it is customary to write $(x,\gamma) = \gamma(x)$**.** $\gamma(x)$ satisfy the following properties

- $(0,\gamma) = (x,0) = 1$
- $(-x,\gamma) = (x,-\gamma) = (x,\gamma)^{-1} = \overline{(x,\gamma)}$

- $(x+y,\gamma)=(x,\gamma)(y,\gamma)$
- $(x,\gamma_1+\gamma_2)=(x,\gamma_1)(x,\gamma_2)$

Definition 1.2 The Fourier transform of $f\in L^1(G)$ is denoted by $\hat{f}(\gamma)$ defined by [4]

$$\hat{f}(\gamma)=\int_G f(x)(-x,\gamma)\,dx, \tag{1.1}$$

and the inverse Fourier transform is defined by

$$f(x)=\int_G \hat{f}(\gamma)(x,\gamma)\,d\gamma\ , x\in G \tag{1.2}$$

Some important properties of the Fourier transform can be proved easily:

- $\left\|\hat{f}\right\|_{L^\infty(G)}\le\|f\|_{L^1(G)}$
- If $f\in L^1(G)$, then $\hat{f}$ is uniformly continuous.
- Parseval formula: If $f\in L^1(G)\cap L^2(G)$, then $\left\|\hat{f}\right\|_{L^2(G)}=\|f\|_{L^2(G)}$
- If the convolution of f and g is defined as

$$(f*g)(x)=\int_G f(x-y)g(y)\,dy \tag{1.3}$$

$$F((f*g))=F(f)F(g) \tag{1.4}$$

The article is divided in three sections. In section 2, we propose the definition of mother wavelet and define the continuous wavelet transform (CWT). In section 3 we prove the Plancherel formula, inversion formula and also define the convolution associated with CWT.

2. Continuous Wavelet Transform on Locally Compact Abelian Group

Similar to $L^2(\mathbb{R})$ [1,3,5], we define the wavelet on locally compact abelian group G and define the generalized continuous wavelet transform.

Definition 2.1: Admissible wavelet on locally compact abelian group

The function $\psi(x)\in L^2(G)$ is said to be an admissible wavelet on locally compact abelian group if $\psi(x)$ satisfies the following admissibility condition:

$$c_\psi = \int_G \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} d\gamma < \infty \qquad (2.1)$$

where $\hat{\psi}$ is the Fourier transform of ψ

Theorem 2.2 If ψ is a mother wavelet and $\phi \in L^1(G)$, then the convolution function $\psi * \phi$ is a mother wavelet.

Proof: Since

$$\int_G |(\psi * \phi)(x)|^2 dx = \int_G \left| \int_G \psi(x-y)\phi(y) dy \right|^2 dx$$

$$\leq \int_G \left(\int_G |\psi(x-y)| |\phi(y)|^{1/2} \left| |\phi(y)|^{1/2} dy \right| \right)^2 dx$$

$$\leq \int_G \left(\int_G |\psi(x-y)| |\phi(y)| dy \int_G |\phi(y)| dy \right) dx$$

$$= \int_G |\phi(y)| dy \int_G \int_G |\psi(x-y)|^2 |\phi(y)| dy dx$$

$$= \left(\int_G |\phi(y)| \right)^2 \int_G |\psi(x)|^2 dx$$

$$= \|\phi\|^2_{L^1(G)} \|\psi\|^2_{L^2(G)}$$

Therefore $(\psi * \phi)(x) \in L^2(G)$. Moreover

$$c_{\psi*\phi} = \int_G \frac{|\psi \hat{*} \phi(\gamma)|^2}{|\gamma|} d\gamma$$

$$= \int_G \frac{|\hat{\psi}(\gamma)|^2 |\hat{\phi}(\gamma)|^2}{|\gamma|} d\gamma$$

$$\leq \|\hat{\phi}\|^2_{L^\infty(G)} \int_G \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|} d\gamma$$

This completes the proof of the theorem.

Definition2.3. Generalized Continuous wavelet transform (CWT) on locally compact

abelian group.

