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A p-ANALOGUE OF THE EXPONENTIAL INTEGRAL FUNCTION AND SOME PROPERTIES

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Abstract. In this paper, we introduced a p-analogue of the exponential integral function and further establish some analytical inequalities involving the function. We employ the Holder's inequality for integrals, the Minkowski's inequality for integral and the Young's inequality for scalars.

Keywords: p-analogue of the exponential integral function; Holder's inequality; Minkowski's inequality; Young's inequality.

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1. INTRODUCTION

The exponential integral was introduced by Legendry in 1811 and was later coined with the Ei notation [1]. The function occurs in a wide variety of application. Examples of applications are cited from diffusion theory and transport problems and the study of the radiative equilibrium of Steller atmosphere [2].

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The exponential integral function is defined as [3]

$$E_1(x)=\int_x^{\infty}\frac{e^{-t}}{t}\;dt \qquad x\in R. \tag{1}$$

It is defined in terms of the Cauchy Principal value due to the singularity of the integrand at zero [3] as

$$Ei(x)=-\int_{-x}^{\infty}\frac{e^{-t}}{t}\;dt=\int_{-\infty}^{x}\frac{e^{t}}{t}\;dt \qquad x>0. \tag{2}$$

The above function should not be confused with $E_1(x)$ because the Risch algorithm shows that $Ei(x)$ is not an elementary function [4]. The two functions are closely related as follows.

$$E_1(-x)=-Ei(x) \qquad x>0. \tag{3}$$

In this paper, our focus is on the usual exponential integral function defined by Schloemich in [5] as

$$E_n(x)=\int_1^{\infty}t^{-n}e^{-tx}\;dt \qquad x>0, n\in N. \tag{4}$$

The following differential equations hold from (4)

$$\frac{d}{dx}E_n(x)=-E_{n-1}(x) \tag{5}$$

and more generally,

$$\frac{d^m}{dx^m}E_n(x)=(-1)^m E_{n-m}(x). \tag{6}$$

The recurrence relation, deduced from equation (4) by means of a suitable integration by parts, is as follows,

$$E_{n+1}(x)=\frac{1}{n}[e^{-t}-xE_n(x)] \tag{7}$$

which generalizes the well-known results when n is an integer.
Other special values of particular interest are the following

$$E_n(0)=\begin{cases}\frac{1}{n-1}, & n>1\\ \infty, & (-\infty<n\leqslant 1)\end{cases} \tag{8}$$

Thus, $E_0(0)=\infty$, $E_1(0)=\infty$, $E_2(0)=1$, $E_3(0)=\frac{1}{2}$, $E_4(0)=\frac{1}{3}$ etc.
The exponential integral function has attracted the attention of several researchers and it has

been investigated in diverse ways (see [6], [7], [8], [9], [10], [11] and the related references therein).

2. Preliminaries

We begin with the following well known results(see for instance [12], [13], [14] or [15]).

Lemma 2.1. *(Holder's Inequality) Let* $p,q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$*. If* $f(t)$ *and* $g(t)$ *are continuous real-valued functions on* $[a,b]$*, then inequality*

$$\int_a^b |f(t)g(t)|dt \leq \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q dt\right)^{\frac{1}{q}}, \tag{9}$$

holds. With equality when $|g(t)|=c|f(t)|^{p-1}$*. If* $p=q=2$*, the inequality becomes Schwarz's Inequality.*

Lemma 2.2. *(Minkowski's Inequality) Let* $p>1$*. If* $f(t)$ *and* $g(t)$ *are continuous real-valued functions on* $[a,b]$*, then inequality*

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}, \tag{10}$$

holds.

Lemma 2.3. *(Young's Inequality) Let* $a,b>0$*,* $p,q>1$*, and* $\frac{1}{p}+\frac{1}{q}=1$*. Then inequality*

$$ab \leq \frac{a^p}{p}+\frac{b^q}{q}, \tag{11}$$

holds.

3. Main results

Definition 3.1. Let $x>0$, $p\in\mathbb{R}^+$, $n\in\mathbb{N}_0$. Then the p-analogue of the exponential integral is defined as

$$E_{n,p}(x) = \int_1^p t^{-n} A_p^{-xt} dt, \tag{12}$$

where, $E_{n,p}(x) \longrightarrow E_n(x)$ as $p\longrightarrow\infty$ and $A_p=(1+\frac{1}{p})^p$.

Lemma 3.2. *The recursive relation*

$$E_{n,p}(x) = \ln A_p^{-x}\left[A_p^{-x} - p^{-n}A_p^{-px} - nE_{n+1,p}(x)\right], \tag{13}$$

holds for $n \in \mathbb{N}_0$.

