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SENSITIVITY ANALYSIS FOR A PARAMETRIC GENERALIZED VARIATIONAL-LIKE INEQUALITY PROBLEM: A P-η-PROXIMAL MAPPING APPROACH

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Abstract. In this paper, using P-η-proximal mapping, we study the existence and sensitivity analysis of solution of a parametric generalized variational-like inequality problem in uniformly smooth Banach space. The approach used in this paper may be treated as an extension and unification of approaches for studying sensitivity analysis of solution for various important classes of variational inequalities (inclusions) given by many authors, see for example [2-7,9-11,13-16,18,19].

Keywords: parametric generalized variational-like inequality; P-η-proximal mapping; existence and sensitivity analysis of solution.

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1. INTRODUCTION

In recent years, much attention has been given to develop general techniques for the sensitivity analysis of solution of various classes of variational inequalities (inclusions). From the

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mathematical and engineering point of view, sensitivity properties of various classes of variational inequalities can provide new insight concerning the problem being studied and can stimulate ideas for solving problems. The sensitivity analysis of solution for variational inequalities have been studied extensively by many authors using quite different techniques. By using the projection technique, Dafermos [3], Ding and Luo [6], Mukherjee and Verma [14], Parida *et al.* [17], Park and Jeong [18] and Yen [19] studied the sensitivity analysis of solution of some classes of variational inequalities with single-valued mappings. By using proximal (resolvent) mapping technique, Adly [1], Agarwal *et al.* [2], Ding [4] and Noor [15] studied the sensitivity analysis of solution of some classes of variational inclusions with single-valued mappings.

Recently, by using projection and proximal mapping techniques, Ding [5], Kazmi and Bhat [9], Kazmi and Khan [10,11], Lim [12], Liu *et al.* [13] and Noor [16] studied the behavior and sensitivity analysis of solution set for some important classes of parametric variational inequalities (inclusions) with single and set-valued mappings. It is worth mentioning that most of the results in this direction have been obtained in the setting of Hilber space.

Inspired by recent research works in this area, in this paper, we consider a parametric generalized variational-like inequality problem (in short, PGVLIP) in uniformly smooth Banach space. Further, using P-η-proximal mapping, we study the existence and sensitivity analysis of solution of PGVLIP. Our results extend, improve, and unify the corresponding results given by many authors, see for example [2-7,9-11,13-16,18,19].

2. Preliminaries

We assume that E is a real Banach space equipped with norm $\|\cdot\|$. Let $\langle\cdot,\cdot\rangle$ denote the dual pair between E and its dual space E^* and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(u) = \{f \in E^* : \langle u, f\rangle = \|u\|^2, \ \|u\| = \|f\|_{E^*}\}, \ u \in E. \tag{2.1}$$

We note that if E is smooth, then J is single-valued and if $E \equiv H$, a Hilbert space, then J is the identity map on H. In sequel, we shall denote a selection of normalized duality mapping J by j.

First, we recall the following concepts and results.

Definition 2.1[9]. Let $P : E \to E^*,\ g : E \to E,$ and $\eta : E \times E \to E$ be single-valued mappings. Then

(i) P is said to be α-strongly η-monotone, if there exists a constant $\alpha > 0$ such that

$$\langle P(u) - P(v), \eta(u,v)\rangle \geq \alpha\|u-v\|^2,\ \forall u,v \in E;$$

