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INEQUALITIES FOR THE RATIONAL FUNCTIONS WITH PRESCRIBED POLES AND RESTRICTED ZEROS

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Abstract. In this paper we shall consider the moduli of all the zeros of $r(z)$ instead of maximum modulus of zeros of $r(z)$ and present a refinement of some results. We shall also prove a result of similar nature.

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1. Introduction

Let $\mathscr{P}_n$ be the class of polynomials $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ of degree at most n. Let D_{k-} denotes the region inside the circle $T_k = \{z; |z| = k > 0\}$ and D_{k+} the region outside T_k. For $a_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, we write

$$W(z) = \prod_{j=1}^{n}(z - a_j) \qquad ; \qquad B(z) = \prod_{j=1}^{n}\left(\frac{1 - \overline{a_j} z_j}{z - a_j}\right)$$

and

$\mathscr{R}_n = \mathscr{R}_n(a_1, a_2, \ldots, a_n) = \left\{\frac{P(z)}{W(z)} : P \in \mathscr{P}_n\right\}$,

then $\mathscr{R}_n$ is the set of all rational functions with poles $a_1, a_2, \ldots, a_n$ at most and with finite limit

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at infinity.We observe that $B(z) \in \mathscr{R}_n$. For f defined on T_k in the complex plane , we set $||f|| = \sup_{z \in T_k} |f(z)|$, the Chebyshev norm f on T_1. Throughout this paper, we also assume that all poles $a_1, a_2, ..., a_n$ are in D_{1+}.The following famous result is due to Bernstein [4].

Theorem 1.1. *If $P \in \mathscr{P}_n$ then $||P'|| \leq n||P||$.*

As a refinement of Theorem 1.1, A. Aziz [1] and Malik [6] proved the following:

Theorem 1.2. *If $P \in \mathscr{P}_n$ and $P^*(z) = z^n \overline{P(\frac{1}{\bar{z}})}$ then $|| \quad |(P^*(z))'| + |P(z)| \quad || = n||P(z)||$.*

The following result was conjectured by Erdös and later proved by Lax [5]

Theorem 1.3. *If $P \in \mathscr{P}_n$ and all the zeros of $P(z)$ lie in $T_1 \cup D_{1+}$ then for $z \in T_1$ we have*

$$||P'|| \leq \frac{n}{2}||P(z)||. \tag{1}$$

Equality in (1) *holds for $P(z) = \alpha z^n + \beta$ with $|\alpha| = |\beta|$.*

Li, Mohapatra and Rodriguez [8] have proved Bernstein-type inequalities similar to Theorem 1.1 and Theorem 1.3 for rational functions with prescribed poles where they replaced z^n by Blaschkes product $B(z)$.Among other things they proved the following generalisation of Theorem 1.3:

Theorem 1.4. *Suppose $r \in \mathscr{R}_n$ and all zeros of r lie in $T_1 \cup D_{1+}$, then for $z \in T_1$, we have*

$$|r'(z)| \leq \frac{1}{2}|B'(z)|||r|| \tag{2}$$

Equality in (2) *holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.*

Aziz and Zargar [3] have proved the following generalization of Theorem 1.4.

Theorem 1.5. *Suppose $r \in \mathscr{R}_n$ and all zeros of r lie in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$, we have*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)}{(k+1)}\frac{|r(z)|^2}{||r||^2}\right\}||r|| \tag{3}$$

Equality in (3) *holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$ where $a > 1, k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.*

2. PRELIMINARIES

For the proof of these Theorems we need the following Lemmas. First Lemma is due to Li, Mohapatra and Rodriguez [8]

Lemma 2.1. *If* $r \in \mathscr{R}_n$ *and* $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$ *then for* $z \in T_1$*, we have*

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)|||r|| \tag{4}$$

Equality in (4) *holds in* $r(z) = uB(z)$ *with* $u \in T_1$.

Lemma 2.2. *If* $z \in T_1$*, then*

$$Re\left(\frac{zW'(z)}{W(z)}\right) = \frac{n - |B(z)|}{2}$$

and

$$Re\left(\frac{z(W^*(z))'}{W^*(z)}\right) = \frac{n + |B(z)|}{2}$$

where $W(z) = \prod_{j=1}^{n}(z - a_j)$ *and* $W*(z) = z^n\overline{W\left(\frac{1}{\bar{z}}\right)}$.

This Lemma is due to Aziz and Zargar [3].

