Available online at http://scik.org
Adv. Inequal. Appl. 2019, 2019:14
https://doi.org/10.28919/aia/4204
ISSN: 2050-7461

COMMON FIXED POINTS FOR TWO SET-VALUED MAPPINGS ON METRIC SPACES

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Abstract. In this paper, we introduce three classes of functions and construct two contractive conditions to discuss and obtain some new common fixed point theorems for two set-valued mappings on non-complete metric spaces.

Keywords: common fixed point; se-valued mapping; Hausdorff metric.

2010 AMS Subject Classification: 47H10, 54C60, 54H25, 55M20.

1. Introduction and preliminaries

Let (X,d) be a metric space and $CB(X)$ the family of all nonempty closed and bounded subset of X.

The following is the famous Banach's fixed point theorem[1]:

Let (X,d) be a complete metric space and $f : X \to X$ a mapping. If f satisfies

$$d(fx, fy) \le hd(x,y), \forall x,y \in X,$$

where $h \in [0,1)$. Then f has a unique common fixed point in X.

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Received July 4, 2019

Later, many generalizations of Banach's fixed point theorem have appeared. For instance, if $f : X \to X$ is a mapping on a complete metric space (X,d) satisfying the following a quasi-contraction

$$d(fx, fy) \leq h \max\{d(x,y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\},\ \forall x, y \in X,$$

where $h \in [0,1)$. Then f has a unique common fixed point in X.(see [2]).

In 1969, Nadlder[3] extended the famous Banach contraction Principle from a single-valued mapping to a set-valued mappings and gave the next fixed point theorem:

Theorem 1.1([3]) Let (X,d) be a complete metric space and $T : X \to CB(X)$. If there exists $h \in [0,1)$ such that

$$\mathscr{H}(Tx, Ty) \leq h\, d(x,y), \forall x, y \in X,$$

where $\mathscr{H}$ denote the Hausdorff metric on $CB(X)$ induced by d, that is,

$$\mathscr{H}(A,B) = \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\},\ \forall\, A, B \in CB(X),$$

where $D(x,B) = \inf_{z\in B} d(x,z)$. Then T has a fixed point in X.

Mizoguchi-Takahashi[4] also gave the following fixed point theorem:

Theorem 1.2([4]) Let (X,d) be a complete metric space and $T : X \to CB(X)$. If

$$\mathscr{H}(Tx, Ty) \leq \xi(d(x,y))\, d(x,y), \forall x, y \in X,$$

where $\xi : [0,\infty) \to [0,1)$ satisfying $\limsup_{s\to t^+} \xi(s) < 1$ for all $t \in [0,\infty)$. Then T has a fixed point in X.

In 2011, Amini-Harandi generalized and improved the corresponding result in [2] from a single-valued mapping to a set-valued mapping, obtained the next result:

Theorem 1.3([5]) Let (X,d) be a complete metric space and $T : X \to CB(X)$ a k-set-valued quasi-contraction with $k < \frac{1}{2}$, that is,

$$\mathscr{H}(Tx, Ty) \leq k \max\{d(x,y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}, \forall\, x, y \in X.$$

Then T has a fixed point in X.

And in 2011, Chen[6] introduced the following definition of ψ-contraction and obtained a fixed point theorem for set-valued mappings:

Theorem 1.4([6]) Let (X,d) be a complete metric space and $T : X \to CB(X)$ a set-valued ψ-contraction, that is,

$$\mathscr{H}(Tx,Ty) \leq \psi(d(x,y),D(x,Tx),D(y,Ty),D(x,Ty),D(y,Tx)), \forall\, x,y \in X,$$

then T has a fixed point in X. Where $\psi : \mathbb{R}^{+5} \to \mathbb{R}^{+}$ is a function satisfying some particular conditions, see [6].