Proof. Using (12) and by means of integration by parts, we have

$$\begin{aligned}
E_{n,p}(x) &= \int_1^p t^{-n}A_p^{-xt}dt \\
&= \left[-\frac{t^{-n}A_p^{-xt}}{\ln A_p^x}\right]_1^p - \frac{n}{\ln A_p^x}\int_1^p t^{-(n+1)}A_p^{-xt}dt \\
&= -\frac{p^{-n}A_p^{-px}}{\ln A_p^x} + \frac{A_p^{-x}}{\ln A_p^x} - \frac{n}{\ln A_p^x}E_{n+1,p}(x) \\
&= \frac{A_p^{-x}}{\ln A_p^x} - \frac{p^{-n}A_p^{-px}}{\ln A_p^x} - \frac{n}{\ln A_p^x}E_{n+1,p}(x) \\
&= \frac{1}{\ln A_p^x}\left[A_p^{-x} - p^{-n}A_p^{-px} - nE_{n+1,p}(x)\right] \\
&= \ln A_p^{-x}\left[A_p^{-x} - p^{-n}A_p^{-px} - nE_{n+1,p}(x)\right],
\end{aligned}$$

which completes the proof.

Theorem 3.3. *Let* $n \in \mathbb{N}_0$, $\eta > 1$, $p \in \mathbb{R}^+$. *Then, the inequality*

$$E_{n,p}\left(\frac{x}{\eta} + \frac{y}{\mu}\right) \le \left(E_{n,p}(x)\right)^{\frac{1}{\eta}}\left(E_{n,p}(y)\right)^{\frac{1}{\mu}}, \tag{14}$$

holds for $x, y > 0$ *and* $\frac{1}{\eta} + \frac{1}{\mu} = 1$.

Proof. Using (12) and Hölder's inequality for integrals, we have

$$\begin{aligned}
E_{n,p}\left(\frac{x}{\eta} + \frac{y}{\mu}\right) &= \int_1^p t^{-n}A_p^{-\left(\frac{x}{\eta}+\frac{y}{\mu}\right)t}dt \\
&= \int_1^p t^{-n\left(\frac{1}{\eta}+\frac{1}{\mu}\right)}A_p^{-\left(\frac{x}{\eta}+\frac{y}{\mu}\right)t}dt \\
&= \int_1^p t^{-\frac{n}{\eta}}A_p^{-\frac{xt}{\eta}}t^{-\frac{n}{\mu}}A_p^{-\frac{yt}{\mu}}dt
\end{aligned}$$

$$\leq \left(\int_1^p \left(t^{-\frac{n}{\eta}} A_p^{-\frac{xt}{\eta}}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}}\right)^{\mu} dt\right)^{\frac{1}{\mu}}$$
$$= \left(\int_1^p t^{-n} A_p^{-xt} dt\right)^{\frac{1}{\eta}} \left(\int_1^p t^{-n} A_p^{-yt} dt\right)^{\frac{1}{\mu}}$$
$$= (E_{n,p}(x))^{\frac{1}{\eta}} (E_{n,p}(y))^{\frac{1}{\mu}}$$

which completes the proof.

Theorem 3.4. *Let $p \in \mathbb{R}^+$ and $m,n \in \mathbb{N}_0$ such that $\eta m, \mu n \in \mathbb{N}_0$. Then, the inequality*

$$E_{m+n,p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) \leq (E_{\eta m,p}(x))^{\frac{1}{\eta}} \left(E_{\mu n,p}(y)\right)^{\frac{1}{\mu}}, \tag{15}$$

holds for $x,y>0$, $\eta>1$ and $\frac{1}{\eta}+\frac{1}{\mu}=1$.

Proof. Using (12) and Hölder's inequality for integrals, we have

$$\begin{aligned} E_{m+n,p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) &= \int_1^p t^{-(m+n)} A_p^{-\left(\frac{x}{\eta}+\frac{y}{\mu}\right)t} dt \\ &= \int_1^p t^{-m} A_p^{-\frac{xt}{\eta}} t^{-n} A_p^{-\frac{yt}{\mu}} dt \\ &\leq \left(\int_1^p \left(t^{-m} A_p^{-\frac{xt}{\eta}}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{-n} A_p^{-\frac{yt}{\mu}}\right)^{\mu} dt\right)^{\frac{1}{\mu}} \\ &= \left(\int_1^p t^{-\eta m} A_p^{-xt} dt\right)^{\frac{1}{\eta}} \left(\int_1^p t^{-\mu n} A_p^{-yt} dt\right)^{\frac{1}{\mu}} \\ &= (E_{\eta m,p}(x))^{\frac{1}{\eta}} \left(E_{\mu n,p}(y)\right)^{\frac{1}{\eta}} \end{aligned}$$

which completes the proof.

Corollary 3.5. *Let $m,n \in \mathbb{N}_0$, $p \in \mathbb{R}^+$. Then, the inequality*

$$\left(E_{m+n,p}\left(\frac{x+y}{2}\right)\right)^2 \leq E_{2m,p}(x) E_{2n,p}(y), \tag{16}$$

holds for $x,y>0$.