3. MAIN RESULTS

Now instead of considering the maximum modulus of zeros of $r(z)$ we shall consider the moduli of all the zeros of $r(z)$ and prove the following refinement of Theorem 1.5.

Theorem 3.1. *Suppose* $r(z) = \frac{P(z)}{W(z)} \in \mathscr{R}_n$. *If* $b_1, b_2, ..., b_m$ *are the zeros of* $r(z)$ *lie in* $T_k \cup D_{k+}$ *where* $k \geq 1$*, then for* $z \in T_1$*, we have*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - 2\left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right)\frac{|r(z)|^2}{||r||^2}\right\}||r||. \tag{5}$$

Proof of Theorem 3.1. Let $r(z) = \frac{P(z)}{W(z)} \in \mathscr{R}_n$. If $b_1, b_2, ..., b_m$ are the zeros of $P(z)$, then $m \leq n$ and $|b_j| \geq k > 1, j = 1, 2, ..., n$ and we have

$$\begin{aligned}\frac{zr'(z)}{r(z)} &= \frac{zP'(z)}{P(z)} - \frac{zW'(z)}{W(z)} \\ &= \sum_{j=1}^{m}\frac{z}{z - b_j} - \frac{zW'(z)}{W(z)}\end{aligned}$$

For $z \in T_1$, this gives with the help of Lemma 2.2, that

$$\begin{aligned} Re\frac{zr'(z)}{r(z)} &= Re\sum_{j=1}^{m}\frac{z}{z-b_j} - Re\frac{zW'(z)}{W(z)} \\ &= Re\sum_{j=1}^{m}\frac{z}{z-b_j} - \left(\frac{n-|B'(z)|}{2}\right) \\ &\leq \sum_{j=1}^{m}\frac{1}{1+|b_j|} - \left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{|B'(z)|}{2} + \left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right) \end{aligned}$$

Hence for $z \in T_1$ we have [[8], p.529],

$$\begin{aligned} \left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|Re\frac{zr'(z)}{r(z)} \\ &\geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|\left\{\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right)\right\} \\ &= \left|\frac{zr'(z)}{r(z)}\right|^2 - 2\left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right)|B'(z)| \end{aligned}$$

This implies for $z \in T_1$,

$$\left\{|r'(z)|^2 - 2\left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right)|r(z)|^2|B'(z)|\right\}^{\frac{1}{2}} \leq |(r^*(z))'| \tag{6}$$

Combining (6) with Lemma 2.1, we get

$$|r'(z)| + \left\{|r'(z)|^2 - 2\left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right)|r(z)|^2|B'(z)|\right\}^{\frac{1}{2}} \leq |B'(z)|||r||$$

or equivalently

$$\begin{aligned} |r'(z)|^2 - 2\left(\sum_{j=1}^{m}\frac{1}{1+|b_j|} - \frac{n}{2}\right)|r(z)|^2|B'(z)| &\leq \left\{|B'(z)|||r|| - |r'(z)|\right\}^2 \\ &= |B'(z)|^2||r||^2 - 2|B'(z)||r'(z)|||r|| + |r'(z)|^2 \end{aligned}$$

which after a simplification yields for $z \in T_1$ that

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| + 2\left(\sum_{j=1}^{m} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\frac{|r(z)|^2}{||r||^2}\right\}||r||$$

this proves Theorem 3.1. □

If $r(z)$ has exactly n zeros in $T_k \cup D_{k+}$ where $k \geq 1$, then we have the following result.

Corollary 3.1. *Suppose $r(z) = \frac{P(z)}{W(z)} \in \mathscr{R}_n$. If $b_1, b_2, \ldots, b_n$ are the zeros of $r(z)$ lie in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$, we have*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \sum_{j=1}^{n}\left(\frac{|b_j|-1}{1+|b_j|}\right)\frac{|r(z)|^2}{||r||^2}\right\}||r||. \tag{7}$$

Remark 3.1. (7) *is refinement of Theorem 1.5. To show this we observe*

$$\frac{(|b_j|-1)}{(|b_j|+1)} \geq \frac{(k-1)}{(k+1)} \quad \text{where} \quad |b_j| \geq k \geq 1 \tag{8}$$

(8) *is true if*

$$(k+1)(|b_j|-1) \geq (k-1)(|b_j|+1), j = 1, 2, \ldots, n$$

which yields that

$$|b_j| \geq k$$

which is clearly true.

Here we shall present the following result which provides a refinement of Theorem 1.5.

