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A COUNTERPART TO JENSEN-MERCER INEQUALITY

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Abstract. The main goal of this paper is to point out some refinements of the reverse of the Jensen-Mercer inequality.

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1. INTRODUCTION

Throughout this paper, for $\alpha, \beta, a, b \in \mathbb{R}$ we always assume $-\infty \leq \alpha < a < b < \beta \leq \infty$. Let $f : (\alpha, \beta) \to \mathbb{R}$ be a convex function. Then for each $x \in (\alpha, \beta)$ there exist $f'_-(x)$ and $f'_+(x)$ and $f'_-(x) \leq f'_+(x)$ (see [5]). Hence, without any loss of generality we may set $f'(x) = f'_+(x)$ for any $x \in (\alpha, \beta)$.

The Jensen-Mercer inequality

$$f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1}$$

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for convex function $f:(\alpha,\beta)\to\mathbb{R}$, real numbers $x_1,\ldots,x_n\in[a,b]$ and positive real numbers $p_1,\ldots,p_n$, where $P_n=\sum_{i=1}^{n}p_i$, was proved in [4]. In [1], it was proved that it remains valid when $x_1,\ldots,x_n\in[a,b]$ and $p_1,\ldots,p_n\in\mathbb{R}$ satisfy the conditions

$$x_1\le x_2\le\cdots\le x_n \quad\text{or}\quad x_1\ge x_2\ge\cdots\ge x_n \tag{2}$$

and

$$0\le P_k=\sum_{i=1}^{k}p_i\le P_n,\ k=1,\ldots,n,\quad P_n>0. \tag{3}$$

Also, under conditions (2) and (3), $\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i$ belongs to $[a,b]$, and consequently $\bar{x}=a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\in[a,b]$.

Furthermore, in [2], under conditions (2) and (3), a reverse of the Jensen-Mercer inequality was obtained in the following form

$$\begin{aligned}0&\le f(a)+f(b)-\frac{1}{P_n}\sum_{i=1}^{n}p_if(x_i)-f(\bar{x})\\&\le f'(a)(a-\bar{x})+f'(b)(b-\bar{x})-\frac{1}{P_n}\sum_{i=1}^{n}p_if'(x_i)(x_i-\bar{x}).\end{aligned} \tag{4}$$

Our goal is to establish refinements of the second inequality in (4).

2. Main results

In [3], the following reverse of the discrete Jensen-Steffensen inequality and its refinements were proved.

Theorem A. *Let $f:(\alpha,\beta)\to\mathbb{R}$ be a convex function and suppose that $\xi_1,\ldots,\xi_m\in[a,b]$, $w_1,\ldots,w_m\in\mathbb{R}$ satisfy conditions*

$$\xi_1\le\xi_2\le\cdots\le\xi_m \quad\text{or}\quad \xi_1\ge\xi_2\ge\cdots\ge\xi_m \tag{5}$$

and

$$0\le W_k=\sum_{i=1}^{k}w_i\le W_m,\ k=1,\ldots,m,\quad W_m>0. \tag{6}$$

Then

$$
\begin{aligned}
0 &\le \overline{\eta} - f\left(\overline{\xi}\right) \\
&\le \inf_{\xi \in (a,b)} \left(f(\xi) - \xi\, \overline{\zeta} \right) + \frac{1}{W_m} \sum_{i=1}^{m} w_i \xi_i f'(\xi_i) - f\left(\overline{\xi}\right) \\
&\le \frac{1}{W_m} \sum_{i=1}^{m} w_i \xi_i f'(\xi_i) - \overline{\xi}\, \overline{\zeta}, \qquad (7)
\end{aligned}
$$

where

$$
\overline{\xi} = \frac{1}{W_m} \sum_{i=1}^{m} w_i \xi_i, \quad \overline{\eta} = \frac{1}{W_m} \sum_{i=1}^{m} w_i f(\xi_i), \quad \overline{\zeta} = \frac{1}{W_m} \sum_{i=1}^{m} w_i f'(\xi_i).
$$

Theorem B. *Suppose that all the conditions of Theorem A are satisfied and additionally assume that f is strictly convex and differentiable on (α, β). Then*

$$
\begin{aligned}
0 &\le \overline{\eta} - f\left(\overline{\xi}\right) \\
&\le \frac{1}{W_m} \sum_{i=1}^{m} w_i \xi_i f'(\xi_i) + f\left((f')^{-1} \left(\overline{\zeta}\right)\right) - \overline{\zeta}\, (f')^{-1} \left(\overline{\zeta}\right) - f\left(\overline{\xi}\right) \\
&\le \frac{1}{W_m} \sum_{i=1}^{m} w_i \xi_i f'(\xi_i) - \overline{\xi}\, \overline{\zeta}. \qquad (8)
\end{aligned}
$$

In the following theorems we show how inequalities (7) and (8) can be used to obtain refinements of the reverse of the Jensen-Mercer inequality, under different conditions on weights $p_1, \ldots, p_n$ and arguments $x_1, \ldots, x_n$.