(ii) g is said to be β-strongly accretive, if there exists a constant $\beta > 0$ and for any $u,v \in E$, $j(u-v) \in J(u-v)$ such that

$$\langle g(u) - g(v), j(u-v)\rangle \geq \beta\|u-v\|^2;$$

(iii) η is said to be τ-Lipschitz continuous, if there exists a constant $\tau > 0$ such that

$$\|\eta(u,v)\| \leq \tau\|u-v\|,\ \forall u,v \in E.$$

Definition 2.2[4]. Let $\eta : E \times E \to E$ be a single-valued mapping. A proper functional $\phi : E \to \mathbb{R} \cup \{+\infty\}$ is said to be η-subdifferentiable at a point $u \in E$, if there exists a point $f^* \in E^*$ such that

$$\phi(v) - \phi(u) \geq \langle f^*, \eta(v,u)\rangle,\ \forall v \in E,$$

where f^* is called η-subgradient of ϕ at u. The set of all η-subgradients of ϕ at u is denoted by $\partial_\eta \phi(u)$. The mapping $\partial_\eta \phi : E \to 2^{E^*}$ defined by

$$\partial_\eta \phi(u) = \{f^* \in E^* : \phi(v) - \phi(u) \geq \langle f^*, \eta(v,u)\rangle,\ \forall v \in E\} \tag{2.2}$$

is said to be η-subdifferential of ϕ at u.

Definition 2.3[8]. A functional $\phi : E \times E \to \mathbb{R} \cup \{+\infty\}$ is said to be 0-diagonally quasi-concave (0-DQCV) in u, if for any finite set $\{u_1, \ldots, u_n\} \subset E$ and for any $v = \sum_{i=1}^n \lambda_i u_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1,\ \min_{1\leq i\leq n} \phi(u_i, v) \leq 0$ holds.

Definition 2.4[9]. Let $\eta : E \times E \to E$ be a single-valued mapping. Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable (may not be convex) and proper functional and let

$P : E \to E^*$ be a nonlinear mapping. If for any given point $u^* \in E^*$ and $\rho > 0$, there exists a unique point $u \in E$ satisfying

$$\langle P(u) - u^*, \eta(v,u)\rangle + \rho\phi(v) - \rho\phi(u) \geq 0,\ \forall v \in E,$$

then the mapping $u^* \mapsto u$, denoted by $P_\rho^{\partial_\eta \phi}(u^*)$, is called P-η-proximal mapping of ϕ. Clearly, $u^* - P(u) \in \rho\partial_\eta\phi(u)$ and then it follow that

$$P_\rho^{\partial_\eta \phi}(u^*) = (P + \rho\partial_\eta\phi)^{-1}(u^*). \tag{2.3}$$

Remark 2.1[9].

(i) If $\eta(v,u) = v - u$, for all $u, v \in E$ and ϕ is a lower semicontinuous and proper functional on E, then the P-η-proximal mapping of ϕ reduces to the P-η-proximal mapping of ϕ discussed by Ding and Xia [7].

(ii) If $E = H$, a Hilbert space, $\eta(v,u) = v - u$, for all $u, v \in H$ and ϕ is a convex, lower semicontinuous and proper functional on E, and P is the identity mapping on H, then the P-η-proximal mapping of ϕ reduces to the usual proximal (resolvent) mapping of ϕ on Hilbert space.

Lemma 2.1[9]. Let E be a real reflexive Banach space; let $\eta : E \times E \to E$ be a continuous mapping such that $\eta(v,v') + \eta(v',v)$, for all $v, v' \in E$; let $P : E \to E^*$ be α-strongly η-monotone continuous mapping; let, for any given $u^* \in E^*$, the function $h(v,u) = \langle u^* - P(u), \eta(v,u)\rangle$ be 0-DQCV in v and let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional on E. Then for any given constant $\rho > 0$ and $u^* \in E^*$, there exists a unique $u \in E$ such that

$$\langle P(u) - u^*, \eta(v,u)\rangle \geq \rho\phi(u) - \rho\phi(v),\ \forall v \in E, \tag{2.4}$$

that is, $u = P_\rho^{\partial_\eta \phi}(u^*)$.

Remark 2.2[9]. Lemma 2.1 shows that for any strongly monotone continuous mapping $P : E \to E^*$ and $\rho > 0$, the P-η-proximal mapping $P_\rho^{\partial_\eta \phi} : E^* \to E$ of a lower semicontinuous, η-subdifferentiable and proper functional ϕ is well defined and for each $u^* \in E^*$, $u = P_\rho^{\partial_\eta \phi}(u^*)$ is the unique solution of the problem (2.4).