On the other hand, In 2009, Wu et al[7] obtained the next common fixed point theorem for set-valued mappings $\{A_i\}$:

Theorem 1.5([7]) Let (X,d) be a complete metric space and $A_i : X \to CB(X)$ satisfy the condition: for any $i, j = 1,2,\cdots, x,y \in X$ and $u \in A_i x$, there exists $v \in A_j y$ such that

$$d(u,v) \leq \Phi(d(x,y),D(x,A_i x),D(y,A_j y),D(x,A_j y),D(y,A_i x)),$$

then $\{A_i\}$ have a common fixed point in X. Where $\Phi : \mathbb{R}^{+5} \to \mathbb{R}^{+}$ is a function satisfying some particular conditions, see [7].

The following is well-known results, see [7]:

Lemma 1.1 If (X,d) is a metric space, $A,B \in CB(X)$, then for any $\varepsilon > 0$ and any $a \in A$, there exists $b \in B$ such that $d(a,b) \leq \mathscr{H}(A,B) + \varepsilon$.

Lemma 1.2 If (X,d) is a metric space, $A \in CB(X)$, then $D(\cdot,A)$ is continuous. Moreover, we have that

(i) $A = \{x \in X | d(x,A) = 0\}$;

(ii) For any $x,y \in X$, $D(x,A) \leq d(x,y) + D(y,A)$.

In this paper, we use the method in [6] to obtain some common fixed point theorems for two set-valued mappings in metric spaces.

2. Common fixed points

Now, we begin with the following definition.

Let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^{+}$ be a functions satisfying the following conditions, where $\mathbb{R}^{+} = [0,+\infty)$:

(A1) ψ is non-decreasing and continuous in each coordinate;

(A2) for all $t > 0$, $\psi(t,t,t,0,2t) < t$, $\psi(t,t,t,2t,0) < t$, $\psi(0,0,t,t,0) < t$, $\psi(0,t,0,0,t) < t$ and $\psi(t,0,0,t,t) < t$.

Example 1.1 Let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ be $\psi(t_1,t_2,t_3,t_4,t_5) = a_1t_1 + a_2t_2 + a_3t_3 + a_4t_4 + a_5t_5$, where a_1,a_2,a_3,a_4,a_5 are non-negative real numbers satisfying $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$, Then ψ satisfies (A1) and (A2).

Theorem 2.1 Let (X,d) be a metric space, $S,T : X \to CB(X)$ two set-valued mappings satisfying the following condition: for all $x,y \in X$,

$$\mathscr{H}(Sx,Ty) < \psi(d(x,y),D(x,Sx),D(y,Ty),D(x,Ty),D(y,Sx)). \tag{1}$$

If $S(X)$ or $T(X)$ is complete, then S and T have a common fixed point in X.

Proof. Note that for each $A,B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathscr{H}(A,B) < \gamma$, there exists $b \in B$ such that $d(a,b) < \gamma$ by Lemma 1.1.

Let $x_0 \in X$ and take any $x_1 \in Sx_0$, then for $x_1 \in Sx_0$, there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq \psi(d(x_0,x_1),D(x_0,Sx_0),D(x_1,Tx_1),D(x_0,Tx_1),D(x_1,Sx_0)).$$

Similarly, for $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$ such that

$$d(x_3,x_2) \leq \psi(d(x_2,x_1),D(x_2,Sx_2),D(x_1,Tx_1),D(x_2,Tx_1),D(x_1,Sx_2)).$$

For $x_3 \in Sx_2$, there exists $x_4 \in Tx_3$ such that

$$d(x_3,x_4) \leq \psi(d(x_2,x_3),D(x_2,Sx_2),D(x_3,Tx_3),D(x_2,Tx_3),D(x_3,Sx_2)).$$

