Available online at http://scik.org
Adv. Inequal. Appl. 2020, 2020:1
https://doi.org/10.28919/aia/4349
ISSN: 2050-7461

m-CONVEX FUNCTIONS ASSOCIATED WITH BOUNDS OF *k*-FRACTIONAL INTEGRALS

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Abstract. In this paper, bounds of Riemann-Liouville k-fractional integrals and bounds of generalized Riemann-Liouville k-fractional integrals have been established. To establish these bounds we use m-convexity and monotonicity of utilized functions. A very simple approach is followed to obtain these results. Also applications of presented results are discussed.

Keywords: convex function; m-convex functions; k-fractional integrals; bounds.

2010 AMS Subject Classification: 26A51, 26A33, 26D15.

1. INTRODUCTION

A function $f:[a,b]\to\mathbb{R}$ is said to be convex, if

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y) \tag{1}$$

holds for all $x,y\in[a,b]$ and $t\in[0,1]$.

If inequality (1) is reversed, then the function f will be the concave on $[a,b]$. Convex functions are very useful because of their different classes including m-convex, h-convex, $(h-m)$-convex,

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Received July 12, 2019

s-convex, and so on has been given over the years.

The concept of m-convex functions is introduced in [17]. Since a lot of integral inequalities has been established due to m-convex functions for more details (see, [1, 4, 6, 7, 8]).

Definition 1.1. A function $f : [0,b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, if

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{2}$$

holds for all $x, y \in [0,b]$, $m \in (0,1]$ and $t \in [0,1]$.

For suitable choice of m, class of m-convex functions reduces to the different known classes of functions defined on $[0,b]$. For example if we choose $m = 1$, then we get the concept of convex function and if we choose $m = 0$, then we get the concept of star-shaped function. We recall that a function $f : [0,b] \to \mathbb{R}$ is called *starshaped* if $f(tx) \leq tf(x)$ holds for all $x \in [0,b]$ and $t \in [0,1]$.

Example 1.1. [6] The function $f : [0,\infty) \to \mathbb{R}$, given by

$$f(x) = \frac{1}{12}\left(x^4 - 5x^3 + 9x^2 - 5x\right)$$

is $\frac{16}{17}$−convex function but it is not m-convex for any $m \in (\frac{16}{17}, 1]$.

Next we give the definition of well known Riemann-Liouville fractional integrals.

Definition 1.2. Let $f \in L_1[a,b]$, then Riemann-Liouville fractional integrals of order $\sigma > 0$ with $a \geq 0$ are defined as:

$$I_{a^+}^{\sigma} f(z) = \frac{1}{\Gamma(\sigma)} \int_a^z (z-t)^{\sigma-1} f(t)dt, \quad z > a \tag{3}$$

and

$$I_{b^-}^{\sigma} f(z) = \frac{1}{\Gamma(\sigma)} \int_z^b (t-z)^{\sigma-1} f(t)dt, \quad z < b. \tag{4}$$

The left-sided and right-sided Riemann-Liouville k-fractional integrals are given in [15].

Definition 1.3. Let $f \in L_1[a,b]$, then Riemann-Liouville k-fractional integrals of order $\sigma, k > 0$ with $a \geq 0$ are defined as:

$$I_{a^+}^{\sigma,k} f(z) = \frac{1}{k\Gamma_k(\sigma)} \int_a^z (z-t)^{\frac{\sigma}{k}-1} f(t)dt, \quad z > a \tag{5}$$

and

$$I_{b^-}^{\sigma,k} f(z) = \frac{1}{k\Gamma_k(\sigma)} \int_z^b (t-z)^{\frac{\sigma}{k}-1} f(t)dt, \quad z < b. \tag{6}$$

A more general definition of the Riemann-Liouville fractional integrals is given in [13].

Definition 1.4. Let $f : [a,b] \to \mathbb{R}$ be a integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\sigma > 0$ are defined as:

$$I^{\sigma}_{g,a^+} f(z) = \frac{1}{\Gamma(\sigma)} \int_a^z (g(z) - g(t))^{\sigma-1} g'(t) f(t) dt, \quad z > a \tag{7}$$

and

$$I^{\sigma}_{g,b^-} f(z) = \frac{1}{\Gamma(\sigma)} \int_z^b (g(t) - g(z))^{\sigma-1} g'(t) f(t) dt, \quad z < b, \tag{8}$$

where $\Gamma(.)$ is the Gamma function.

Generalized Riemann-Liouville k-fractional integrals is given in [14].

Definition 1.5. Let $f : [a,b] \to \mathbb{R}$ be a integrable function. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). The left-sided and right-sided k-fractional integrals of a function f with respect to another function g on $[a,b]$ of order $\sigma, k > 0$ are defined as:

$$I^{\sigma,k}_{g,a^+} f(z) = \frac{1}{k\Gamma_k(\sigma)} \int_a^z (g(z) - g(t))^{\frac{\sigma}{k}-1} g'(t) f(t) dt, \quad z > a \tag{9}$$

and

$$I^{\sigma,k}_{g,b^-} f(z) = \frac{1}{k\Gamma_k(\sigma)} \int_z^b (g(t) - g(z))^{\frac{\sigma}{k}-1} g'(t) f(t) dt, \quad z < b, \tag{10}$$

where $\Gamma_k(.)$ is the k-Gamma function.

These are compact forms of fractional integrals in the sense that a lot of fractional integrals can obtained as desired. Surprisingly such particular form of fractional integrals are independently investigated by many authors in recent years. These are comprise in the following remark.

Remark 1.1. In the above Definition 1.5.

(i) If we take $k = 1$, then we get the Definition 1.4 of Riemann-Liouville fractional integrals with respect to an increasing function.

(ii) If we take $g(z) = z$, then we get the Definition 1.3 of Riemann-Liouville k-fractional integrals.