Lemma 2.2[9]. Let E be a real reflexive Banach space and let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(v,v^{'}) + \eta(v^{'},v)$, for all $v,v^{'} \in E$; let $P : E \to E^*$ be α-strongly η-monotone continuous mapping; let, for any given $u^* \in E^*$, the function $h(v,u) = \langle u^* - P(u), \eta(v,u)\rangle$ be 0-DQCV in v; let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional on E and let $\rho > 0$ be any given constant. Then P-η-proximal mapping $P_{\rho}^{\partial_\eta \phi}$ of ϕ is $\frac{\tau}{\delta}$-Lipschitz continuous.

Throughout the rest of the paper unless otherwise stated, let E be a real uniformly smooth Banach space with $\rho_E(q) \leq cq^2$ for some $c > 0$, where the modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$, defined below in the Lemma 2.3.

Lemma 2.3[4,11]. Let E be a real uniformly smooth Banach space and let $J : E \to E^*$ be the normalized duality mapping. Then, for all $u,v \in E$, we have

(a) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, J(u+v)\rangle$;

(b) $\langle u-v, J(u)-J(v)\rangle \leq 2d^2\rho_E(4\|u-v\|/d)$, where $d = \sqrt{(\|u\|^2 + \|v\|^2)/2}$,
$\rho_E(q) = \sup\left\{\frac{(\|u\|+\|v\|)}{2} - 1 : \ \|u\| = 1, \ \|v\| = q\right\}$.

3. Formulation of Problem

Let $T,A,S : E \to E^*$, $g : E \to E$, $\eta : E \times E \to E$, $N : E^* \times E^* \times E^* \to E^*$ be nonlinear mappings. Assume that $\phi : E \times E \to R \cup \{+\infty\}$ is a lower semicontinuous, η-subdifferentiable (may not be convex) and proper functional such that $g(u) \in \partial_\eta \phi(u,z)$, for all $u,z \in E$, then we consider the following generalized variational-like inequality problem (in short, GVLIP): find $u \in E$ such that

$$\langle N(T(u),A(u),S(u)), \eta(v,g(u))\rangle + \phi(v,u) - \phi(g(u),u) \geq 0, \ \forall v \in E. \tag{3.1}$$

For a suitable choices of mappings $T,A,S,N,P,g,P \circ g,\phi,\eta$ and the space E, it is easy to see that GVLIP (3.1) includes a number of known classes of variational inequalities studied by many authors as special cases, see for example [3,6,11,12,14,17-19].

Next, we consider the parametric problem corresponding to GVLIP (3.1).

Let Ω be a nonempty open subset of E in which the parameter λ takes the values. Let $T, A, S : E \times \Omega \to E^*$, $g : E \times \Omega \to E$, $\eta : E \times E \to E$, $N : E^* \times E^* \times E^* \times \Omega \to E^*$ be single-valued mappings. Assume that $\phi : E \times E \times \Omega \to R \cup \{+\infty\}$ is a lower semicontinuous, η-subdifferentiable (may not be convex) and proper functional such that $g(u,\lambda) \in \partial_\eta \phi(u,v,\lambda)$, for all $u, v \in E$, $\lambda \in \Omega$. We consider the following parametric generalized variational-like inequality problem (in short, PGVLIP): find $u \in E$ such that

$$\langle N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda),\eta(v,g(u,\lambda))\rangle + \phi(v,u,\lambda) - \phi(g(u,\lambda),u,\lambda) \geq 0,\ \forall v \in E. \tag{3.2}$$

The aim of this paper is to study the existence and sensitivity analysis of the solution for PGVLIP (3.2), and the conditions on these mappings $T, A, S, N, P, g, P \circ g, \phi, \eta$, under which the solution of PGVLIP (3.2) is nonempty and Lipschitz continuous with respect to the parameter $\lambda \in \Omega$.