Proof. This follows from Theorem 3.4 by letting $\eta=\mu=2$.

Theorem 3.6. *Let $p \in \mathbb{R}^+$, $m,n \in \mathbb{N}_0$ such that $\frac{m}{\eta} + \frac{n}{\mu} \in \mathbb{N}_0$. Then, the inequality*

$$E_{\frac{m}{\eta}+\frac{n}{\mu},p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) \leq \left(E_{m,p}(x)\right)^{\frac{1}{\eta}}\left(E_{n,p}(y)\right)^{\frac{1}{\mu}}, \tag{17}$$

holds for $\eta > 1$, $x,y > 0$, $\frac{1}{\eta}+\frac{1}{\mu} = 1$.

Proof. Using (12) and Hölder's inequality for integrals, we have

$$\begin{aligned}
E_{\frac{m}{\eta}+\frac{n}{\mu},p}\left(\frac{x}{\eta}+\frac{y}{\mu}\right) &= \int_1^p t^{-\left(\frac{m}{\eta}+\frac{n}{\mu}\right)} A_p^{-(\frac{x}{\eta}+\frac{y}{\mu})t}\,dt \\
&= \int_1^p t^{-\frac{m}{\eta}} A_p^{-\frac{xt}{\eta}} t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}}\,dt \\
&\leq \left(\int_1^p \left(t^{-\frac{m}{\eta}} A_p^{-\frac{xt}{\eta}}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{-\frac{n}{\mu}} A_p^{-\frac{yt}{\mu}}\right)^{\mu} dt\right)^{\frac{1}{\mu}} \\
&= \left(\int_1^p t^{-m} A_p^{-xt}\,dt\right)^{\frac{1}{\eta}} \left(\int_1^p t^{-n} A_p^{-yt}\,dt\right)^{\frac{1}{\mu}} \\
&= \left(E_{m,p}(x)\right)^{\frac{1}{\eta}}\left(E_{n,p}(y)\right)^{\frac{1}{\mu}}
\end{aligned}$$

which completes the proof.

Corollary 3.7. *Let $m,n \in N_0$, $p \in R^+$. Then, the inequality*

$$\left(E_{\frac{m+n}{2},p}\left(\frac{x+y}{2}\right)\right)^2 = E_{m,p}(x)E_{n,p}(y), \tag{18}$$

holds for $x,y > 0$.

Proof. This follows from Theorem 3.6 by letting $\eta = \mu = 2$.

Theorem 3.8. *Let $m,n \in \mathbb{N}_0$, $\alpha \in \mathbb{Z}^+$, and $p \in \mathbb{R}^+$. Then, the inequality*

$$\left[E_{m,p}(x)+E_{n,p}(y)\right]^{\frac{1}{\alpha}} \leq \left[E_{m,p}(x)\right]^{\frac{1}{\alpha}} + \left[E_{n,p}(y)\right]^{\frac{1}{\alpha}}, \tag{19}$$

holds for $x,y > 0$.

Proof. Using (12), the Minkowski's inequality for integrals and $a^{\alpha}+b^{\alpha} \leq (a+b)^{\alpha}$, for $a,b \geq 0$ and $\alpha \in \mathbf{Z}^{+}$, we have

$$\begin{aligned}
[E_{m,p}(x)+E_{n,p}(y)]^{\frac{1}{\alpha}} &= \left[\int_1^p t^{-m}A_p^{-xt}dt + \int_1^p t^{-n}A_p^{-yt}dt\right]^{\frac{1}{\alpha}} \\
&= \left[\int_1^p \left(\left(t^{-\frac{m}{\alpha}}A_p^{-\frac{xt}{\alpha}}\right)^{\alpha} + \left(t^{-\frac{n}{\alpha}}A_p^{-\frac{yt}{\alpha}}\right)^{\alpha}\right) dt\right]^{\frac{1}{\alpha}} \\
&\leq \left[\int_1^p \left(\left(t^{-\frac{m}{\alpha}}A_p^{-\frac{xt}{\alpha}}\right) + \left(t^{-\frac{n}{\alpha}}A_p^{-\frac{yt}{\alpha}}\right)\right)^{\alpha} dt\right]^{\frac{1}{\alpha}} \\
&\leq \left[\int_1^p \left[t^{-\frac{m}{\alpha}}A_p^{-\frac{xt}{\alpha}}\right]^{\alpha} dt\right]^{\frac{1}{\alpha}} + \left[\int_1^p \left[t^{-\frac{n}{\alpha}}A_p^{-\frac{yt}{\alpha}}\right]^{\alpha} dt\right]^{\frac{1}{\alpha}} \\
&= \left[\int_1^p t^{-m}A_p^{-xt}dt\right]^{\frac{1}{\alpha}} + \left[\int_1^p t^{-n}A_p^{-yt}dt\right]^{\frac{1}{\alpha}} \\
&= [E_{m,p}(x)]^{\frac{1}{\alpha}} + [E_{n,p}(y)]^{\frac{1}{\alpha}}
\end{aligned}$$

which completes the proof.