(iii) If we take $g(z) = z$ and $k = 1$, then we get the Definition 1.2 of Riemann-Liouville fractional integrals.

(iv) If we take $g(z) = \frac{z^\rho}{\rho}$, $\rho > 0$ and $k = 1$, then we get the definition of Katugampola fractional integrals given in [2].

(v) If we take $g(z) = \frac{z^{\tau+s}}{\tau+s}$ and $k = 1$, then we get the definition of generalized conformable fractional integrals defined by T. U. Khan et al. in [12].

(vi) If we take $g(z) = \frac{(z-a)^s}{s}$, $s > 0$ in (9) and $g(z) = -\frac{(b-z)^s}{s}$, $s > 0$ in (10), then we get the definition of conformable (k,s)- fractional integrals defined by S. Habib et al. in [10].

(vii) If we take $g(z) = \frac{z^{1+s}}{1+s}$, then we get the definition of conformable fractional integrals defined by Sarikaya et al. in [16].

(viii) If we take $g(z) = \frac{(z-a)^s}{s}$, $s > 0$ in (9) and $g(z) = -\frac{(b-z)^s}{s}$, $s > 0$ in (10) with $k = 1$, then we get the definition of conformable fractional integrals defined by F. Jarad et al. in [11].

In Section 2, we give the bounds of the left-sided and right-sided Riemann-Liouville k-fractional integrals. To establish these bounds we use m-convexity of utilized function. In Section 3, we give the bounds of the left-sided and right-sided generalized Riemann-Liouville k-fractional integrals. To establish these bounds we use m-convexity and monotonicity of utilized functions. The presented results are also have some connections with already published [3, 5, 9] results. In Section 4, we give the applications of the established results of Section 2 and Section 3.

2. Bounds of Riemann-Liouville k-fractional integrals

In this section, we give the bounds of Riemann-Liouville k-fractional integrals. To establish these bounds we use m-convexity of utilized function. First we give the following bounds of the sum of the left-sided $I_{a^+}^{\sigma,k} f$ and right-sided $I_{b^-}^{\sigma,k} f$ Riemann-Liouville k-fractional integrals.

Theorem 2.1. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a positive function such that $0 \leq a < b$. If f is m-convex, then the following inequality holds:

$$2k \left(\Gamma_k(\sigma) I_{a^+}^{\sigma,k} f(z) + \Gamma_k(\tau) I_{b^-}^{\tau,k} f(z) \right)$$

(11)

$$\leq \frac{(z-a)^{\frac{\sigma}{k}}}{mz-a}\Big[f(a)((2m-1)z-a)+mf(z)(z-a)\Big]$$
$$+\frac{(b-z)^{\frac{\tau}{k}}}{mb-z}\Big[f(b)(mb-(2-m)z)+mf(z)(b-z)\Big]$$

for all $z\in[a,b]$, $\sigma,\tau\geq 1$ and $k>0$.

Proof. Consider the function f over the interval $[a,z]$. Then for $\sigma\geq 1$ and $k>0$, the following inequality holds:

$$(z-t)^{\frac{\sigma}{k}-1}\leq (z-a)^{\frac{\sigma}{k}-1} \tag{12}$$

for all $t\in[a,z]$ and $z\in[a,b]$.

By using m-convexity of f, we have

$$f(t)\leq \frac{mz-t}{mz-a}f(a)+m\frac{t-a}{mz-a}f(z). \tag{13}$$

From (12) and (13), one can has

$$\int_a^z (z-t)^{\frac{\sigma}{k}-1}f(t)dt$$
$$\leq \frac{(z-a)^{\frac{\sigma}{k}-1}}{mz-a}\Big[f(a)\int_a^z (mz-t)dt+mf(z)\int_a^z (t-a)dt\Big].$$

By using the (5) of Definition 1.3 and after simplification, we get

$$2k\Gamma_k(\sigma)I_{a^+}^{\sigma,k}f(z)\leq \frac{(z-a)^{\frac{\sigma}{k}}}{mz-a}\Big[f(a)((2m-1)z-a)+mf(z)(z-a)\Big]. \tag{14}$$

Now consider the function f over the interval $[z,b]$. Then for $\tau\geq 1$ and $k>0$, the following inequality holds:

$$(t-z)^{\frac{\tau}{k}-1}\leq (b-z)^{\frac{\tau}{k}-1} \tag{15}$$

for all $t\in[z,b]$ and $z\in[a,b]$.

Again by using m-convexity of f, we have

$$f(t)\leq \frac{mt-z}{mb-z}f(b)+m\frac{b-t}{mb-z}f(z). \tag{16}$$

From (15) and (16), one can has

$$\begin{aligned}(17)\qquad &\int_z^b (t-z)^{\frac{\tau}{k}-1} f(t)dt \\ &\leq \frac{(b-z)^{\frac{\tau}{k}-1}}{mb-z}\left[f(b)\int_z^b (mt-z)dt + mf(z)\int_z^b (b-t)dt\right].\end{aligned}$$

By using the (6) of Definition 1.3 and after simplification, we get

$$(18)\qquad 2k\Gamma_k(\tau) I_{b^-}^{\tau,k} f(z) \leq \frac{(b-z)^{\frac{\tau}{k}}}{mb-z}\left[f(b)(mb-(2-m)z)+mf(z)(b-z)\right].$$

From inequalities (14) and (18), we get (11) which is required inequality. □

Corollary 2.1. By taking $\sigma = \tau$ in (11), then we get the following inequality:

$$\begin{aligned}(19)\qquad &2k\Gamma_k(\sigma)\left(I_{a^+}^{\sigma,k} f(z) + I_{b^-}^{\sigma,k} f(z)\right) \\ &\leq \frac{(z-a)^{\frac{\sigma}{k}}}{mz-a}\left[f(a)((2m-1)z-a)+mf(z)(z-a)\right] \\ &+ \frac{(b-z)^{\frac{\sigma}{k}}}{mb-z}\left[f(b)(mb-(2-m)z)+mf(z)(b-z)\right].\end{aligned}$$

If we take $k=1$ in (11), then we get the following inequality for Riemann-Liouville fractional integrals.