4. Existence and Sensitivity Analysis of Solution

First, we prove the following technical lemma.

Lemma 4.1. $u \in E$ is the solution of PGVLIP (3.2) if and only if it satisfies the relation

$$g(u,\lambda) = P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}[P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda)], \tag{4.1}$$

where $P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)} = (P + \rho \partial_\eta \phi(\cdot,u,\lambda))^{-1}$ is the P-η-proximal mapping of ϕ for each fixed $u \in E$, $\lambda \in \Omega$, $P : E \to E^*$, $P \circ g(\cdot,\lambda)$ denotes P composition $g(\cdot,\lambda)$, and $\rho > 0$ is a constant.

Proof. Assume that $u \in E$ satisfies (4.1), that is,

$$g(u,\lambda) = P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}[P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda)]. \tag{4.2}$$

Since $P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)} = (P + \rho \partial_\eta \phi(\cdot,u,\lambda))^{-1}$, the above relation holds if and only if

$$P \circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda) \in P \circ g(u,\lambda) + \rho \partial_\eta \phi(g(u,\lambda),u,\lambda). \tag{4.3}$$

By the definition of η-subdifferential of $\phi(g(u,\lambda),u,\lambda)$, the above inclusion holds if and only if

$$\phi(v,u,\lambda) - \phi(g(u,\lambda),u,\lambda) \geq \langle N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda),\eta(v,g(u,\lambda))\rangle,\ \forall v \in E, \tag{4.4}$$

that is, $u \in E$ is the solution of PGVLIP (3.2). This completes the proof.

Now, assume that for some $\bar{\lambda} \in \Omega$, PGVLIP (3.2) has a solution $\bar{u}$ and K is a closed sphere in E centered at $\bar{u}$. We are interested in investigating those conditions under which, for each λ in a neighborhood of $\bar{\lambda}$, PGVLIP (3.2) has a unique solution $u(\lambda)$ near $\bar{u}$ and the solution $u(\lambda)$ is Lipschitz continuous.

Next, we define the following concepts.

Definition 4.1. A mapping $g : K \times \Omega \to E$ is said to be

(i) locally β-strongly accretive, if there exists a constant $\beta > 0$ such that

$$\langle g(u,\lambda) - g(v,\lambda), J(u-v)\rangle \geq \beta\|u-v\|^2,\ \forall u,v \in K,\ \lambda \in \Omega;$$

(ii) locally (L_g, l_g)-mixed Lipschitz continuous, if there exist constants $L_g, l_g > 0$ such that

$$\|g(u,\lambda) - g(v,\tilde{\lambda})\| \leq L_g\|u-v\| + l_g\|\lambda - \tilde{\lambda}\|,\ \forall u,v \in K,\ \lambda,\tilde{\lambda} \in \Omega.$$

Definition 4.2. Let $P : E \to E^*,\ g : K \times \Omega \to E,\ T,A,S : K \times \Omega \to E^*,\ N : E^* \times E^* \times E^* \times \Omega \to E^*$. Then N is said to be

(i) locally α-strongly $P \circ g$-accretive with respect to T, A and S, if there exists a constant $\alpha > 0$ such that

$$\langle N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda) - N(T(v,\lambda),A(v,\lambda),S(v,\lambda),\lambda), J^*(P\circ g(u,\lambda) - P\circ g(v,\lambda))\rangle$$

$$\geq \alpha\|u-v\|^2,\ \forall u,v \in K,\ \lambda \in \Omega,$$

where $J^* : E^* \to E$ is a normalized duality mapping;

(ii) locally $(L_{(N,1)}, L_{(N,2)}, L_{(N,3)}, l_N)$-mixed Lipschitz continuous, if there exist constants $L_{(N,1)}, L_{(N,2)}, L_{(N,3)}, l_N > 0$ such that

$$\|N(u_1,v_1,w_1,\lambda) - N(u_2,v_2,w_2,\tilde{\lambda})\| \leq L_{(N,1)}\|u_1-u_2\| + L_{(N,2)}\|v_1-v_2\| + L_{(N,3)}\|w_1-w_2\|$$

$$+ l_N\|\lambda - \tilde{\lambda}\||,\ \forall u_1,u_2,v_1,v_2,w_1,w_2 \in K,\ \lambda,\tilde{\lambda} \in \Omega.$$