By the mathematical induction and the above observation, we can construct a sequence $\{x_n\}$ satisfying that for $x_{2n+1} \in Sx_{2n}$, there exists $x_{2n+2} \in Tx_{2n+1}$ such that

$$\begin{aligned}
&d(x_{2n+1},x_{2n+2})\\
\leq&\psi(d(x_{2n},x_{2n+1}),D(x_{2n},Sx_{2n}),D(x_{2n+1},Tx_{2n+1}),D(x_{2n},Tx_{2n+1}),D(x_{2n+1},Sx_{2n}))\\
\leq&\psi(d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+2}),(x_{2n+1},x_{2n+1}))\\
\leq&\psi(d(x_{2n},x_{2n+1}),d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2}),d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2}),0)
\end{aligned} \tag{2}$$

and for $x_{2n+2} \in Tx_{2n+1}$, there exists $x_{2n+3} \in Sx_{2n+2}$ such that

$$
\begin{aligned}
&d(x_{2n+3}, x_{2n+2}) \\
\leq&\psi(d(x_{2n+2}, x_{2n+1}), D(x_{2n+2}, Sx_{2n+2}), D(x_{2n+1}, Tx_{2n+1}), D(x_{2n+2}, Tx_{2n+1}), D(x_{2n+1}, Sx_{2n+2})) \\
\leq&\psi(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+2}, x_{2n+2}), d(x_{2n+1}, x_{2n+3})) \\
\leq&\psi(d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n+3}), d(x_{2n+1}, x_{2n+2}), 0, d(x_{2n+1}, x_{2n+2}) + d(x_{2n+2}, x_{2n+3})).
\end{aligned} \tag{3}
$$

If $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$ for some $n \in \mathbb{N}$, then $d(x_{2n+1}, x_{2n+2}) > 0$, hence by (2), (A1) and (A2),

$$
\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) \\
\leq&\psi(d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2}), 2d(x_{2n+1}, x_{2n+2}), 0) \\
<&d(x_{2n+1}, x_{2n+2}),
\end{aligned}
$$

which is a contradiction. Hence

$$d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}), \ \forall\, n \in \mathbb{N}.$$

Similarly, by (A1), (A2) and (3),

$$d(x_{2n+3}, x_{2n+2}) \leq d(x_{2n+1}, x_{2n+2}), \ \forall\, n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n).$$

Let $c_m = d(x_m, x_{m+1})$ for all $m \in \mathbb{N}$, then $\{c_m\}$ is a decreasing sequence and bounded below, hence there exists $c \geq 0$ such that $\lim_{m\to\infty} c_m = c$. If $c > 0$, then using (2), we obtain

$$
\begin{aligned}
c \leq& c_{2n+1} = d(x_{2n+1}, x_{2n+2}) \\
\leq&\psi(d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}), 0) \\
=&\psi(c_{2n}, c_{2n}, c_{2n+1}, c_{2n} + c_{2n+1}, 0) \\
\leq&\psi(c_{2n}, c_{2n}, c_{2n+1}, 2c_{2n}, 0).
\end{aligned}
$$

Let $n \to \infty$ on the two-sides of the above, then by (A1) and (A2), we obtain

$$c \leq \psi(c, c, c, 2c, 0) < c.$$

This contradiction shows $\lim_{m\to\infty} c_m = c = 0$.

Next, we will prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exists $\gamma > 0$ such that for all $k \in \mathbb{N}$, there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ satisfying

(i) $m(k)$ is even and $n(k)$ is odd;

(ii) $d(x_{m(k)}, x_{n(k)}) \geq \gamma$;

(iii) $m(k)$ the smallest even number satisfying the conditions (i) and (ii).