Corollary 2.2. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a positive function such that $0\leq a<b$. If f is m-convex, then for $\sigma,\tau\geq 1$, the following inequality holds:

$$\begin{aligned}(20)\qquad &2\left(\Gamma(\sigma) I_{a^+}^{\sigma} f(z) + \Gamma(\tau) I_{b^-}^{\tau} f(z)\right) \\ &\leq \frac{(z-a)^{\sigma}}{mz-a}\left[f(a)((2m-1)z-a)+mf(z)(z-a)\right] \\ &+ \frac{(b-z)^{\tau}}{mb-z}\left[f(b)(mb-(2-m)z)+mf(z)(b-z)\right].\end{aligned}$$

Proof. Proof is on the same lines as the proof of Theorem 2.1. □

Corollary 2.3. By taking $\sigma = \tau$ in (20), then we get the following inequality:

$$
\begin{aligned}
&2\Gamma(\sigma)\left(I_{a^+}^{\sigma} f(z) + I_{b^-}^{\sigma} f(z)\right) \\
&\leq \frac{(z-a)^{\sigma}}{mz-a}\left[f(a)((2m-1)z-a) + mf(z)(z-a)\right] \\
&+ \frac{(b-z)^{\sigma}}{mb-z}\left[f(b)(mb-(2-m)z) + mf(z)(b-z)\right].
\end{aligned} \tag{21}
$$

Remark 2.1. (i) If we take $k = m = 1$ in (11), then [5, Theorem 1] is obtained.

(ii) If we take $k = m = 1$ in (19), then [5, Corollary 1] is obtained.

(iii) If we take $m = 1$ in (20), then [5, Theorem 1] is obtained.

(iv) If we take $m = 1$ in (21), then [5, Corollary 1] is obtained.

Next result is the generalization of some known result.

Theorem 2.2. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function such that $0 \leq a < b$. If $|f'|$ is m-convex, then the following inequality holds:

$$
\begin{aligned}
&\left|\Gamma_k(\sigma+k) I_{a^+}^{\sigma,k} f(z) + \Gamma_k(\tau+k) I_{b^-}^{\tau,k} f(z) - \left((z-a)^{\frac{\sigma}{k}} f(a) + (b-z)^{\frac{\tau}{k}} f(b)\right)\right| \\
&\leq \frac{(z-a)^{\frac{\sigma}{k}+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a) + m|f'(z)|(z-a)\right] \\
&+ \frac{(b-z)^{\frac{\tau}{k}+1}}{2(mb-z)}\left[|f'(b)|(mb-(2-m)z) + m|f'(z)|(b-z)\right]
\end{aligned} \tag{22}
$$

for all $z \in [a,b]$ and $\sigma, \tau, k > 0$.

Proof. For $\sigma, k > 0$ the following inequality holds:

$$
(z-t)^{\frac{\sigma}{k}} \leq (z-a)^{\frac{\sigma}{k}} \tag{23}
$$

for all $t \in [a,z]$ and $z \in [a,b]$.

By using m-convexity of f, we have

$$
|f'(t)| \leq \frac{mz-t}{mz-a}|f'(a)| + m\frac{t-a}{mz-a}|f'(z)| \tag{24}
$$

from (24), we can write

$$
f'(t) \leq \frac{mz-t}{mz-a}|f'(a)| + m\frac{t-a}{mz-a}|f'(z)|. \tag{25}
$$

From (23) and (25), one can has

$$\begin{aligned}(26)\qquad &\int_a^z (z-t)^{\frac{\sigma}{k}} f'(t)dt\\ &\leq \frac{(z-a)^{\frac{\sigma}{k}}}{mz-a}\left[|f'(a)|\int_a^z (mz-t)dt + m|f'(z)|\int_a^z (t-a)dt\right]\\ &= \frac{(z-a)^{\frac{\sigma}{k}+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right],\end{aligned}$$

and

$$\begin{aligned}\int_a^z (z-t)^{\frac{\sigma}{k}} f'(t)dt &= f(t)(z-t)^{\frac{\sigma}{k}}|_a^z + \frac{\sigma}{k}\int_a^z (z-t)^{\frac{\sigma}{k}-1} f(t)dt\\ &= -f(a)(z-a)^{\frac{\sigma}{k}} + \Gamma_k(\sigma+k) I_{a^+}^{\sigma,k} f(z).\end{aligned}$$

Therefore (26) takes the form

$$\begin{aligned}(27)\qquad &\Gamma_k(\sigma+k) I_{a^+}^{\sigma,k} f(z) - f(a)(z-a)^{\frac{\sigma}{k}}\\ &\leq \frac{(z-a)^{\frac{\sigma}{k}+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right].\end{aligned}$$

Also from (24), we can write

$$(28)\qquad f'(t) \geq -\left(\frac{mz-t}{mz-a}|f'(a)| + m\frac{t-a}{mz-a}|f'(z)|\right).$$

Following the same way as we did for (23) and (25), similarly we have

$$\begin{aligned}(29)\qquad &f(a)(z-a)^{\frac{\sigma}{k}} - \Gamma_k(\sigma+k) I_{a^+}^{\sigma,k} f(z)\\ &\leq \frac{(z-a)^{\frac{\sigma}{k}+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right].\end{aligned}$$

From (27) and (29), we get

$$\begin{aligned}(30)\qquad &\left|\Gamma_k(\sigma+k) I_{a^+}^{\sigma,k} f(z) - f(a)(z-a)^{\frac{\sigma}{k}}\right|\\ &\leq \frac{(z-a)^{\frac{\sigma}{k}+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right].\end{aligned}$$

Now for $\tau, k > 0$, the following inequality holds:

$$(31)\qquad (t-z)^{\frac{\tau}{k}} \leq (b-z)^{\frac{\tau}{k}}$$

for all $t \in [z,b]$ and $z \in [a,b]$.