Using the technique of Dafermos [3], we consider the mapping $F(\cdot,\lambda) : K \times \Omega \to E$ defined by

$$F(u,\lambda) := u - g(u,\lambda) + P_{\rho}^{\partial_{\eta}\phi(\cdot,u,\lambda)}[P\circ g(u,\lambda) - \rho N(T(u,\lambda),A(u,\lambda),S(u,\lambda),\lambda)]. \quad (4.5)$$

Remark 4.1. It follows from Lemma 4.1 that the fixed point of the mapping F defined by (4.5) is the solution of PGVLIP (3.2).

Now, we show that the mapping $F(u,\lambda)$ defined by (4.5) is a contraction mapping with respect to u uniformly in $\lambda \in \Omega$.

Theorem 4.1. Let E be a real uniformly smooth Banach space with $\rho_E(q) \leq cq^2$ for some $c > 0$. Let the mapping $g : K \times \Omega \to E$ be locally β-strongly accretive and locally (L_g, l_g)-mixed Lipschitz continuous. Let $T, A, S : K \times \Omega \to E^*$ be the mappings such that T, A and S are locally Lipschitz continuous in the first argument with constants L_T, L_A and L_S, respectively; let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(u,v) + \eta(v,u) = 0$, for all $u, v \in E$ and let $P : E \to E^*$ be δ-strongly η-monotone continuous mapping; the function $h(v,u) = \langle u^* - P(u), \eta(v,u)\rangle$ be 0-DQCV in v. Let $\phi : E \times E \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, η-subdifferentiable and proper functional such that $g(u,\lambda) \in \partial_\eta \phi(u,v,\lambda)$, for all $u, v \in E,\ \lambda \in \Omega$; let $P \circ g : K \times \Omega \to E^*$ be locally $(L_{P\circ g}, l_{P\circ g})$-mixed Lipschitz continuous. Let $N : E^* \times E^* \times E^* \times \Omega \to E^*$ be locally α-strongly $P \circ g$-accretive with respect to T, A and S, and locally $(L_{(N,1)}, L_{(N,2)}, L_{(N,3)}, l_N)$-mixed Lipschitz continuous. Suppose that there exist some real constants $k_1 > 0$ and $\rho > 0$ such that

$$\|P_\rho^{\partial_\eta \phi(\cdot,u_1,\lambda)}(z) - P_\rho^{\partial_\eta \phi(\cdot,u_2,\lambda)}(z)\| \leq k_1 \|u_1 - u_2\|,\ \forall u_1, u_2 \in E, z \in E^*, \lambda \in \Omega. \tag{4.6}$$

Then for each $u_1, u_2 \in E,\ \lambda \in \Omega$,

$$\|F(u_1,\lambda) - F(u_2,\lambda)\| \leq \theta \|u_1 - u_2\|, \tag{4.7}$$

where $\theta := l + \frac{\tau}{\delta} t(\rho) \in (0,1)$; $l := k_1 + \sqrt{1 - 2\beta + 64cL_g^2}$;

$t(\rho) := \sqrt{L_{P\circ g}^2 - 2\rho\alpha + 64c\rho^2 L_N^2}$; $L_N := (L_{(N,1)} L_T + L_{(N,2)} L_A + L_{(N,3)} L_S)$, that is,

F is θ-contraction uniformly in $\lambda \in \Omega$.