Theorem 1. *Let $f : (\alpha, \beta) \to \mathbb{R}$ be a convex function and $x_1, \ldots, x_n \in [a,b]$, $p_1, \ldots, p_n \in \mathbb{R}$ be such that conditions (2) and (3) are fulfilled. Then*

$$
\begin{aligned}
0 &\le f(a) + f(b) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f(\overline{x}) \\
&\le \inf_{x \in (a,b)} (f(x) - x\overline{z}) + a f'(a) + b f'(b) - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i f'(x_i) - f(\overline{x}) \\
&\le f'(a)(a - \overline{x}) + f'(b)(b - \overline{x}) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f'(x_i)(x_i - \overline{x}), \qquad (9)
\end{aligned}
$$

where $\overline{x} = a + b - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$ and $\overline{z} = f'(a) + f'(b) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f'(x_i)$.

Proof. For $m = n + 2$ we define

$$\begin{array}{llllll} \xi_1 = a, & \xi_2 = x_1, & \xi_3 = x_2, & \dots & \xi_{m-1} = x_n, & \xi_m = b \\ w_1 = 1, & w_2 = -\frac{p_1}{P_n}, & w_2 = -\frac{p_2}{P_n}, & \dots & w_{m-1} = -\frac{p_n}{P_n}, & w_m = 1 \end{array}. \tag{10}$$

It is obvious that $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_m$ if $x_1 \leq x_2 \leq \cdots \leq x_n$ or $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_m$ if $x_1 \geq x_2 \geq \cdots \geq x_n$ and that

$$0 \leq W_k = \sum_{i=1}^{k} w_i \leq W_m, \ \ k = 1, 2, \dots, m, \quad W_m = 1 > 0.$$

Hence, we can apply Theorem A thus obtaining inequalities (9). □

In the same way, by applying Theorem B, we prove the following theorem.

Theorem 2. *Suppose that all the conditions of Theorem 1 are satisfied and additionally assume that f is strictly convex and differentiable on (α, β). Then*

$$\begin{aligned} 0 &\leq f(a) + f(b) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f(\bar{x}) \\ &\leq a f'(a) + b f'(b) - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i f'(x_i) + f\left(\left(f'\right)^{-1}(\bar{z})\right) - \bar{z}\left(f'\right)^{-1}(\bar{z}) - f(\bar{x}) \\ &\leq f'(a)(a - \bar{x}) + f'(b)(b - \bar{x}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f'(x_i)(x_i - \bar{x}), \end{aligned} \tag{11}$$

where $\bar{x} = a + b - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ *and* $\bar{z} = f'(a) + f'(b) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f'(x_i)$.

Theorem 3. *Let $f : (\alpha, \beta) \to \mathbb{R}$ be a convex function, $p_1, \dots, p_n$ positive real numbers and $x_1, \dots, x_n \in [a, b]$. Then inequalities (9) hold.*

Proof. When $p_1, \dots, p_n$ are positive, condition (3) is permutation invariant, that is, in that case (3) does not depend on the order of $p_1, \dots, p_n$. Because of that, we can take any $x_1, \dots, x_n \in [a, b]$ and rearrange them in the way that, after substitutions (10), conditions (5) and (6) are fulfilled. Hence, we can apply Theorem A. □

Analogously, we can apply Theorem B and prove the following theorem.

Theorem 4. *Let $f : (\alpha, \beta) \to \mathbb{R}$ be a differentiable strictly convex function, $p_1, \dots, p_n$ positive real numbers and $x_1, \dots, x_n \in [a, b]$. Then inequalities (11) hold.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S. Abramovich, M. Klaričić Bakula, M. Matić, J. Pečarić, A variant of Jensen-Steffensen's inequality and quasi-arithmetic means, J. Math. Anal. Appl. 307 (3) (2005), 370-376.

[2] M. Klaričić Bakula, M. Matić, J. Pečarić, On some general inequalities related to Jensen's inequality, Inequalities and Applications, Basel: Birkhauser Verlag AG, 2009. p. 233-243

[3] M. Klaričić Bakula, M. Matić, J. Pečarić, Generalizations of the Jensen-Steffensen and related inequalities, Cent. Eur. J. Math. 7 (4) (2009), 787-803.

[4] A. McD. Mercer, A Variant of Jensen's Inequality, J. Inequal. Pure Appl. Math. 4 (4) (2003), Article 73.

[5] A.W. Roberts, D.E. Varberg, Convex Functions, Academic Press, New York-London, 1973