Again by using m-convexity of $|f'|$, we have

$$|f'(t)| \leq \frac{mt-z}{mb-z}|f'(b)| + m\frac{b-t}{mb-z}|f'(z)|. \tag{32}$$

Following the same procedure as we have done for (23) and (24) one can obtain from (31) and (32), the following inequality:

$$\begin{aligned} &\left|\Gamma_k(\tau+k) I_{b^-}^{\tau,k} f(z) - f(b)(b-z)^{\frac{\tau}{k}}\right| \\ &\leq \frac{(b-z)^{\frac{\tau}{k}+1}}{2(mb-z)}\left[|f'(b)|(mb-(2-m)z) + m|f'(z)|(b-z)\right]. \end{aligned} \tag{33}$$

From inequalities (30) and (33) via triangular inequality, we get (22) which is required inequality. □

Corollary 2.4. By taking $\sigma = \tau$ in (22), then we get the following inequality:

$$\begin{aligned} &\left|\Gamma_k(\sigma+k)(I_{a^+}^{\sigma,k} f(z) + I_{b^-}^{\sigma,k} f(z)) - \left((z-a)^{\frac{\sigma}{k}} f(a) + (b-z)^{\frac{\sigma}{k}} f(b)\right)\right| \\ &\leq \frac{(z-a)^{\frac{\sigma}{k}+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a) + m|f'(z)|(z-a)\right] \\ &+ \frac{(b-z)^{\frac{\sigma}{k}+1}}{2(mb-z)}\left[|f'(b)|(mb-(2-m)z) + m|f'(z)|(b-z)\right]. \end{aligned} \tag{34}$$

If we take $k = 1$ in (22), then we get the following inequality for Riemann-Liouville fractional integrals.

Corollary 2.5. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function such that $0 \leq a < b$. If $|f'|$ is m-convex, then for $\sigma, \tau > 0$ the following inequality holds:

$$\begin{aligned} &\left|\Gamma(\sigma+1) I_{a^+}^{\sigma} f(z) + \Gamma(\tau+1) I_{b^-}^{\tau} f(z) - ((z-a)^{\sigma} f(a) + (b-z)^{\tau} f(b))\right| \\ &\leq \frac{(z-a)^{\sigma+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a) + m|f'(z)|(z-a)\right] \\ &+ \frac{(b-z)^{\tau+1}}{2(mb-z)}\left[|f'(b)|(mb-(2-m)z) + m|f'(z)|(b-z)\right]. \end{aligned} \tag{35}$$

Proof. Proof is on the same lines as the proof of Theorem 2.2. □

Corollary 2.6. By taking $\sigma=\tau$ in (35), then we get the following inequality:

$$\begin{aligned}
(36)\qquad &\left|\Gamma(\sigma+1)(I^{\sigma}_{a^+}f(z)+I^{\sigma}_{b^-}f(z))-((z-a)^{\sigma}f(a)+(b-z)^{\sigma}f(b))\right|\\
&\leq \frac{(z-a)^{\sigma+1}}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right]\\
&+\frac{(b-z)^{\sigma+1}}{2(mb-z)}\left[|f'(b)|(mb-(2-m)z)+m|f'(z)|(b-z)\right].
\end{aligned}$$

Remark 2.2. (i) If we take $k=m=1$ in (22), then [5, Theorem 2] is obtained.

(ii) If we take $k=m=1$ in (34), then [5, Corollary 4] is obtained.

(iii) If we take $m=1$ in (35), then [5, Theorem 2] is obtained.

(iv) If we take $m=1$ in (36), then [5, Corollary 4] is obtained.

3. Bounds of generalized Riemann-Liouville k-fractional integrals

In this section, we give the bounds of the generalized Riemann-Liouville k-fractional integrals. To establish these bounds we use m-convexity and monotonicity of utilized functions. These bounds are the generalization of the bounds proved in Section 2. First we give the bounds of the sum of the left-sided $I^{\sigma,k}_{g,a^+}f$ and right-sided $I^{\sigma,k}_{g,b^-}f$ generalized Riemann-Liouville k-fractional integrals.

Theorem 3.1. Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be the functions such that $f\in L[a,b]$ with $0\leq a<b$. Also let f be positive and m-convex, and g be differentiable and strictly increasing function with $g'\in L[a,b]$. Then the following inequality holds:

$$\begin{aligned}
(37)\qquad &k\left(\Gamma_k(\sigma)I^{\sigma,k}_{g,a^+}f(z)+\Gamma_k(\tau)I^{\tau,k}_{g,b^-}f(z)\right)\\
&\leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}-1}}{mz-a}\Bigg[f(a)\left((m-1)zg(z)-(mz-a)g(a)\right)\\
&+m(z-a)f(z)g(z)-(mf(z)-f(a))\int_a^z g(t)dt\Bigg]\\
&+\frac{((g(b)-g(z))^{\frac{\tau}{k}-1}}{mb-z}\Bigg[f(b)\left((mb-z)g(b)-(m-1)zg(z)\right)\\
&-m(b-z)f(z)g(z)-m(f(b)-f(z))\int_z^b g(t)dt\Bigg]
\end{aligned}$$

for all $z\in[a,b]$, $\sigma,\tau\geq 1$ and $k>0$.