Proof. For all $u_1, u_2 \in E,\ \lambda \in \Omega$, using condition(4.6), locally $(L_{P\circ g}, l_{P\circ g})$-mixed Lipschitz continuity of $P \circ g$ and locally L_T-Lipschitz continuity of T, we have

$$\|F(u_1,\lambda) - F(u_2,\lambda)\|$$

$$= \|u_1 - g(u_1,\lambda) + P_\rho^{\partial_\eta \phi(\cdot,u_1,\lambda)}[P \circ g(u_1,\lambda) - \rho N(T(u_1,\lambda), A(u_1,\lambda), S(u_1,\lambda), \lambda)]$$

$$-\Big[u_2-g(u_2,\lambda)+P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_2,\lambda)-\rho N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)]\Big]\Big\|$$

$$\begin{aligned}
&\le \|u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda))\|\\
&\quad+\Big\|P_\rho^{\partial_\eta\phi(\cdot,u_1,\lambda)}[P\circ g(u_1,\lambda)-\rho N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)]\\
&\qquad-P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_1,\lambda)-\rho N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)]\Big\|\\
&\quad+\Big\|P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_1,\lambda)-\rho N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)]\\
&\qquad-P_\rho^{\partial_\eta\phi(\cdot,u_2,\lambda)}[P\circ g(u_2,\lambda)-\rho N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)]\Big\|\\
&\le \|u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda))\|+k_1\|u_1-u_2\|+\frac{\tau}{\delta}\Big[\Big\|P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)\\
&\quad-\rho[N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)]\Big\|\Big]. \qquad (4.8)
\end{aligned}$$

Using Lemma 2.3, locally β-strongly accretiveness and locally (L_g,l_g)-mixed Lipschitz continuity of g, we have

$$\begin{aligned}
&\|u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda))\|^2\\
&\le \|u_1-u_2\|^2-2\langle g(u_1,\lambda)-g(u_2,\lambda),J(u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda)))\rangle\\
&\le \|u_1-u_2\|^2-2\langle g(u_1,\lambda)-g(u_2,\lambda),J(u_1-u_2)\rangle\\
&\quad+2\langle g(u_1,\lambda)-g(u_2,\lambda),J(u_1-u_2)-J(u_1-u_2-(g(u_1,\lambda)-g(u_2,\lambda)))\rangle\\
&\le (1-2\beta)\|u_1-u_2\|^2+64c\|g(u_1,\lambda)-g(u_2,\lambda)\|^2\\
&\le (1-2\beta+64cL_g^2)\|u_1-u_2\|^2. \qquad (4.9)
\end{aligned}$$

Since N is locally α-strongly $P\circ g$-accretive with respect to T, A and S, and locally $(L_{(N,1)},L_{(N,2)},L_{(N,3)},l_N)$-mixed Lipschitz continuous; T,A,S are locally L_T-Lipschitz continuous, locally L_A-Lipschitz continuous and locally L_S-Lipschitz continuous, respectively, we have

$$\begin{aligned}
&\|N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)\|\\
&\le L_{(N,1)}\|T(u_1,\lambda)-T(u_2,\lambda)\|+L_{(N,2)}\|A(u_1,\lambda)-A(u_2,\lambda)\|+L_{(N,3)}\|S(u_1,\lambda)-S(u_2,\lambda)\|\\
&\le L_{(N,1)}L_T\|u_1-u_2\|+L_{(N,2)}L_A\|u_1-u_2\|+L_{(N,3)}L_S\|u_1-u_2\|\\
&\le (L_{(N,1)}L_T+L_{(N,2)}L_A+L_{(N,3)}L_S)\|u_1-u_2\|. \qquad (4.10)
\end{aligned}$$

Since $P\circ g$ is locally $(L_{P\circ g}, l_{P\circ g})$-mixed Lipschitz continuous, using Lemma 2.3, we have

$$\|P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)-\rho[N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)]\|^2$$

$$\leq \|P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)\|^2$$

$$-2\rho\langle N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda),J^*(P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda))\rangle$$

$$+2\rho\langle N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda),J^*(P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda))$$

$$-J^*(P\circ g(u_1,\lambda)-P\circ g(u_2,\lambda)-\rho[N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)])\rangle$$

$$\leq (L^2_{P\circ g}-2\rho\alpha)\|u_1-u_2\|^2$$

$$+64c\rho^2\|N(T(u_1,\lambda),A(u_1,\lambda),S(u_1,\lambda),\lambda)-N(T(u_2,\lambda),A(u_2,\lambda),S(u_2,\lambda),\lambda)\|^2. \tag{4.11}$$