By (iii), we have

$$d(x_{m(k)-2}, x_{n(k)}) < \gamma,\ \forall k = 1, 2, \cdots$$

and

$$\gamma \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{n(k)}, x_{m(k)-2}) + c_{m(k)-2} + c_{m(k)-1} < \gamma + c_{m(k)-2} + c_{m(k)-1}.$$

Letting $k \to \infty$ on the above, we obtain

$$\gamma = \lim_{k\to\infty} d(x_{m(k)-2}, x_{n(k)}) = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}). \tag{4}$$

We also have

$$\begin{aligned}
&\gamma \\
\leq& d(x_{m(k)}, x_{n(k)}) \leq \mathscr{H}(Tx_{m(k)-1}, Sx_{n(k)-1}) = \mathscr{H}(Sx_{n(k)-1}, Tx_{m(k)-1}) \\
\leq& \psi(d(x_{n(k)-1}, x_{m(k)-1}), D(x_{n(k)-1}, Sx_{n(k)-1}), D(x_{m(k)-1}, Tx_{m(k)-1}), D(x_{n(k)-1}, Tx_{m(k)-1}), D(x_{m(k)-1}, Sx_{n(k)-1})) \\
\leq& \psi(d(x_{n(k)-1}, x_{m(k)-1}), d(x_{n(k)-1}, x_{n(k)}), d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{m(k)}), d(x_{m(k)-1}, x_{n(k)})) \\
\leq& \psi(c_{n(k)-1} + d(x_{n(k)}, x_{m(k)}) + c_{m(k)-1}, c_{n(k)-1}, c_{m(k)-1}, c_{n(k)-1} + d(x_{n(k)}, x_{m(k)}), c_{m(k)-1} + d(x_{m(k)}, x_{n(k)})).
\end{aligned}$$

Letting $k \to \infty$ on the above, then using (4) and $c = 0$, we obtain the following contradiction

$$\gamma \leq \psi(\gamma, 0, 0, \gamma, \gamma) < \gamma.$$

Hence $\{x_n\}$ is a Cauchy sequence.

Suppose that $S(X)$ is complete. Since $x_{2n+1} \in Sx_{2n} \subset S(X)$ for all $n \in \mathbb{N}$, there exists $u \in S(X)$ such that $x_{2n+1} \to u$ as $n \to \infty$. Hence

$$d(x_{2n+2}, u) \leq d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, u) = c_{2n+1} + d(x_{2n+1}, u)$$

implies that $x_{2n+2} \to u$ as $n \to \infty$. By Lemma 1.2,

$$\begin{aligned} D(u,Tu) &= \lim_{n\to\infty} D(x_{2n+3},Tu) \\ &\leq \lim_{n\to\infty} \mathscr{H}(Sx_{2n+2},Tu) \\ &\leq \lim_{n\to\infty} \psi(d(x_{2n+2},u),d(x_{2n+2},x_{2n+3}),D(u,Tu),D(x_{2n+2},Tu),d(u,x_{2n+3})). \end{aligned}$$

Let $n \to \infty$ on the two-sides of the above, then

$$D(u,Tu) \leq \psi(0,0,D(u,Tu),D(u,Tu),0),$$

hence $D(u,Tu) = 0$ by (A2), so $u \in Tu$ by Lemma 1.2 again. Similarly,

$$\begin{aligned} D(u,Su) &= \lim_{n\to\infty} D(x_{2n+2},Su) \\ &\leq \lim_{n\to\infty} \mathscr{H}(Tx_{2n+1},Su) \\ &= \lim_{n\to\infty} \mathscr{H}(Su,Tx_{2n+1}) \\ &\leq \lim_{n\to\infty} \psi(d(u,x_{2n+1}),D(u,Su),d(x_{2n+1},x_{2n+2}),d(u,x_{2n+2}),d(x_{2n+1},Su)). \end{aligned}$$

Let $n \to \infty$ on the two sides of the above, then

$$D(u,Su) \leq \psi(0,D(u,Su),0,0,D(u,Su)),$$

hence $D(u,Su) = 0$ by (A2), so $u \in Su$. Therefore, u is the common fixed point of S and T.

If $T(X)$ is is complete, then since $x_{2n} \in T(X)$, there exists $u \in TX$ such that $x_{2n} \to u$ as $n \to \infty$. But

$$d(x_{2n+1},u) \leq d(x_{2n+1},x_{2n}) + d(x_{2n},u) = c_{2n} + d(x_{2n},u),$$

hence $x_{2n+1} \to u$ as $n \to \infty$. Hence we are easy to prove the same conclusion for the case that $T(X)$ is complete.