Proof. As we know that the function g is differentiable and strictly increasing, therefore $(g(z)-g(t))^{\frac{\sigma}{k}-1} \leq (g(z)-g(a))^{\frac{\sigma}{k}-1}$ holds true. Then for $g'(z)>0$, $\sigma \geq 1$ and $k>0$, the following inequality holds:

$$g'(t)(g(z)-g(t))^{\frac{\sigma}{k}-1} \leq g'(t)(g(z)-g(a))^{\frac{\sigma}{k}-1} \tag{38}$$

for all $t \in [a,z]$ and $z \in [a,b]$.

From (13) and (38), one can has

$$\begin{aligned}&\int_a^z (g(z)-g(t))^{\frac{\sigma}{k}-1} f(t)g'(t)dt \\ &\leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}-1}}{mz-a}\left[f(a)\int_a^z (mz-t)g'(t)dt + mf(z)\int_a^z (t-a)g'(t)dt\right].\end{aligned}$$

By using the (9) of Definition 1.5 and after simplification, we get

$$\begin{aligned}&k\Gamma_k(\sigma) I_{g,a^+}^{\sigma,k} f(z) \\ &\leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}-1}}{mz-a}\Big[f(a)\left((m-1)zg(z)-(mz-a)g(a)\right) \\ &+(z-a)f(z)g(z)-(mf(z)-f(a))\int_a^z g(t)dt\Big].\end{aligned} \tag{39}$$

Now for $\tau \geq 1$ and $k>0$, the following inequality holds:

$$g'(t)(g(t)-g(z))^{\frac{\tau}{k}-1} \leq g'(t)(g(b)-g(z))^{\frac{\tau}{k}-1} \tag{40}$$

for all $t \in [z,b]$ and $z \in [a,b]$.

From (16) and (40), one can has

$$\begin{aligned}&\int_z^b (g(t)-g(z))^{\frac{\tau}{k}-1} f(t)g'(t)dt \\ &\leq \frac{(g(b)-g(z))^{\frac{\tau}{k}-1}}{mb-z}\left[f(b)\int_z^b (mt-z)g'(t)dt + mf(z)\int_z^b (b-t)g'(t)dt\right].\end{aligned}$$

By using the (10) of Definition 1.5 and after simplification, we get

$$\begin{aligned}&k\Gamma_k(\tau) I_{g,b^-}^{\tau,k} f(z) \\ &\leq \frac{((g(b)-g(z))^{\frac{\tau}{k}-1}}{mb-z}\Big[f(b)\left((mb-z)g(b)-(m-1)zg(z)\right) \\ &-m(b-z)f(z)g(z)-m(f(b)-f(z))\int_z^b g(t)dt\Big].\end{aligned} \tag{41}$$

From inequalities (39) and (41), we get (37) which is required inequality. □

Corollary 3.1. By taking $\sigma = \tau$ in (37), then we get the following inequality:

$$
\begin{aligned}
&k\Gamma_k(\sigma)\left(I^{\sigma,k}_{g,a^+}f(z)+I^{\sigma,k}_{g,b^-}f(z)\right)\\
&\leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}-1}}{mz-a}\Bigg[f(a)\left((m-1)zg(z)-(mz-a)g(a)\right)\\
&+(z-a)f(z)g(z)-(mf(z)-f(a))\int_a^z g(t)dt\Bigg]\\
&+\frac{((g(b)-g(z))^{\frac{\sigma}{k}-1}}{mb-z}\Bigg[f(b)\left((mb-z)g(b)-(m-1)zg(z)\right)\\
&-m(b-z)f(z)g(z)-m(f(b)-f(z))\int_z^b g(t)dt\Bigg].
\end{aligned}
\tag{42}
$$

If we take $k = 1$ in (37), then we get the following inequality for generalized Riemann-Liouville fractional integrals.

Corollary 3.2. Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be functions such that $f \in L[a,b]$ with $0 \leq a < b$. Also let f be positive and m-convex, and g be differentiable and strictly increasing function with $g' \in L[a,b]$. Then for $\sigma, \tau \geq 1$, the following inequality holds:

$$
\begin{aligned}
&\Gamma(\sigma)I^{\sigma}_{g,a^+}f(z)+\Gamma(\tau)I^{\tau}_{g,b^-}f(z)\\
&\leq \frac{(g(z)-g(a))^{\sigma-1}}{mz-a}\Bigg[f(a)\left((m-1)zg(z)-(mz-a)g(a)\right)\\
&+(z-a)f(z)g(z)-(mf(z)-f(a))\int_a^z g(t)dt\Bigg]\\
&+\frac{((g(b)-g(z))^{\tau-1}}{mb-z}\Bigg[f(b)\left((mb-z)g(b)-(m-1)zg(z)\right)\\
&-m(b-z)f(z)g(z)-m(f(b)-f(z))\int_z^b g(t)dt\Bigg].
\end{aligned}
\tag{43}
$$

Proof. Proof is on the same lines as the proof of Theorem 3.1. □

Corollary 3.3. By taking $\sigma=\tau$ in (43), then we get the following inequality:

$$\begin{aligned}
&\Gamma(\sigma)\left(I^{\sigma}_{g,a^+}f(z)+I^{\sigma}_{g,b^-}f(z)\right)\\
&\leq \frac{(g(z)-g(a))^{\sigma-1}}{mz-a}\bigg[f(a)\left((m-1)zg(z)-(mz-a)g(a)\right)\\
&+(z-a)f(z)g(z)-(mf(z)-f(a))\int_a^z g(t)dt\bigg]\\
&+\frac{((g(b)-g(z))^{\sigma-1}}{mb-z}\bigg[f(b)\left((mb-z)g(b)-(m-1)zg(z)\right)\\
&-m(b-z)f(z)g(z)-m(f(b)-f(z))\int_z^b g(t)dt\bigg].
\end{aligned}\tag{44}$$

Remark 3.1. (i) If we take $g(z)=z$ in (37), then Theorem 2.1 is obtained.