Now, from (4.8)-(4.11), we have

$$\|F(u_1,\lambda)-F(u_2,\lambda)\| \leq \theta\|u_1-u_2\|, \tag{4.12}$$

where $\theta := l+\frac{\tau}{\delta}t(\rho)$; $l := k_1+\sqrt{1-2\beta+64cL_g^2}$;

$t(\rho) := \sqrt{L^2_{P\circ g}-2\rho\alpha+64c\rho^2L_N^2}$; $L_N := (L_{(N,1)}L_T+L_{(N,2)}L_A+L_{(N,3)}L_S)$.

Next, we have to show that $\theta<1$. It is clear that $t(\rho)$ assumes its minimum value for $\tilde{\rho} = \alpha/64cL_N^2$ with $t(\tilde{\rho}) = \sqrt{L^2_{P\circ g}-(\alpha^2/64cL_N^2)}$, where $L_N := (L_{(N,1)}L_T+L_{(N,2)}L_A+L_{(N,3)}L_S)$. For $\rho=\tilde{\rho}$, $l+\frac{\tau}{\delta}t(\rho)<1 \to l<1$, then it follows that $\theta<1$ for all ρ satisfying (4.7). Hence, it follows that F defined by (4.5) is a θ-contraction mapping uniformly in $\lambda\in\Omega$. Therefore, invoking Banach contraction principle, F admits a unique fixed point, say $u(\lambda)$, which in turn is a solution of PGVLIP (3.2). This completes the proof.

Remark 4.2. From Theorem 4.1, it is clear that the mapping F defined by (4.5) has a unique fixed point $u(\lambda)$, that is, $u(\lambda)=F(u,\lambda)$.

It also follows from our assumption that the function $\tilde{u}$ for $\lambda=\tilde{\lambda}$ is a solution of PGVLIP (3.2). Again by using Theorem 4.1, we observe that for $\lambda=\tilde{\lambda}$, $\tilde{u}$ is a fixed point of $F(u,\lambda)$ and it is a fixed point of $F(u,\tilde{\lambda})$. Consequently, we conclude that

$$u(\tilde{\lambda})=\tilde{u}=F(u(\tilde{\lambda}),\tilde{\lambda}). \tag{4.13}$$

Next, using Theorem 4.1, we show the Lipschitz continuity of the solution $u(\lambda)$ of PGVLIP (3.2).

Theorem 4.2. Let the mappings $T, A, S, N, P, g, h, P \circ g$ be same as in Theorem 4.1 and let conditions (4.6)-(4.7) of Theorem 4.1 hold. Suppose that $\lambda \to P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}$ is k_2-Lipschitz continuous at $\lambda = \tilde{\lambda}$, then the function $u(\lambda)$ is Lipschitz continuous at $\lambda = \tilde{\lambda}$.

Proof. For all $\lambda, \tilde{\lambda} \in \Omega$, using (4.13) and Theorem 4.1, we have

$$\begin{aligned}\|u(\lambda) - u(\tilde{\lambda})\| &= \|F(u(\lambda), \lambda) - F(u(\tilde{\lambda}), \tilde{\lambda})\| \\ &\le \|F(u(\lambda), \lambda) - F(u(\tilde{\lambda}), \lambda)\| + \|F(u(\tilde{\lambda}), \lambda) - F(u(\tilde{\lambda}), \tilde{\lambda})\| \\ &\le \theta \|u(\lambda) - u(\tilde{\lambda})\| + \|F(u(\tilde{\lambda}), \lambda) - F(u(\tilde{\lambda}), \tilde{\lambda})\|, \end{aligned} \tag{4.14}$$