Now, we consider another type common fixed point theorem.

Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a increasing and continuous function with $\phi(t) < \frac{1}{4}t$ for all $t > 0$ and $\phi(0) = 0$.

Let $\varphi : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ be a decreasing and continuous in each coordinate such that $\varphi(x,y) = 0$ if and only if $x = y = 0$.

Theorem 2.2 Suppose (X,d) is a metric space and two mappings $S,T : X \to CB(X)$ satisfy that for all $x,y \in X$,

$$\mathscr{H}(Sx,Ty) < \frac{1}{2}d(x,y) + \phi(D(x,Ty)+D(y,Sx)) - \varphi(D(x,Ty),D(y,Sx)). \tag{5}$$

If $S(X)$ or $T(X)$ is complete, Then S and T have an common fixed point in X.

Proof. Note that for each $A,B \in CB(X)$, $a \in A$ and $\gamma > 0$ with $\mathscr{H}(A,B) < \gamma$, there exists $b \in B$ such that $d(a,b) < \gamma$.

Let $x_0 \in X$ and take $x_1 \in Sx_0$, then there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq \frac{1}{2}d(x_0,x_1) + \phi(D(x_0,Tx_1)+D(x_1,Sx_0)) - \varphi(D(x_0,Tx_1),D(x_1,Sx_0));$$

For $x_2 \in Tx_1$, there exists $x_3 \in Sx_2$ such that

$$d(x_3,x_2) \leq \frac{1}{2}d(x_2,x_1) + \phi(D(x_2,Tx_1)+D(x_1,Sx_2)) - \varphi(D(x_2,Tx_1),D(x_1,Sx_2)).$$

Generally, for $x_{2n+1} \in Sx_{2n}$, there exists $x_{2n+2} \in Tx_{2n+1}$ such that

$$\begin{aligned} & d(x_{2n+1},x_{2n+2}) \\ \leq\ & \frac{1}{2}d(x_{2n},x_{2n+1}) + \phi(D(x_{2n},Tx_{2n+1})+D(x_{2n+1},Sx_{2n})) - \varphi(D(x_{2n},Tx_{2n+1}),D(x_{2n+1},Sx_{2n})) \end{aligned}$$

and for $x_{2n+2} \in Tx_{2n+1}$, there exists $x_{2n+3} \in Sx_{2n+2}$ such that

$$\begin{aligned} & d(x_{2n+3},x_{2n+2}) \\ \leq\ & \frac{1}{2}d(x_{2n+2},x_{2n+1}) + \phi(D(x_{2n+2},Tx_{2n+1})+D(x_{2n+1},Sx_{2n+2})) - \varphi(D(x_{2n+2},Tx_{2n+1}),D(x_{2n+1},Sx_{2n+2})). \end{aligned}$$

For any $n \in \mathbb{N}$,

$$\begin{aligned} & d(x_{2n+1},x_{2n+2}) \\ \leq & \frac{1}{2}d(x_{2n},x_{2n+1}) + \phi(d(x_{2n},x_{2n+2})) - \varphi(d(x_{2n},x_{2n+2}),0) \\ \leq & \frac{1}{2}d(x_{2n},x_{2n+1}) + \frac{1}{4}d(x_{2n},x_{2n+2}) - \varphi(d(x_{2n},x_{2n+2}),0) \\ \leq & \frac{1}{2}d(x_{2n},x_{2n+1}) + \frac{1}{4}d(x_{2n},x_{2n+2}) \\ \leq & \frac{1}{2}d(x_{2n},x_{2n+1}) + \frac{1}{4}[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})], \end{aligned} \tag{6}$$