(ii) If we take $g(z)=z$ and $k=1$ in (37), then Corollary 2.2 is obtained.

(iii) If we take $g(z)=z$ and $k=m=1$ in (37), then [5, Theorem 1] is obtained.

(iv) If we take $k=m=1$ in (37), then [9, Theorem 1] is obtained.

Next result is the generalization of some known result.

Theorem 3.2. Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be functions with $0\leq a<b$. Also let f be differentiable and $|f'|$ is m-convex, and g be also differentiable and strictly increasing with $g'\in L[a,b]$. Then the following inequality holds:

$$\begin{aligned}
&\bigg|\Gamma_k(\sigma+k)I^{\sigma,k}_{g,a^+}f(z)+\Gamma_k(\tau+k)I^{\tau,k}_{g,b^-}f(z)\\
&-\left((g(z)-g(a))^{\frac{\sigma}{k}}f(a)+(g(b)-g(z))^{\frac{\tau}{k}}f(b)\right)\bigg|\\
&\leq\frac{(g(z)-g(a))^{\frac{\sigma}{k}}(z-a)}{2(mz-a)}\Big[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\Big]\\
&+\frac{(g(b)-g(z))^{\frac{\tau}{k}}(b-z)}{2(mb-z)}\Big[|f'(b)|(mb-(2-m)z)+m|f'(z)|(b-z)\Big]
\end{aligned}\tag{45}$$

for all $z\in[a,b]$, and $\sigma,\tau,k>0$.

Proof. As we know that the function g is differentiable and strictly increasing, therefore for $\sigma,k>0$, the following inequality holds:

$$(g(z)-g(t))^{\frac{\sigma}{k}}\leq(g(z)-g(a))^{\frac{\sigma}{k}}\tag{46}$$

for all $t \in [a,z]$ and $z \in [a,b]$.

From (25) and (46), one can has

$$\begin{aligned}(47)\qquad & \int_a^z (g(z)-g(t))^{\frac{\sigma}{k}} f'(t)dt \\ & \leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}}}{mz-a}\left[|f'(a)|\int_a^z (mz-t)dt + m|f'(z)|\int_a^z (t-a)dt\right] \\ & = \frac{(g(z)-g(a))^{\frac{\sigma}{k}}(z-a)}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right],\end{aligned}$$

and

$$\begin{aligned}\int_a^z (g(z)-g(t))^{\frac{\sigma}{k}} f'(t)dt &= f(t)(g(z)-g(t))^{\frac{\sigma}{k}}|_a^z + \frac{\sigma}{k}\int_a^z (g(z)-g(t))^{\frac{\sigma}{k}-1} f(t)g'(t)dt \\ &= -f(a)(g(z)-g(a))^{\frac{\sigma}{k}} + \Gamma_k(\sigma+k) I_{g,a^+}^{\sigma,k} f(z).\end{aligned}$$

Therefore (47) takes the form

$$\begin{aligned}(48)\qquad & \Gamma_k(\sigma+k) I_{g,a^+}^{\sigma,k} f(x) - f(a)(g(z)-g(a))^{\frac{\sigma}{k}} \\ & \leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}}(z-a)}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right].\end{aligned}$$

Similarly from (28) and (46), one can get

$$\begin{aligned}(49)\qquad & f(a)(g(z)-g(a))^{\frac{\sigma}{k}} - \Gamma_k(\sigma+k) I_{g,a^+}^{\sigma,k} f(x) \\ & \leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}}(z-a)}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right].\end{aligned}$$

From (48) and (49), we get

$$\begin{aligned}(50)\qquad & \left|\Gamma_k(\sigma+k) I_{g,a^+}^{\sigma,k} f(z) - f(a)(g(z)-g(a))^{\frac{\sigma}{k}}\right| \\ & \leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}}(z-a)}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right].\end{aligned}$$

Now for $\tau, k > 0$, the following inequality holds:

$$(51)\qquad (g(t)-g(z))^{\frac{\tau}{k}} \leq (g(b)-g(z))^{\frac{\tau}{k}}$$

for all $t \in [z,b]$ and $z \in [a,b]$.

Following the same procedure as we have done for (25), (28) and (46) one can get from (32) and (51), the following inequality:

$$\begin{aligned}(52)\qquad &\left|\Gamma_k(\tau+k)I^{\tau,k}_{g,b^-}f(z)-f(b)(g(b)-g(z))^{\frac{\tau}{k}}\right|\\ &\leq \frac{(g(b)-g(z))^{\frac{\tau}{k}}(b-z)}{2(mb-z)}\Big[|f'(b)|(mb-(2-m)z)+m|f'(z)|(b-z)\Big].\end{aligned}$$

From inequalities (50) and (52) via triangular inequality, we get (45) which is required inequality. □

Corollary 3.4. By taking $\sigma=\tau$ in (45), then we get the following inequality:

$$\begin{aligned}&\left|\Gamma_k(\sigma+k)(I^{\sigma,k}_{g,a^+}f(z)+I^{\sigma,k}_{g,b^-}f(z))-\Big((g(z)-g(a))^{\frac{\sigma}{k}}f(a)+(g(b)-g(z))^{\frac{\sigma}{k}}f(b)\Big)\right|\\ &\leq \frac{(g(z)-g(a))^{\frac{\sigma}{k}}(z-a)}{2(mz-a)}\Big[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\Big]\\ &+\frac{(g(b)-g(z))^{\frac{\sigma}{k}}(b-z)}{2(mb-z)}\Big[|f'(b)|(mb-(2-m)z)+m|f'(z)|(b-z)\Big].\end{aligned}$$

If we take $k=1$ in (45), then we get the following inequality for generalized Riemann-Liouville fractional integrals.