Theorem 3.2. *Suppose $r(z) = \frac{P(z)}{W(z)} \in \mathscr{R}_n$, where r has exactly n poles at $a_1, a_2, \ldots, a_n$ and all the zeros of $r(z)$ lie in $T_k \cup D_{k+}$ where $k \geq 1$, then for $z \in T_1$, we have*

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{(n(k+1)-2m)}{(k+1)}\frac{|r(z)|^2}{||r(z)||^2}\right\}||r(z)||. \tag{9}$$

where m is number of zeros of r.

Equality in (9) *holds for $r(z) = \frac{(z+k)^m}{(z-a)^n}$ where $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.*

Proof of Theorem 3.2

Let $r(z)=\frac{P(z)}{W(z)}\in\mathscr{R}_n$. If $b_1,b_2,...,b_m$ are all the zeros of $P(z)$, then $m\leq n$ and $|b_j|\geq k>1$, $j=1,2,...,m$ and we have

$$\begin{aligned}\frac{zr'(z)}{r(z)}&=\frac{zP'(z)}{P(z)}-\frac{zW'(z)}{W(z)}\\&=\sum_{j=1}^{m}\frac{z}{z-b_j}-\frac{zW'(z)}{W(z)}.\end{aligned}$$

For $z\in T_{k_+}$, this gives with the help of Lemma 2.2, that

$$\begin{aligned}Re\frac{zr'(z)}{r(z)}&=Re\sum_{j=1}^{m}\frac{z}{z-b_j}-Re\frac{zW'(z)}{W(z)}\\&=Re\sum_{j=1}^{m}\frac{z}{z-b_j}-\left(\frac{n-|B'(z)|}{2}\right)\end{aligned}\tag{10}$$

Now it can be easily verified that for $z\in T_1$, $|b|>k>1$,

$$Re\left(\frac{z}{z-b}\right)\leq\frac{1}{1+k}$$

Using this in (10), we get for $z\in T_1$

$$\begin{aligned}Re\frac{zr'(z)}{r(z)}&\leq\frac{m}{1+k}-\left(\frac{n-|B'(z)|}{2}\right)\\&=\frac{m+n-n}{1+k}-\left(\frac{n-|B'(z)|}{2}\right)\\&=\frac{n}{1+k}-\left(\frac{n-m}{1+k}+\frac{n-|B'(z)|}{2}\right)\\&=-\left(\frac{n(k-1)}{2(k+1)}+\frac{n-m}{1+k}\right)+\frac{|B'(z)|}{2}\\&=\frac{|B'(z)|}{2}-\frac{1}{1+k}\left(\frac{n(k+1)-2m}{2}\right).\end{aligned}$$

Hence for $z \in T_1$, we have [Li, Mohapatra and Rodriguez [8] p-529]

$$\begin{aligned}\left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - 2|B'(z)|Re\frac{zr'(z)}{r(z)} \\ &\geq |B'(z)|^2 + \left|\frac{zr'(z)}{r(z)}\right|^2 - |B'(z)|\left(|B'(z)| - \frac{n(k+1)-2m}{1+k}\right) \\ &= \left|\frac{zr'(z)}{r(z)}\right|^2 + \frac{n(k+1)-2m}{1+k}|B'(z)|\end{aligned}$$

This implies for $z \in T_1$,

$$\left\{|r'(z)|^2 + \frac{n(k+1)-2m}{k+1}|B'(z)||r(z)|^2\right\}^{\frac{1}{2}} \leq |(r^*(z))'| \tag{11}$$

Combining (11) with Lemma 2.1, we get

$$|r'(z)| + \left\{|r'(z)|^2 + \frac{n(k+1)-2m}{k+1}|B'(z)||r(z)|^2\right\}^{\frac{1}{2}} \leq |B'(z)|||r(z)||$$

or equivalently,

$$\begin{aligned}|r'(z)|^2 + \frac{n(k+1)-2m}{k+1}|B'(z)||r(z)|^2 &\leq \left(|B'(z)|||r(z)|| - |r'(z)|\right)^2 \\ &= |B'(z)|^2||r(z)||^2 - 2|B'(z)||r'(z)|||r(z)|| + |r'(z)|^2.\end{aligned}$$

Which after simplification yields for $z \in T_1$ that

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{(n(k+1)-2m)}{(k+1)}\frac{|r(z)|^2}{||r(z)||^2}\right\}||r(z)||.$$

The desired result follows. □

Remark 3.2. *If $r(z)$ has exactly n zeros in $T_k \cup D_{k+}$, then we get Theorem 1.5.*

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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