hence we obtain that

$$d(x_{2n+1},x_{2n+2}) \leq d(x_{2n},x_{2n+1}). \tag{7}$$

Similarly, since

$$\begin{aligned}
&d(x_{2n+3},x_{2n+2})\\
\leq&\frac{1}{2}d(x_{2n+2},x_{2n+1})+\phi(d(x_{2n+1},x_{2n+3}))-\varphi(0,d(x_{2n+1},x_{2n+3}))\\
\leq&\frac{1}{2}d(x_{2n+2},x_{2n+1})+\frac{1}{4}d(x_{2n+1},x_{2n+3})-\varphi(0,d(x_{2n+1},x_{2n+4}))\\
\leq&\frac{1}{2}d(x_{2n+2},x_{2n+1})+\frac{1}{4}[d(x_{2n+2},x_{2n+1})+d(x_{2n+3},x_{2n+2})],
\end{aligned} \tag{8}$$

so

$$d(x_{2n+2},x_{2n+3}) \leq d(x_{2n+1},x_{2n+2}). \tag{9}$$

Combining (7) and (9), we have that for all $n \in \mathbb{N}$,

$$d(x_{n+1},x_{n+2}) \leq d(x_n,x_{n+1}).$$

Let $c_m = d(x_m,x_{m+1})$ for all $m \in \mathbb{N}$, then $\{c_m\}$ is a non-increasing sequence and bounded below, so there exists $\xi \geq 0$ such that $\lim_{m\to\infty} c_m = \xi$.

In view of (6),

$$\begin{aligned}
&\xi \leq c_{2n+1} = d(x_{2n+1},x_{2n+2})\\
\leq&\frac{1}{2}d(x_{2n},x_{2n+1})+\frac{1}{4}d(x_{2n},x_{2n+2})\\
\leq&\frac{1}{2}d(x_{2n},x_{2n+1})+\frac{1}{4}[d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})].
\end{aligned} \tag{10}$$

Let $n \to \infty$ on the above, then $\xi \leq \frac{1}{2}\xi + \frac{1}{4}\lim_{n\to\infty} d(x_{2n},x_{2n+2}) \leq \frac{1}{2}\xi + \frac{1}{4}[\xi+\xi] = \xi$, hence we have that

$$\lim_{n\to\infty} d(x_{2n},x_{2n+2}) = 2\xi.$$

By (6) again,

$$\xi \leq d(x_{2n+1},x_{2n+2}) \leq \frac{1}{2}d(x_{2n},x_{2n+1})+\frac{1}{4}d(x_{2n},x_{2n+2})-\varphi(d(x_{2n},x_{2n+2}),0).$$

Let $n \to \infty$ on the above, then

$$\xi \leq \frac{1}{2}\xi+\frac{1}{4}\times 2\xi-\varphi(2\xi,0) \leq \xi,$$

so

$$\varphi(2\xi,0)=0,$$

hence

$$\xi=0.$$

This prove that $\lim_{n\to\infty} d(x_n,x_{n+1})=\lim_{n\to\infty} c_n=0$.

If $\{x_n\}$ is not a Cauchy sequence, then there exists $\gamma>0$ such that for all $k\in\mathbb{N}$, there exist $m(k),n(k)\in\mathbb{N}$ with $m(k)>n(k)\geq k$ satisfying (i),(ii) and (iii) in Theorem 2.1 and (4) holds.