where θ is given by (4.7). Using (4.5), Lemma 2.2 and the conditions on mappings $T, A, S, N, P, g, h, P \circ g$ and $P_\rho^{\partial_\eta \phi(\cdot,u,\lambda)}$, we have

$$\begin{aligned}&\|F(u(\tilde{\lambda}), \lambda) - F(u(\tilde{\lambda}), \tilde{\lambda})\| \\ &= \Big\| u(\tilde{\lambda}) - g(u(\tilde{\lambda}), \lambda) + P_\rho^{\partial_\eta \phi(\cdot,u(\tilde{\lambda}),\lambda)}[P \circ g(u(\tilde{\lambda}), \lambda) - \rho N(T(u(\tilde{\lambda})), A(u(\tilde{\lambda})), S(u(\tilde{\lambda})), \lambda)] \\ &\quad - \Big[u(\tilde{\lambda}) - g(u(\tilde{\lambda}), \tilde{\lambda}) + P_\rho^{\partial_\eta \phi(\cdot,u(\tilde{\lambda}),\tilde{\lambda})}[P \circ g(u(\tilde{\lambda}), \tilde{\lambda}) - \rho N(T(u(\tilde{\lambda})), A(u(\tilde{\lambda})), S(u(\tilde{\lambda})), \tilde{\lambda})]\Big] \Big\| \\ &\le l_g \|\lambda - \tilde{\lambda}\| + k_2 \|\lambda - \tilde{\lambda}\| + \frac{\tau}{\delta}\Big[\|P \circ g(u(\tilde{\lambda}), \lambda) - P \circ g(u(\tilde{\lambda}), \tilde{\lambda})\| \\ &\quad + \rho \|N(T(u(\tilde{\lambda})), A(u(\tilde{\lambda})), S(u(\tilde{\lambda})), \lambda) - N(T(u(\tilde{\lambda})), A(u(\tilde{\lambda})), S(u(\tilde{\lambda})), \tilde{\lambda})\Big\|\Big] \\ &\le (l_g + k_2)\|\lambda - \tilde{\lambda}\| + \frac{\tau}{\delta}[l_{P\circ g}\|\lambda - \tilde{\lambda}\| + \rho l_N \|\lambda - \tilde{\lambda}\|] \\ &\le \Big[l_g + k_2 + \frac{(l_{P\circ g} + \rho l_N)\tau}{\delta}\Big]\|\lambda - \tilde{\lambda}\|. \end{aligned} \tag{4.15}$$

Combining (4.14) and (4.15), we have

$$\|u(\lambda) - u(\tilde{\lambda})\| = \theta \|u(\lambda) - u(\tilde{\lambda})\| + \Big[l_g + k_2 + \frac{(l_{P\circ g} + \rho l_N)\tau}{\delta}\Big]\|\lambda - \tilde{\lambda}\|, \tag{4.16}$$

which implies

$$\|u(\lambda) - u(\tilde{\lambda})\| \le \Big[\frac{(l_g + k_2)\delta + (l_{P\circ g} + \rho l_N)\tau}{\delta(1-\theta)}\Big]\|\lambda - \tilde{\lambda}\|. \tag{4.17}$$

Since $\theta \in (0,1)$, by the condition (4.7), $\theta_1 := ((l_g + k_2)\delta + (l_{P\circ g} + \rho l_N)\tau)/\delta(1-\theta) > 0$. Hence, it follows from (4.17) that $u(\lambda)$ is θ_1-lipschitz continuous at $\lambda = \tilde{\lambda}$. This completes the proof.

Remark 4.3. Since the PGVLIP (3.2) includes many known classes of parametric variational inequalities as special cases, Theorems 4.1-4.2 improve and generalize the known results given in [3,6,11,14,17-20].

Conflict of Interests

The authors declare that there is no conflict of interests.

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