For $\psi(x) \in L^2(G)$ and $a, b \in G, a \neq 0$, we define the unitary linear operator:

$$U_a^b : L^2(G) \to L^2(G),$$

by

$$U_a^b(\psi(x)) = \psi_{a,b}(x) = \frac{1}{|a|^{1/2}} \psi\left(\frac{x-b}{a}\right) \tag{2.2}$$

ψ is called mother wavelet and $\psi_{a,b}(x)$ are called daughter wavelets, where a is a dilation parameter, b is a translation parameter.

The Fourier transform of $\psi_{a,b}(x)$ is given by

$$\hat{\psi}_{a,b}(\gamma) = |a|^{\frac{1}{2}} \hat{\psi}(a\gamma)(-b, \gamma) \tag{2.3}$$

where $\hat{\psi}$ is the Fourier transform of ψ.

The CWT on locally compact abelian group

$$G_\psi : L^2(G) \to L^2(G \times G),$$

of a function $f \in L^2(G)$ with respect to a mother wavelet ψ is defined by

$$f \mapsto G_\psi f(a,b) = (f, \psi_{a,b})_{L^2(G)}$$

$$= \int_G f(x) \overline{\psi_{a,b}(x)} dx$$

$$= \int_G f(x) \frac{1}{|a|^{1/2}} \overline{\psi\left(\frac{x-b}{a}\right)} dx \tag{2.4}$$

3. Main Properties of the CWT

This section describes important properties of the CWT, such as the Plancherel, inversion formula and associated convolution first, we establish the Plancherel theorem.

Theorem 3.1.(CWT Plancherel) Let $f, g \in L^2(G)$. Then we have

$$\left(\left(G_\psi f\right)(a,b),\left(G_\psi g\right)(a,b)\right)_{L^2(G\times G)}=c_\psi\left(f,g\right)_{L^2(G)} \tag{3.1}$$

where c_ψ is given in (2.1).

Proof: By using Parseval formula for Fourier we can write the wavelet transform as

$$\left(G_\psi f\right)(a,b)=\int_G f(x)\frac{1}{|a|^{1/2}}\overline{\psi\left(\frac{x-b}{a}\right)}dx$$

$$=\left(f,\psi_{a,b}\right)$$

$$=\left(\hat f,\hat\psi_{a,b}\right)$$

$$=\int_G \hat f(\gamma)|a|^{1/2}\,\overline{\hat\psi(a\gamma)}\overline{(-b,\gamma)}d\gamma \tag{3.2}$$

Therefore

$$\overline{G_\psi(f)(a,b)}=\int_G \overline{\hat f(\gamma)}|a|^{1/2}\,\hat\psi(a\gamma)(-b,\gamma)\,d\gamma \tag{3.3}$$

Now, by using above (3.2) and (3.3) we get

$$\int_G\int_G G_\psi(f)(a,b)\overline{G_\psi(g)(a,b)}\frac{dadb}{|a|^2}$$

$$=\int_G\int_G |a|\frac{dadb}{|a|^2}\int_G \hat f(\gamma)\overline{\hat\psi(a\xi)(-b,\gamma)}d\gamma\times\int_G \overline{\hat g(\mu)}\psi(a\mu)(-b,\mu)\,d\mu$$

$$=\int_G\int_G \frac{dadb}{|a|}\int_G \overline{\overline{\hat f(\gamma)}\hat\psi(a\gamma)(-b,\gamma)\,d\gamma}\times\int_G \overline{\hat g(\mu)}\psi(a\mu)(-b,\mu)\,d\mu$$

$$=\int_G\int_G \overline{F\left(\overline{\hat f(\gamma)}\hat\psi(a\gamma)\right)}(b)\,F\left(\overline{\hat g(\mu)}\psi(a\mu)\right)(b)\frac{dadb}{|a|}$$

$$=\int_G\int_G \hat f(\gamma)\overline{\hat\psi(a\gamma)}\overline{\hat g(\gamma)}\psi(a\gamma)\frac{d\gamma da}{|a|}$$

$$=\int_G \hat f(\gamma)\overline{\hat g(\gamma)}\left(\int_G \overline{\hat\psi(a\gamma)}\psi(a\gamma)\frac{da}{|a|}\right)d\gamma$$

$$= \int_G \hat{f}(\gamma)\overline{\hat{g}(\gamma)}\left(\int_G \frac{|\hat{\psi}(a\gamma)|^2}{|a|} da\right) d\gamma$$