We have

$$\begin{aligned}
&\gamma\leq d(x_{m(k)},x_{n(k)})\\
\leq&\mathscr{H}(Tx_{m(k)-1},Sx_{n(k)-1})\\
=&\mathscr{H}(Sx_{n(k)-1},Tx_{m(k)-1})\\
\leq&\frac{1}{2}d(x_{n(k)-1},x_{m(k)-1})+\phi(D(x_{n(k)-1},Tx_{m(k)-1})+D(x_{m(k)-1},Sx_{n(k)-1}))\\
&-\varphi(D(x_{n(k)-1},Tx_{m(k)-1}),D(x_{m(k)-1},Sx_{n(k)-1}))\\
\leq&\frac{1}{2}[c_{n(k)-1}+d(x_{n(k)},x_{m(k)})+c_{m(k)-1}]+\phi(d(x_{n(k)-1},x_{m(k)})+d(x_{m(k)-1},x_{n(k)}))\\
&-\varphi(d(x_{n(k)-1},x_{m(k)}),d(x_{m(k)-1},x_{n(k)}))\\
\leq&\frac{1}{2}[c_{n(k)-1}+d(x_{n(k)},x_{m(k)})+c_{m(k)-1}]+\phi([c_{n(k)-1}+d(x_{n(k)},x_{m(k)})]+[d(x_{m(k)},x_{n(k)})+c_{m(k)-1}])\\
&-\varphi([c_{n(k)-1}+d(x_{n(k)},x_{m(k)})],[d(x_{m(k)},x_{n(k)})+c_{m(k)-1}]).
\end{aligned}$$

Let $k\to\infty$ on the above, then we obtain that

$$\gamma\leq\frac{1}{2}\gamma+\phi(2\gamma)-\varphi(\gamma,\gamma)\leq\frac{1}{2}\gamma+\frac{1}{4}(2\gamma)-\varphi(\gamma,\gamma)\leq\gamma,$$

hence $\varphi(\gamma,\gamma)=0$, so $\gamma=0$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence.

Since $S(X)$ or $T(X)$ is complete, there exists $z\in S(X)$ or $z\in T(X)$ such that $x_n\to z$ as $n\to\infty$ (for detail, see the proof process of Theorem 2.1).

Finally, by Lemma 1.2 and (5),

$$
\begin{aligned}
D(z,Tz) &= \lim_{n\to\infty} D(x_{2n+1},Tz)\\
&\leq \lim_{n\to\infty} \mathscr{H}(Sx_{2n},Tz)\\
&\leq \lim_{n\to\infty}[\frac{1}{2}d(x_{2n},z)+\phi(D(x_{2n},Tz)+D(z,Sx_{2n}))-\varphi(D(x_{2n},Tz),D(z,Sx_{2n}))]\\
&\leq \lim_{n\to\infty}[\frac{1}{2}d(x_{2n},z)+\phi(D(x_{2n},Tz)+d(z,x_{2n+1}))-\varphi(D(x_{2n},Tz),d(z,x_{2n+1}))]\\
&= \phi(D(z,Tz))-\varphi(D(z,Tz),0)\\
&\leq \frac{1}{4}D(z,Tz),
\end{aligned}
$$

hence $D(z,Tz)=0$, therefore $z\in Tz$ since Tz is closed.

Similarly, by Lemma 1.2 and (5),

$$
\begin{aligned}
D(z,Sz) &= \lim_{n\to\infty} D(x_{2n+2},Sz)\\
&\leq \lim_{n\to\infty} \mathscr{H}(Tx_{2n+1},Sz)\\
&= \lim_{n\to\infty} \mathscr{H}(Sz,Tx_{2n+1})\\
&\leq \lim_{n\to\infty}[\frac{1}{2}d(z,x_{2n+1})+\phi(D(z,Tx_{2n+1})+D(x_{2n+1},Sz))-\varphi(D(z,Tx_{2n+1}),D(x_{2n+1},Sz))]\\
&\leq \lim_{n\to\infty}[\frac{1}{2}d(z,x_{2n+1})+\phi(d(z,x_{2n+2})+D(x_{2n+1},Sz))-\varphi(d(z,x_{2n+2}),D(x_{2n+1},Sz))]\\
&= \phi(D(z,Sz))-\varphi(0,D(z,Sz))\\
&\leq \frac{1}{4}D(z,Sz),
\end{aligned}
$$

hence $D(z,Sz)=0$, therefore $z\in Sz$ since Sz is closed. This complete that z is the common fixed point of S and T.

Conflict of Interests

The authors declare that there is no conflict of interests.

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