Corollary 3.5. Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be functions with $0\leq a<b$. Also let f be differentiable and $|f'|$ is m-convex, and g be also differentiable and strictly increasing with $g'\in L[a,b]$. Then for $\sigma,\tau>0$, the following inequality holds:

$$\begin{aligned}(53)\qquad &\Big|\Gamma(\sigma+1)I^{\sigma}_{g,a^+}f(z)+\Gamma(\tau+1)I^{\tau}_{g,b^-}f(z)\\ &-((g(z)-g(a))^{\sigma}f(a)+(g(b)-g(z))^{\tau}f(b))|\\ &\leq \frac{(g(z)-g(a))^{\sigma}(z-a)}{2(mz-a)}\Big[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\Big]\\ &+\frac{(g(b)-g(z))^{\tau}(b-z)}{2(mb-z)}\Big[|f'(b)|(mb-(2-m)z)+m|f'(z)|(b-z)\Big].\end{aligned}$$

Proof. Proof is on the same lines as the proof of Theorem 3.2. □

Corollary 3.6. By taking $\sigma = \tau$ in (53), then we get the following inequality:

$$\begin{aligned}
(54)\quad & \left|\Gamma(\sigma+1)\left(I^{\sigma}_{g,a^+}f(z)+I^{\sigma}_{g,b^-}f(z)\right)-((g(z)-g(a))^{\sigma}f(a)+(g(b)-g(z))^{\sigma}f(b))\right| \\
& \leq \frac{(g(z)-g(a))^{\sigma}(z-a)}{2(mz-a)}\left[|f'(a)|((2m-1)z-a)+m|f'(z)|(z-a)\right] \\
& +\frac{(g(b)-g(z))^{\sigma}(b-z)}{2(mb-z)}\left[|f'(b)|(mb-(2-m)z)+m|f'(z)|(b-z)\right].
\end{aligned}$$

Remark 3.2. (i) If we take $g(z) = z$ in (45), then Theorem 2.2 is obtained.

(ii) If we take $g(z) = z$ and $k = 1$ in (45), then Corollary 2.5 is obtained.

(iii) If we take $g(z) = z$ and $k = m = 1$ in (45), then [5, Theorem 2] is obtained.

(iv) If we take $k = m = 1$ in (45), then [9, Theorem 2] is obtained.

4. APPLICATIONS

In this section, we give the some applications of the results proved in Section 2 and Section 3. First we apply results of Section 2 to obtain the following relations.

Theorem 4.1. Under the assumptions of Theorem 2.1, we have

$$\begin{aligned}
(55)\quad & 2k\left(\Gamma_k(\sigma)I^{\sigma,k}_{a^+}f(b)+\Gamma_k(\tau)I^{\tau,k}_{b^-}f(a)\right) \\
& \leq \frac{(b-a)^{\frac{\sigma}{k}}}{mb-a}\left[f(a)((2m-1)b-a)+mf(b)(b-a)\right] \\
& +\frac{(b-a)^{\frac{\tau}{k}}}{mb-a}\left[f(b)(mb-(2-m)a)+mf(a)(b-a)\right].
\end{aligned}$$

Proof. If we take $z = a$ and $z = b$ in (11), then adding resulting inequalities, we get (55) which is required inequality. □

Corollary 4.1. By taking $\sigma = \tau$ in (55), then we get following inequality:

$$\begin{aligned}
(56)\quad & 2k\Gamma_k(\sigma)\left(I^{\sigma,k}_{a^+}f(b)+I^{\sigma,k}_{b^-}f(a)\right) \\
& \leq \frac{(b-a)^{\frac{\sigma}{k}}}{mb-a}\left[f(a)((2m-1)b-a)+f(b)(mb-(2-m)a)+m(f(a)+f(b))(b-a)\right].
\end{aligned}$$

Corollary 4.2. [5, 9] If we take $\sigma = k = m = 1$ in (56), then we get the following inequality:

$$(57)\quad \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

Theorem 4.2. Under the assumptions of Theorem 2.2, we have

$$\begin{aligned}
(58)\qquad & \left|\Gamma_k(\sigma+k)I_{a^+}^{\sigma,k}f\left(\frac{a+b}{2}\right)+\Gamma_k(\tau+k)I_{b^-}^{\tau,k}f\left(\frac{a+b}{2}\right)\right.\\
& \left.-\left(\left(\frac{b-a}{2}\right)^{\frac{\sigma}{k}}f(a)+\left(\frac{b-a}{2}\right)^{\frac{\tau}{k}}f(b)\right)\right|\\
& \leq \frac{(b-a)^{\frac{\sigma}{k}+1}}{2^{\frac{\sigma}{k}+2}(m(a+b)-2a)}\left[|f'(a)|((2m-1)(a+b)-2a)+m\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)\right]\\
& +\frac{(b-a)^{\frac{\tau}{k}+1}}{2^{\frac{\tau}{k}+2}(2mb-(a+b))}\left[|f'(b)|(2mb-(2-m)(a+b))+m\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)\right].
\end{aligned}$$

Proof. If we take $z=\frac{a+b}{2}$ in (22), then we get (58) which is required inequality. □