$$= \int_G \hat{f}(\gamma)\overline{\hat{g}(\gamma)}\left(\int_G \frac{|\hat{\psi}(\eta)|^2}{\eta} d\eta\right) d\gamma$$

$$= c_\psi\left(\hat{f},\hat{g}\right)_{L^2(G)}$$

$$= c_\psi\left(f,g\right)_{L^2(G)} \tag{3.4}$$

Theorem 3.2. :(Inversion Formula) Let $f \in L^2(G)$. Then we have

$$f(x) = \frac{1}{c_\psi}\int_G\int_G G_\psi(f)(a,b)\psi_{a,b}(x)\frac{dadb}{|a|^2} \tag{3.5}$$

where c_ψ is given in (2.1).

Proof: Let $h(x) \in L^2(G)$ be any function, then by using above theorem, we have

$$c_\psi\left(f,g\right)_{L^2(G)} = \int_G\int_G \left(G_\psi f\right)(a,b)\overline{G_\psi(h)(a,b)}\frac{dadb}{|a|^2}$$

$$= \int_G\int_G \left(G_\psi f\right)(a,b)\overline{\int_G h(x)\overline{\psi_{a,b}(x)}dx}\frac{dadb}{|a|^2}$$

$$= \int_G\int_G\int_G \left(G_\psi f\right)(a,b)\psi_{a,b}(x)\overline{h(x)}\frac{dadbdx}{|a|^2}$$

$$= \left(\int_G\int_G \left(G_\psi f\right)(a,b)\psi_{a,b}(x)\frac{dadb}{|a|^2}, h(x)\right)$$

Hence the result follows.

If $f = h$,

$$\|f\|^2_{L^2(G)} = \int_G\int_G \left|\left(G_\psi f\right)(a,b)\right|^2 \frac{dadb}{|a|^2} \tag{3.6}$$

Moreover, the wavelet transform is isometry from $L^2(G)$ to $L^2(G \times G)$.

3.3. Associated convolution for CWT on locally compact abelian group

Using Pathak and Pathak techniques [5], we define the basic function $W(x,y,z)$, translation τ_x and associated convolution # operator for CWT.

The basic function $W(x,y,z)$ for (2.4) is defined as

$$G_\phi\left[W(x,y,z)\right](a,b)=\int_G W(x,y,z)\overline{\phi_{a,b}(t)}dt$$

$$=\overline{\psi_{a,b}(z)}\overline{(a,\gamma)(b,\gamma)(y)} \tag{3.7}$$

where ψ,ϕ and γ are three wavelets satisfying certain conditions (2.1).

Now, by using (3.5) we get,

$$W(x,y,z)=c_\phi^{-1}\int_G\int_G\overline{\psi_{a,b}(z)}\overline{(a,\gamma)(b,\gamma)(y)}\phi_{a,b}(x)|a|^{-2}\,dadb \tag{3.8}$$

The translation τ_x is defined as [5]

$$(\tau_x h)(y)=h^*(x,y)=\int_G W(x,y,z)h(z)dz$$

$$=c_\phi^{-1}\int_G\int_G\int_G\overline{\psi_{a,b}(z)}\overline{(a,\gamma)(b,\gamma)(y)}\phi_{a,b}(x)h(z)|a|^{-2}\,dadbdz$$

The associated convolution is defined as

$$(h\#g)(x)=\int_G h^*(x,y)g(y)dy$$

$$=\int_G\int_G W(x,y,z)h(z)g(y)dydz$$

$$=c_\phi^{-1}\int_G\int_G\int_G\int_G\overline{\psi_{a,b}(z)}\overline{(a,\gamma)(b,\gamma)(y)}\phi_{a,b}(x)h(z)g(y)|a|^{-2}\,dadbdzdy \tag{3.9}$$

by using the inversion formula we can write the above equation as

$$(h\#g)(x)=c_\phi^{-1}\int_G\int_G(G_\psi h)(a,b)(G_\gamma g)(a,b)\varphi_{a,b}(x)|a|^{-2}\,dadb$$

$$=G_\phi^{-1}\left[(G_\psi h)(a,b)(G_\chi g)(a,b)\right](x);$$

So that

$$G_\phi[h\#g](a,b)=(G_\psi h)(a,b)(G_\gamma g)(a,b)(x) \tag{3.10}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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