Theorem 3.9. *Let $n \in \mathbb{N}_0$ and $p \in \mathbb{R}^{+}$. Then, the inequality*

$$E_{n,p}(xy) \geq E_{n,p}^{\frac{1}{\eta}}\left(\frac{\eta x^{q_1}}{q_1}\right) E_{n,p}^{\frac{1}{\mu}}\left(\frac{\mu y^{q_2}}{q_2}\right), \tag{20}$$

holds for $x>0$, $y>0$, $0<\eta<1$, $q_1>1$, $\frac{1}{q_1}+\frac{1}{q_2}=1$ and $\frac{1}{\eta}+\frac{1}{\mu}=1$.

Proof. Using (12), the reverse Hölder's inequality for integrals, the Young's inequality and the fact that $E_{n,p}(x)$ is decreasing, we have

$$\begin{aligned}
E_{n,p}(xy) \geq E_{n,p}\left(\frac{x^{q_1}}{q_1}+\frac{y^{q_2}}{q_2}\right) &= \int_1^p t^{-n}A_p^{-\left(\frac{x^{q_1}}{q_1}+\frac{y^{q_2}}{q_2}\right)t}dt \\
&= \int_1^p t^{-n\left(\frac{1}{\eta}+\frac{1}{\mu}\right)}A_p^{-\left(\frac{x^{q_1}}{q_1}+\frac{y^{q_2}}{q_2}\right)t}dt \\
&= \int_1^p \left(t^{-\frac{n}{\eta}}A_p^{-\frac{x^{q_1}t}{q_1}}t^{-\frac{n}{\mu}}A_p^{-\frac{y^{q_2}t}{q_2}}\right)dt
\end{aligned}$$

$$\geq \left(\int_1^p \left(t^{-\frac{n}{\eta}} A_p^{-\frac{x^{q_1} t}{q_1}}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{-\frac{n}{\mu}} A_p^{-\frac{y^{q_2} t}{q_2}}\right)^{\mu} dt\right)^{\frac{1}{\mu}}$$
$$= \left(\int_1^p t^{-n} A_p^{-\frac{\eta x^{q_1} t}{q_1}} dt\right)^{\frac{1}{\eta}} \left(\int_1^p t^{-n} A_p^{-\frac{\mu y^{q_2} t}{q_2}} dt\right)^{\frac{1}{\mu}}$$
$$= E_{n,p}^{\frac{1}{\eta}}\left(\frac{\eta x^{q_1}}{q_1}\right) E_{n,p}^{\frac{1}{\mu}}\left(\frac{\mu y^{q_2}}{q_2}\right)$$

which completes the proof.

Theorem 3.10. *Let $n \in N_0$, and $p \in \mathbb{R}^+$. Then, the inequality*

$$E_{n,p}(xy) \geq (E_{n,p}(\eta x))^{\frac{1}{\eta}} (E_{n,p}(\mu y))^{\frac{1}{\mu}}, \tag{21}$$

holds for $x > 0$, $0 < y < 1$, $0 < \eta < 1$, $\frac{1}{\eta} + \frac{1}{\mu} = 1$ and $x + y \geq xy$.

Proof. Using (12), the reverse Holder's inequality for integrals and the fact that $E_{n,p}(x)$ is decreasing for $x > 0$, we have

$$E_{n,p}(xy) \geq E_{n,p}(x+y) = \int_1^p t^{-n\left(\frac{1}{\eta}+\frac{1}{\mu}\right)} A_p^{-(x+y)t} dt$$
$$= \int_1^p t^{-\frac{n}{\eta}} A_p^{-xt} t^{-\frac{n}{\mu}} A_p^{-yt} dt$$
$$\geq \left(\int_1^p \left(t^{-\frac{n}{\eta}} A_p^{-xt}\right)^{\eta} dt\right)^{\frac{1}{\eta}} \left(\int_1^p \left(t^{-\frac{n}{\mu}} A_p^{-yt}\right)^{\mu} dt\right)^{\frac{1}{\mu}}$$
$$= \left(\int_1^p t^{-n} A_p^{-\eta xt} dt\right)^{\frac{1}{\eta}} \left(\int_1^p t^{-n} A_p^{-\mu yt} dt\right)^{\frac{1}{\mu}}$$
$$= (E_{n,p}(\eta x))^{\frac{1}{\eta}} (E_{n,p}(\mu y))^{\frac{1}{\mu}}$$

which completes the proof.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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