Corollary 4.3. By taking $\sigma=\tau$ in (58), then we get the following inequality:

$$\begin{aligned}
(59)\qquad & \left|\Gamma_k(\sigma+k)\left(I_{a^+}^{\sigma,k}f\left(\frac{a+b}{2}\right)+I_{b^-}^{\sigma,k}f\left(\frac{a+b}{2}\right)\right)-\left(\frac{b-a}{2}\right)^{\frac{\sigma}{k}}(f(a)+f(b))\right|\\
& \leq \frac{(b-a)^{\frac{\sigma}{k}+1}}{2^{\frac{\sigma}{k}+2}}\left[\frac{1}{(m(a+b)-2a)}\Big(|f'(a)|((2m-1)(a+b)-2a)\Big)\right.\\
& +\frac{1}{(2mb-(a+b))}\Big(|f'(b)|(2mb-(2-m)(a+b))\Big)\\
& \left.+m\left|f'\left(\frac{a+b}{2}\right)\right|\left(\frac{1}{(m(a+b)-2a)}+\frac{1}{(2mb-(a+b))}\right)(b-a)\right].
\end{aligned}$$

Corollary 4.4. [5, 9] If we take $\sigma=k=m=1$ in (59), then we get the following inequality:

$$(60)\qquad \left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{8}\left[|f'(a)|+|f'(b)|+2\left|f'\left(\frac{a+b}{2}\right)\right|\right].$$

Remark 4.1. If $f'\left(\frac{a+b}{2}\right)=0$, then from (60) we get [3, Theorem 2.2]. If $f'(z)\leq 0$, then from (60) we get refinement of [3, Theorem 2.2].

Next we apply results of Section 3 to obtain the following relations.

Theorem 4.3. Under the assumptions of Theorem 3.1, we have

$$
\begin{aligned}
&k\left(\Gamma_k(\sigma)I^{\sigma,k}_{g,a^+}f(b)+\Gamma_k(\tau)I^{\tau,k}_{g,b^-}f(a)\right)\\
&\leq \frac{(g(b)-g(a))^{\frac{\sigma}{k}-1}}{mb-a}\Bigg[f(a)\left((m-1)bg(b)-(mb-a)g(a)\right)\\
&+m(b-a)f(b)g(b)-(mf(b)-f(a))\int_a^b g(t)dt\Bigg]\\
&+\frac{((g(b)-g(a))^{\frac{\tau}{k}-1}}{mb-a}\Bigg[f(b)\left((mb-a)g(b)-(m-1)ag(a)\right)\\
&-m(b-a)f(a)g(a)-m(f(b)-f(a))\int_a^b g(t)dt\Bigg].
\end{aligned}
\tag{61}
$$

Proof. If we take $z=a$ and $z=b$ in (37), then adding resulting inequalities, we get (61) which is required inequality. □

Corollary 4.5. By taking $\sigma=\tau$ in (61), then we get the following inequality:

$$
\begin{aligned}
&k\Gamma_k(\sigma)\left(I^{\sigma,k}_{g,a^+}f(b)+I^{\sigma,k}_{g,b^-}f(a)\right)\\
&\leq \frac{(g(b)-g(a))^{\frac{\sigma}{k}-1}}{mb-a}\Bigg[f(a)\left((m-1)bg(b)-(mb-a)g(a)\right)\\
&+f(b)\left((mb-a)g(b)-(m-1)ag(a)\right)+m(b-a)(f(b)g(b)-f(a)g(a))\\
&-(2mf(b)-(m+1)f(a))\int_a^b g(t)dt\Bigg].
\end{aligned}
\tag{62}
$$

Remark 4.2. (i) If we take $g(z)=z$ in (61), then Theorem 4.1 is obtained.

(ii) If we take $k=m=1$ in (61), then [9, Theorem 4] is obtained.

(iii) If we take $\sigma=k=m=1$ and $g(z)=z$ in (62), then Corollary 4.2 is obtained.

Theorem 4.4. Under the assumptions of Theorem 3.2, we have

$$
\begin{aligned}
&\Bigg|\Gamma_k(\sigma+k)I^{\sigma,k}_{g,a^+}f\left(\frac{a+b}{2}\right)+\Gamma_k(\tau+k)I^{\tau,k}_{g,b^-}f\left(\frac{a+b}{2}\right)\\
&-\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\sigma}{k}}f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\tau}{k}}f(b)\right)\Bigg|
\end{aligned}
$$

(63)

$$\leq \frac{\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\sigma}{k}}(b-a)}{4(m(a+b)-2a)}\left[|f'(a)|((2m-1)(a+b)-2a)+m\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)\right]$$
$$+\frac{\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\tau}{k}}(b-a)}{4(2mb-(a+b))}\left[|f'(b)|(2mb-(2-m)(a+b))+m\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)\right].$$

Proof. If we take $z=\frac{a+b}{2}$ in (45), then we get (63) which is required inequality. □

Corollary 4.6. By taking $\sigma=\tau$ in (63), then we get the following inequality:

$$\begin{aligned}(64)\quad &\left|\Gamma_k(\sigma+k)\left(I^{\sigma,k}_{g,a^+}f\left(\frac{a+b}{2}\right)+I^{\sigma,k}_{g,b^-}f\left(\frac{a+b}{2}\right)\right)\right.\\ &\left.-\left(\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\sigma}{k}}f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\sigma}{k}}f(b)\right)\right|\\ &\leq \frac{\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\frac{\sigma}{k}}(b-a)}{4(m(a+b)-2a)}\left[|f'(a)|((2m-1)(a+b)-2a)+m\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)\right]\\ &+\frac{\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\frac{\sigma}{k}}(b-a)}{4(2mb-(a+b))}\left[|f'(b)|(2mb-(2-m)(a+b))+m\left|f'\left(\frac{a+b}{2}\right)\right|(b-a)\right].\end{aligned}$$

Remark 4.3. (i) If we take $g(z)=z$ in (63), then Theorem 4.2 is obtained.

(ii) If we take $k=m=1$ in (63), then [9, Theorem 5] is obtained.

(iii) If we take $\sigma=k=m=1$ and $g(z)=z$ in (64), then Corollary 4.4 is obtained.

ACKNOWLEDGEMENT

This research work is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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